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# RECTANGULAR PARTIAL $b$-METRIC SPACES

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**Abstract.** In this article concept of rectangular partial $b$-metric space have been introduced. It is shown that rectangular $b$-metric can be achieved from rectangular partial $b$-metric. Moreover equivalence of completeness of both the spaces have been achieved. An analog to Cantor intersection theorem has been established in such spaces. A variant of Banach fixed point theorem and Kannan fixed point theorem are also proved in the language of rectangular partial $b$-metric spaces.

**Keywords:**rectangular partial $b$-metric; cantor intersection theorem; fixed point.

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## 1. INTRODUCTION

A partial metric is a generalization of metric space by replacing the condition $d(x,x) = 0$ by the condition $d(x,y) \geq d(x,x)$ for all $x,y$, introduced by S. G. Matthews [1]. Later many generalization partial metric space appeared. In this sequel S. Shukla [3] defined partial-$b$ metric space, S. Souayah [6] defined partial $S_b$-metric space, A. Gupta and P. Gautam [7] introduced the concept of quasi partial $b$-metric space and they presented some fixed point theorems in these spaces.

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R. George et. al [4] introduced the concept of rectangular $b$-metric space by replacing triangular inequality by three term expression and proved some fixed point theorems and Bakhtin [8] generalize the concept of metric space and defined $b$-metric space. In this paper we define rectangular partial $b$-metric space generalizing the concept of partial metric spaces and rectangular $b$-metric spaces.

## 2. PRELIMINARIES

Lets begin with some definitions.

**Definition 2.1.** [4] *A mapping $d : X \times X \longrightarrow [0,\infty)$, where $X$ is a non empty set, is said to be rectangular b-metric if whenever $x$, $y$, $z \in X$ the following conditions hold:*

(1) $x = y \Leftrightarrow d(x,y) = 0$;

(2) $d(x,y) = d(y,x)$;

(3) *there exists a real number $s \geq 1$ such that*

$$d(x,y) \leq s[d(x,u) + d(u,v) + d(v,y)] \quad \forall x,y \in X \text{ and } u,v \in X \setminus \{x,y\}$$

*Then $d$ is called a rectangular b-metric and $(X,d)$ is called a rectangular b-metric space with coefficient $s \geq 1$.*

**Definition 2.2.** [4] *In a rectangular b-metric space $(X,d)$*

- *A sequence $\{x_n\}$ in $(X,d)$ is said to be convergent to $x \in X$ such that for any $\varepsilon > 0$, $\exists$ a positive integer $N$ so that $d(x_n,x) < \varepsilon \; \forall \; n \geq N$.*
- *A sequence $\{x_n\}$ in a rectangular b-metric space $(X,d)$ is said to be Cauchy sequence if for any $\varepsilon > 0$, $\exists$ a positive integer $N$ such that $d(x_n,x_m) < \varepsilon \; \forall \; m,n \geq N$.*
- *A rectangular b-metric space is called complete if every Cauchy sequence is convergent therein.*

## 3. MAIN RESULTS

Now we define

**Definition 3.1.** *A mapping $p_b^r : X \times X \longrightarrow [0,\infty)$, where $X$ is a non empty set, is said to be rectangular partial b-metric if whenever $x$, $y$, $z,w \in X$ the following conditions hold:*

(1) $x = y \Leftrightarrow p_b^r(x,x) = p_b^r(x,y) = p_b^r(y,y)$;

(2) $p_b^r(x,y) = p_b^r(y,x)$;

(3) $p_b^r(x,y) \geq p_b^r(x,x)$;

(4) *there exists a real number* $s \geq 1$ *such that*

$$\begin{aligned} p_b^r(x,y) \quad &\leq \quad s[p_b^r(x,z) + p_b^r(z,w) + p_b^r(w,y) - p_b^r(z,z) - p_b^r(w,w)] + \\ &\quad \frac{1-s}{2}[p_b^r(x,x) + p_b^r(y,y)], \ \ z,w \in X \setminus x,y \end{aligned}$$

*and the ordered pair* $(X, p_b^r)$ *is called partial rectangular b-metric space. The number s is called coefficient of* $(X, p_b^r)$.

**Example 3.1.** *Let* $(X,d)$ *be a rectangular metric space. Let* $p_b^r(x,y) = d(x,y)^q + k$ *where* $q > 1$. *Then* $p_b^r$ *is a rectangular partial b-metric space with coefficient* $3^{q-1}$. *Conditions* (1), (2), (3) *satisfied automatically. We now check* (4).

$$\begin{aligned} p_b^r(x,y) &= d(x,y)^q + k \\ &\leq (d(x,w) + d(w,z) + d(z,y))^q + k \\ &\leq 3^{q-1}(d(x,w)^q + d(w,z)^q + d(z,y)^q) + k \\ &= 3^{q-1}(d(x,w)^q + k + d(w,z)^q + k + d(z,y)^q + k - k) - 2(3^{q-1}k) + k \\ &\leq 3^{q-1}(p_b^r(x,w) + p_b^r(w,z) + p_b^r(z,y)) + \frac{1-3^{q-1}}{2}(p_b^r(x,x) + p_b^r(y,y)) \end{aligned}$$

**Definition 3.2.** $(i)$ *A sequence* $\{x_n\}$ *in a rectangular partial b-metric space* $(X, p_b^r)$ *convergent to* $x \in X$ *if* $\lim_{n\to\infty} p_b^r(x_n,x) = p_b^r(x,x) = \lim_{n\to\infty} p_b^r(x_n,x_n)$.

$(ii)$ *A sequence* $\{x_n\}$ *in* $(X, p_b^r)$ *is a Cauchy sequence if* $\lim_{n,m\to\infty} p_b^r(x_n,x_m)$ *exists.*

$(iii)$ *A rectangular partial b-metric space is said to be complete if every Cauchy sequence* $\{x_n\}$ *in* $(X, p_b^r)$ *is convergent.*

We define open ball in $(X, p_b^r)$ by $B_{p_b^r}(x,\varepsilon) = \{y \in$X: $p_b^r(x,y) < p_b^r(x,x) + \varepsilon\}$ and closed ball by $B_{p_b^r}[x,\varepsilon] = \{y \in$X: $p_b^r(x,y) \leq p_b^r(x,x) + \varepsilon\}$.

## 3.1. Relation with Rectangular $b$-metric Spaces.

**Lemma 3.1.** *Let $(X, p_b^r)$ be a rectengular partial b-metric space with coefficient $s \geq 1$. Then $d_{p_b^r}(x,y) = 2p_b^r(x,y) - p_b^r(x,x) - p_b^r(y,y)$ is a rectengular b-metric on X with the same coefficient and a sequence $\{x_n\}$ is convergent to x in $(X, p_b^r)$ iff $\{x_n\}$ is convergent to x in $(X, d_{p_b^r})$.*

*Proof.*

$$\begin{aligned}
d_{p_b^r}(x,y) &= 2p_b^r(x,y) - p_b^r(x,x) - p_b^r(y,y) \\
&\leq 2s[p_b^r(x,z) + p_b^r(z,w) + p_b^r(w,y) - p_b^r(z,z) - p_b^r(w,w)] \\
&\quad +(1-s)(p_b^r(x,x) + p_b^r(y,y)) - p_b^r(x,x) - p_b^r(y,y) \\
&= s[2p_b^r(x,z) - p_b^r(x,x) - p_b^r(z,z)] \\
&\quad +s[2p_b^r(z,w) - p_b^r(z,z) - p_b^r(w,w)] \\
&\quad +s[2p_b^r(w,y) - p_b^r(w,w) - p_b^r(y,y)] \\
&= s[d_{p_b^r}(x,z) + d_{p_b^r}(z,w) + d_{p_b^r}(w,y)]
\end{aligned}$$

Other parts can be easily proved. □

**Theorem 3.1.** *(a) A sequence $\{x_n\}$ is a Cauchy sequence in $(X, p_b^r)$ iff $\{x_n\}$ is a Cauchy sequence in $(X, d_{p_b^r})$.*

*(b) $(X, p_b^r)$ is complete if and only if $(X, d_{p_b^r})$ is complete.*

*Proof.* Let $\{x_n\}$ be Cauchy sequence in $(X, p_b^r)$. So $\lim_{n,m\to\infty} p_b^r(x_n,x_m) = l$. Let $\varepsilon > 0$, then there exists a natural number $M$ such that

$\mid p_b^r(x_n,x_m) - l \mid < \frac{\varepsilon}{4}$ *for all* $n,m \geq M$.

$$\begin{aligned}
Now \mid d_{p_b^r}(x_n,x_m) \mid &= \mid 2p_b^r(x_n,x_m) - p_b^r(x_n,x_n) - p_b^r(x_m,x_m) \mid \\
&= \mid 2p_b^r(x_n,x_m) - 2l - p_b^r(x_n,x_n) + l - p_b^r(x_m,x_m) + l \mid \\
&\leq \mid p_b^r(x_n,x_m) - l \mid + \mid p_b^r(x_n,x_m) - l \mid \\
&\quad + \mid p_b^r(x_n,x_n) - l \mid + \mid p_b^r(x_m,x_m) - l \mid \\
&< \varepsilon \qquad\qquad \text{for all } n,m \geq M.
\end{aligned}$$

So $\{x_n\}$ is a Cauchy sequence in $(X, d_{p_b^r})$.

Conversely let $\{x_n\}$ be a Cauchy sequence in $(X, d_{p_b^r})$. Let $\varepsilon = \frac{1}{2}$, then there exists $n_0 \in \mathbb{N}$ such that

$d_{p_b^r}(x_n, x_m) < \frac{1}{2}$ for all $n, m \geq n_0$.
$\Rightarrow 2p_b^r(x_n, x_{n_0}) - p_b^r(x_{n_0}, x_{n_0}) - p_b^r(x_n, x_n) < \frac{1}{2}$
$\Rightarrow p_b^r(x_n, x_{n_0}) - p_b^r(x_{n_0}, x_{n_0}) < \frac{1}{2}$.
Now $p_b^r(x_n, x_n) < p_b^r(x_n, x_{n_0}) < p_b^r(x_{n_0}, x_{n_0}) + \frac{1}{2}$.
So $\{p_b^r(x_n, x_n)\}$ is a bounded sequence in $\mathbb{R}$. Hence $\lim\limits_{k\to\infty} p_b^r(x_{n_k}, x_{n_k}) = l_1$.
Since $\{x_n\}$ is a Cauchy sequence in $(X, d_{p_b^r})$, for a $\varepsilon > 0$, there exists $n_\varepsilon$ such that

$$d_{p_b^r}(x_n, x_m) < \varepsilon \quad \forall n, m \geq n_\varepsilon$$

Then for all $n, m \geq n_\varepsilon$

$$\begin{aligned} p_b^r(x_n, x_n) - p_b^r(x_m, x_m) &\leq p_b^r(x_n, x_m) - p_b^r(x_m, x_m) \\ &\leq d_{p_b^r}(x_n, x_m) \\ &< \varepsilon. \end{aligned}$$

So $\{p_b^r(x_n, x_n)\}$ is a Cauchy sequence in $\mathbb{R}$.
$\Rightarrow \lim\limits_{n\to\infty} p_b^r(x_n, x_n) = l_1$.

$$\begin{aligned} Now \ \mid p_b^r(x_n, x_m) - l_1 \mid &= \mid p_b^r(x_n, x_m) - p_b^r(x_n, x_n) + p_b^r(x_n, x_n) - l_1 \mid \\ &\leq d_{p_b^r}(x_n, x_m) + \mid p_b^r(x_n, x_n) - l_1 \mid \end{aligned}$$

$\Rightarrow \lim\limits_{n,m\to\infty} p_b^r(x_n, x_m) = l_1$.
So $\{x_n\}$ is a Cauchy sequence in $(X, p_b^r)$.

Now we prove that completeness of $(X, d_{p_b^r})$ implies completeness of $(X, p_b^r)$. Let $\{x_n\}$ be a Cauchy sequence in $(X, p_b^r)$. Then $\{x_n\}$ is Cauchy sequence in $(X, d_{p_b^r})$. Since $(X, d_{p_b^r})$ is complete there exists a point $x \in X$ such that $\lim\limits_{n\to\infty} d_{p_b^r}(x_n, x) = 0$ and by Lemma 3.1 $(X, p_b^r)$ is complete.

Now we prove the converse. Let $(X, p_b^r)$ be complete. We will show that $(X, d_{p_b^r})$ is complete. Let $\{x_n\}$ be a Cauchy sequence in $(X, d_{p_b^r})$. Then $\{x_n\}$ is Cauchy sequence in $(X, p_b^r)$. Since

$(X,p_b^r)$ is complete there exists $y \in X$ such that

$$\lim_{n\to\infty} p_b^r(x_n,y) = p_b^r(y,y) = \lim_{n\to\infty} p_b^r(x_n,x_n).$$

Now Lemma 3.1 implies $\lim_{n\to\infty} d_{p_b^r}(x_n,y) = 0$. Hence $(X,d_{p_b^r})$ is complete. □

**Lemma 3.2.** *Let $(X,p_b^r)$ be a rectangular partial b-metric space with the coefficient $s \geq 1$ and suppose that $\{x_n\}$ and $\{y_n\}$ are convergent to x and y respectively. Then we have $\frac{1}{s}p_b^r(x,y) \leq \lim_{n\to\infty}\inf p_b^r(x_n,y_n) \leq \lim_{n\to\infty}\sup p_b^r(x_n,y_n) \leq sp_b^r(x,x)$.*

## 3.2. Cantor Intersection Theorem.

**Lemma 3.3.** *Let $(X,p_b^r)$ be a rectangular partial b-metric space and A be any subset of X. Then $p_b^r(\overline{A}) \leq sp_b^r(A)$*
*where $p_b^r(A) = \sup\{p_b^r(x,y) - p_b^r(x,x) : \forall x,y \in A\}$.*

*Proof.* Let $x,y \in \overline{A}$, then there exists $\{x_n\}$, $\{y_n\}$ in $A$ such that $\{x_n\}$ converges to $x$ and $\{y_n\}$ converges to $y$. i.e.,

$$\lim_{n\to\infty} p_b^r(x_n,x) = p_b^r(x,x) = \lim_{n\to\infty} p_b^r(x_n,x_n)$$
$$\text{and } \lim_{n\to\infty} p_b^r(y_n,y) = p_b^r(y,y) = \lim_{n\to\infty} p_b^r(y_n,y_n).$$

Let $p_b^r(x,x) \geq p_b^r(y,y)$

Now

$$\begin{aligned} p_b^r(x,y) - p_b^r(x,x) &\leq s[p_b^r(x,x_n) + p_b^r(x_n,y_n) + p_b^r(y_n,y) - p_b^r(x_n,x_n) - p_b^r(y_n,y_n)] \\ &\quad + \frac{1-s}{2}[p_b^r(x,x) + p_b^r(y,y)] - p_b^r(x,x) \\ &\leq s[p_b^r(x,x_n) + p_b^r(x_n,y_n) + p_b^r(y_n,y) - p_b^r(x_n,x_n) - p_b^r(y_n,y_n)] \\ &\quad + \frac{1-s}{2}[2p_b^r(y,y)] - p_b^r(x,x) \end{aligned}$$

Taking limit $n \to \infty$ in the above equation we get

$$\begin{aligned} p_b^r(x,y) - p_b^r(x,x) &\leq s[p_b^r(x,x) + p_b^r(A) + p_b^r(y,y) - p_b^r(x,x)] \\ &\quad + p_b^r(y,y) - sp_b^r(y,y) - p_b^r(x,x) \end{aligned}$$

$\Rightarrow \sup\{p_b^r(x,y) - p_b^r(x,x) : \forall x,y \in \overline{A}\} \leq sp_b^r(A)$.

$\Rightarrow p_b^r(\overline{A}) \leq sp_b^r(A)$.

Similarly if $p_b^r(x,x) < p_b^r(y,y)$ we can show that $p_b^r(\overline{A}) \leq sp_b^r(A)$. □

**Theorem 3.2.** *A ractangular partial b-metric space $(X, p_b^r)$ is complete if and only if for every sequence $\{F_n\}$ of closed sets in $(X, p_b^r)$ satisfying:*

*(a) $F_{n+1} \subset F_n \ \forall n \in \mathbb{N}$ and*

*(b) $p_b^r(F_n) \longrightarrow 0$ as $n \to \infty$.*

*Then $\bigcap\limits_{n=1}^{\infty} F_n$ is singleton.*

*Proof.* Let $\{x_n\}$ be a Cauchy sequence in $(X, p_b^r)$. Then $\lim\limits_{n\to\infty} p_b^r(x_{n+p}, x_n) = \alpha$. i.e. for any $\varepsilon > 0$, $\exists$ a natural number $v$ such that

$\mid p_b^r(x_{n+p}, x_n) - \alpha \mid < \frac{\varepsilon}{2} \ \ \forall n \geq v$.

Let $F_n = \{x_{n+p-1} : p \in \mathbb{N}\}$. Then $F_{n+1} \subset F_n \ \Rightarrow \overline{F_{n+1}} \subset \overline{F_n}$. Now

$$\begin{aligned} \mid p_b^r(x_{n+p}, x_n) - p_b^r(x_n, x_n) \mid &\leq \mid p_b^r(x_{n+p}, x_n) - \alpha \mid + \mid p_b^r(x_n, x_n) - \alpha \mid \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \quad \forall n \geq v. \end{aligned}$$

Also

$$\begin{aligned} \mid p_b^r(x_{n+p}, x_n) - p_b^r(x_{n+p}, x_{n+p}) \mid &\leq \mid p_b^r(x_{n+p}, x_n) - \alpha \mid + \mid p_b^r(x_{n+p}, x_{n+p}) - \alpha \mid \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \quad \forall n \geq v. \end{aligned}$$

So $\lim\limits_{n\to\infty} [p_b^r(x_{n+p}, x_n) - \max\{p_b^r(x_n, x_n), p_b^r(x_{n+p}, x_{n+p})\} : \forall x_{n+p}, x_n \in F_n] = 0$.

$\Rightarrow p_b^r(F_n) \longrightarrow 0$ as $n \to \infty$,

$\Rightarrow p_b^r(\overline{F_n}) \longrightarrow 0$ as $n \to \infty$ [using Lemma 3.3.]

So $\bigcap\limits_{n=1}^{\infty} \overline{F_n} \neq \phi$. Let $x \in \bigcap\limits_{n=1}^{\infty} \overline{F_n} \Rightarrow x \in \overline{F_n} \ \ \forall n \in \mathbb{N}$. Also $x_n \in F_n \subset \overline{F_n}$.

Then $0 \leq p_b^r(x_n, x) - p_b^r(x, x) \leq p_b^r(\overline{F_n})$.

Taking limit and using Sandwitch theorem we get

$$\lim_{n\to\infty} p_b^r(x_n, x) = p_b^r(x, x). \tag{1}$$

Similarly we can show that $0 \leq p_b^r(x,x_n) - p_b^r(x_n,x_n) \leq p_b^r(\overline{F_n})$.

Using Sandwitch theorem we get

$$\lim_{n\to\infty} p_b^r(x_n,x_n) = p_b^r(x,x). \tag{2}$$

From (1) and (2) we have

$\lim_{n\to\infty} p_b^r(x_n,x) = p_b^r(x,x) = \lim_{n\to\infty} p_b^r(x_n,x_n)$.

Hence $(X,p_b^r)$ is complete.

Conversely let $(X,p_b^r)$ be a complete rectangular partial $b$-metric space satisfying condition $(a)$ and $(b)$.

Let us consider $x_n \in F_n \quad \forall n \in \mathbb{N}$. Since $F_{n+1} \subset F_n \Rightarrow x_m \in F_n \ \ \forall m \geq n$. Now

$0 \leq p_b^r(x_n,x_m) - p_b^r(x_n,x_n) \leq p_b^r(F_n)$

and $0 \leq p_b^r(x_m,x_n) - p_b^r(x_m,x_m) \leq p_b^r(F_n)$.

$\Rightarrow 0 \leq 2p_b^r(x_n,x_m) - p_b^r(x_n,x_n) - p_b^r(x_m,x_m) \leq 2p_b^r(F_n)$

$\Rightarrow 0 \leq d_{p_b^r}(x_n,x_m) \leq 2p_b^r(F_n)$

Using condition $(b)$ and by Sandwitch theorem we have $\{x_n\}$ is a Cauchy sequence in $(X,d_{p_b^r})$. Since $(X,p_b^r)$ is complete by Theorem 3.1 we can say that $(X,d_{p_b^r})$ is complete. Hence $\exists\, x \in X$ such that $d_{p_b^r}(x_n,x) \longrightarrow 0$ *as* $n \longrightarrow \infty$. This implies $\{x_n\}$ converges to $x$ in $(X,p_b^r)$. Therefore $x \in F_n$ as $F_n$ is closed in $(X,p_b^r)$. Thus $x \in F_n \ \forall n \in \mathbb{N}$. Let $y \in \bigcap_{n=1}^{\infty} F_n \Rightarrow y \in F_n \ \ \forall n \in \mathbb{N}$.
$\Rightarrow 0 \leq p_b^r(x,y) - p_b^r(x,x) \leq p_b^r(F_n)$. Taking limit and using Sandwitch theorem we get $p_b^r(x,y) = p_b^r(x,x)$. Similarly we can get $p_b^r(x,y) = p_b^r(y,y)$. Hence $p_b^r(x,y) = p_b^r(x,x) = p_b^r(y,y) \Rightarrow x = y$. Thus we have proved $\bigcap_{n=1}^{\infty} F_n$ is singleton. □

## 3.3. Fixed Point Theorems.

**Theorem 3.3.** *Let $(X,p_b^r)$ be a complete rectangular partial b-metric space with coefficient $s \geq 1$ and $T : X \longrightarrow X$ be a mapping satisfying the following condition*

$$p_b^r(Tx,Ty) \leq \lambda\, p_b^r(x,y) \ \forall x,y \in X,\ \lambda \in [0,1). \tag{3}$$

*Then $T$ has a unique fixed point $u \in X$ with $p_b^r(u,u) = 0$.*

*Proof.* First we show that the fixed point of $T$ is unique and if $u$ be a fixed point of $T$ then $p_b^r(u,u)=0$. Let $u,v$ be two distinct fixed point of $T$. i.e., $Tu=u$ and $Tv=v$.

$p_b^r(u,v)=p_b^r(Tu,Tv)\leq \lambda p_b^r(u,v)<p_b^r(u,v)$.

Hence $p_b^r(u,v)=0\Rightarrow u=v$. Therefore $T$ has a unique fixed point.

Since $\lambda\in[0,1)$, we can choose $n_0\in\mathbb{N}$ such that for a given $0<\varepsilon<1$, we have $\lambda^{n_0}<\frac{\varepsilon}{8s}$. Let $T^{n_0}\equiv F$ and $F^kx_0=x_k$ $\forall k\in\mathbb{N}$, where $x_0\in X$. Then for all $x,y\in X$,

$$p_b^r(Fx,Fy)=p_b^r(T^{n_0}x,T^{n_0}y)\leq \lambda^{n_0}p_b^r(x,y) \tag{4}$$

For any $k\in\mathbb{N}$, we have

$$\begin{aligned} p_b^r(x_{k+1},x_k) &= p_b^r(Fx_k,Fx_{k-1}) \\ &\leq \lambda^{n_0}p_b^r(x_k,x_{k-1}) \\ &\leq \lambda^{kn_0}p_b^r(x_1,x_0)\longrightarrow 0 \text{ as } k\longrightarrow\infty. \end{aligned}$$

Similarly, $p_b^r(x_{k+2},x_k)\longrightarrow 0$ as $k\longrightarrow\infty$. So we can choose $l\in\mathbb{N}$ such that

$p_b^r(x_l,x_{l+1})<\frac{\varepsilon}{8s}$ and $p_b^r(x_l,x_{l+2})<\frac{\varepsilon}{8s}$

We show that if $z\in B_{p_b^r}[x_l,\frac{\varepsilon}{2}]$ then $Fz\in B_{p_b^r}[x_l,\frac{\varepsilon}{2}]$.

Let $A=\{y\in X: y\rho x_l\}$. Since $x_l\in B_{p_b^r}[x_l,\frac{\varepsilon}{2}]$, $B_{p_b^r}[x_l,\frac{\varepsilon}{2}]\neq\phi$.

Let $z\in B_{p_b^r}[x_l,\frac{\varepsilon}{2}]$. Then

$$\begin{aligned} p_b^r(Fx_l,Fx_z) &\leq \lambda^{n_0}p_b^r(x_l,x_z) \\ &< \frac{\varepsilon}{8s}p_b^r(x_z,x_l) \\ &\leq \frac{\varepsilon}{8s}[\frac{\varepsilon}{2}+p_b^r(x_l,x_l)] \\ &< \frac{\varepsilon}{8s}[1+p_b^r(x_l,x_l)]. \end{aligned}$$

Therefore

$$
\begin{aligned}
p_b^r(x_l, Fz) &\leq s[p_b^r(x_l, Fx_{l+1}) + p_b^r(Fx_{l+1}, Fx_l) + p_b^r(Fx_l, Fz) - \\
&\quad p_b^r(Fx_{l+1}, Fx_{l+1}) - p_b^r(Fx_l, Fx_l)] \\
&\quad + \frac{1-s}{2}[p_b^r(x_l, x_l) + p_b^r(Fz, Fz)] \\
&< s[2\frac{\varepsilon}{8s} + \frac{\varepsilon}{8s}(1 + p_b^r(x_l, x_l))] \\
&< \frac{\varepsilon}{2} + p_b^r(x_l, x_l).
\end{aligned}
$$

Hence $Fz \in B_{p_b^r}[x_l, \frac{\varepsilon}{2}]$ and consequently $Fz \in A$. Since $x_l \in A$ therefore $Fx_l \in A$. Repeating this above process $F^n x_l \in A$ $\forall n \in \mathbb{N}$. i.e., $x_m \in A \quad \forall m \geq l$. Let $m > n \geq l$ and $n = l + i$. Then

$$
\begin{aligned}
p_b^r(x_n, x_m) &= p_b^r(Fx_{n-1}, Fx_{m-1}) \\
&\leq \lambda^{n_0} p_b^r(x_{n-1}, x_{m-1}) \\
&\leq \lambda^{2n_0} p_b^r(x_{n-2}, x_{m-2}) \\
&\quad . \\
&\quad . \\
&\quad . \\
&\leq \lambda^{in_0} p_b^r(x_{n-i}, x_{m-i}) \\
&< p_b^r(x_l, x_{m-i}) \\
&< \frac{\varepsilon}{2} + p_b^r(x_l, x_l) < \varepsilon.
\end{aligned}
$$

Thus $\{x_n\}$ is a Cauchy sequence in $(X, p_b^r)$. By completeness of $(X, p_b^r)$ there exists $u \in X$ such that

$$
\lim_{n \to \infty} p_b^r(x_n, u) = \lim_{n,m \to \infty} p_b^r(x_n, x_m) = p_b^r(u, u) = 0 \tag{5}
$$

For all $n \in \mathbb{N}$,

$$\begin{aligned} p_b^r(u,Fu) &\leq s[p_b^r(u,x_n)+p_b^r(x_n,x_{n+1})+p_b^r(x_{n+1},Fu) \\ &\quad -p_b^r(x_n,x_n)-p_b^r(x_{n+1},x_{n+1})] \\ &\leq s[p_b^r(u,x_n)+p_b^r(x_n,x_{n+1})+\lambda^{n_0}p_b^r(x_n,u)] \end{aligned}$$

Using equation(4) and (5) we have $p_b^r(u,Fu)=0$. Hence $Fu=u$. i.e., $T^{n_0}u=u$. Since $\{T^n u\}$ is a Cauchy sequence with $\lim_{n,m\to\infty} p_b^r(u_n,u_m)=0$, we have $Tu=u$. □

**Example 3.2.** *Let $X=\{0,1,2,3\}$ and define $p_b^r : X\times X \longrightarrow [0,\infty)$by*

$$p_b^r(x,y)=\begin{cases} x^2 & \text{if } x=y\neq 0 \\ 2(x^2+y^2) & \text{if } x,y\notin\{2,3\}, x\neq y \\ x^2+y^2 & \text{if } x,y\in\{2,3\}, x\neq y \\ \frac{1}{2} & \text{if } x=y=0 \end{cases}$$

*Then $(X,p_b^r)$ is a rectangular partial b-metric space with coefficient $s=2$. Define $T : X\longrightarrow X$ by $T0=0$, $T1=0$, $T2=1$, $T3=1$. Then $T$ satisfies the condition of Theorem 3.3 and 0 is the unique fixed point of $T$.*

**Theorem 3.4.** *Let $(X,p_b^r)$ be a complete rectangular partial-b metric space with coefficient $s\geq 1$ and $T : X\longrightarrow X$ be a mapping satisfying the following condition*

$$p_b^r(Tx,Ty)\leq \lambda[p_b^r(x,Tx)+p_b^r(y,Ty)] \tag{6}$$

*for all $x,y\in X$, where $\lambda\in[0,\frac{1}{s+1})$. Then $T$ has a unique fixed point $u\in X$ with $p_b^r(u,u)=0$.*

*Proof.* Let $x_0\in X$ and define a sequence $x_{n+1}=Tx_n$ $\forall n\in\mathbb{N}\cup\{0\}$.

Let $p_n=p_b^r(x_n,x_{n+1})$. From condition (6) it follows that

$p_b^r(x_n,x_{n+1})=p_b^r(Tx_{n-1},Tx_n)\leq\lambda[p_b^r(x_{n-1},x_n)+p_b^r(x_n,x_{n+1})]$

$\Rightarrow p_n\leq\lambda[p_{n-1}+p_n]$

$\Rightarrow p_n\leq\frac{\lambda}{1-\lambda}p_{n-1}=kp_{n-1}$ where $k=\frac{\lambda}{1-\lambda}$.

Proceeding in this way we have

$$p_n\leq k^n p_0$$

Now,

$$
\begin{aligned}
p_b^r(x_n,x_{n+2}) &= p_b^r(Tx_{n-1},Tx_{n+1}) \\
&\leq \lambda[p_b^r(x_{n-1},Tx_{n-1})+p_b^r(x_{n+1},Tx_{n+1})] \\
&= \lambda[p_b^r(x_{n-1},x_n)+p_b^r(x_{n+1},x_{n+2})] \\
&= \lambda[p_{n-1}+p_{n+1}] \\
&\leq \lambda[k^{n-1}p_0+k^{n+1}p_0] \\
&= k^{n-1}\lambda[1+k^2]p_0 \\
&= tk^{n-1}p_0.
\end{aligned}
$$

Where $t=\lambda[1+k^2]$. Now we show that $\lim_{n\to\infty} p_b^r(x_n,x_{n+p})=0$. Here we consider two cases. First when $p$ is odd, say $p=2m+1$. Then

$$
\begin{aligned}
p_b^r(x_n,x_{n+2m+1}) &\leq s[p_b^r(x_n,x_{n+1})+p_b^r(x_{n+1},x_{n+2})+p_b^r(x_{n+2},x_{n+2m+1}) \\
&\quad -p_b^r(x_{n+1},x_{n+1})-p_b^r(x_{n+2},x_{n+2})] \\
&\quad +\frac{1-s}{2}[p_b^r(x_n,x_n)+p_b^r(x_{n+2m+1},x_{n+2m+1})] \\
&\leq s[p_b^r(x_n,x_{n+1})+p_b^r(x_{n+1},x_{n+2})+p_b^r(x_{n+2},x_{n+2m+1})] \\
&\leq s[p_b^r(x_n,x_{n+1})+p_b^r(x_{n+1},x_{n+2})]+ \\
&\quad s^2[p_b^r(x_{n+2},x_{n+3})+p_b^r(x_{n+3},x_{n+4})+p_b^r(x_{n+4},x_{n+2m+1})] \\
&\leq s[p_n+p_{n+1}]+s^2[p_{n+2}+p_{n+3}]+s^3[p_{n+4}+p_{n+5}]+\ldots \\
&\quad +s^m p_{n+2m} \\
&\leq s[k^n+k^{n+1}]p_0+s^2[k^{n+2}+k^{n+3}]p_0+s^3[k^{n+4}+k^{n+5}]p_0+\ldots \\
&\quad +s^m k^{n+2m}p_0 \\
&\leq sk^n[1+sk^2+s^2k^4+\ldots]p_0+sk^{n+1}[1+sk^2+s^2k^4+\ldots]p_0 \\
&= sk^n\frac{1+k}{1-sk^2}p_0
\end{aligned}
$$

Thus

$$p_b^r(x_n, x_{n+2m+1}) \leq sk^n \frac{1+k}{1-sk^2} p_0$$

Now let $p$ is even. i.e., $p = 2m$

$$\begin{aligned}
p_b^r(x_n, x_{n+2m}) &\leq s[p_b^r(x_n, x_{n+1}) + p_b^r(x_{n+1}, x_{n+2}) + p_b^r(x_{n+2}, x_{n+2m}) \\
&\quad - p_b^r(x_{n+1}, x_{n+1}) - p_b^r(x_{n+2}, x_{n+2})] \\
&\quad + \frac{1-s}{2}[p_b^r(x_n, x_n) + p_b^r(x_{n+2m}, x_{n+2m})] \\
&\leq s[p_b^r(x_n, x_{n+1}) + p_b^r(x_{n+1}, x_{n+2}) + p_b^r(x_{n+2}, x_{n+2m})] \\
&\leq s[p_b^r(x_n, x_{n+1}) + p_b^r(x_{n+1}, x_{n+2})] + \\
&\quad s^2[p_b^r(x_{n+2}, x_{n+3}) + p_b^r(x_{n+3}, x_{n+4}) + p_b^r(x_{n+4}, x_{n+2m})] \\
&\leq s[p_n + p_{n+1}] + s^2[p_{n+2} + p_{n+3}] + s^3[p_{n+4} + p_{n+5}] + \ldots \\
&\quad + s^{m-1} p_b^r(x_{n+2m-2}, x_{2m}) \\
&\leq s[k^n + k^{n+1}]p_0 + s^2[k^{n+2} + k^{n+3}]p_0 + s^3[k^{n+4} + k^{n+5}]p_0 + \ldots \\
&\quad + s^{m-1}[k^{2m-4} + k^{2m-3}]p_0 + s^{m-1} t k^{2m-3} p_0 \\
&\leq sk^n[1 + sk^2 + s^2k^4 + \ldots]p_0 + sk^{n+1}[1 + sk^2 + s^2k^4 + \ldots]p_0 \\
&\quad + s^{m-1} t k^{2m-3} p_0 \\
&= sk^n \frac{1+k}{1-sk^2} p_0 + s^{m-1} t k^{2m-3} p_0
\end{aligned}$$

Thus

$$p_b^r(x_n, x_{n+2m}) \leq sk^n \frac{1+k}{1-sk^2} p_0 + s^{m-1} t k^{n+2m-3} p_0$$

Hence $\lim_{n,m\to\infty} p_b^r(x_n, x_m) = 0$. i.e., $\{x_n\}$ is a Cauchy sequence in $(X, p_b^r)$. By completeness of $(X, p_b^r)$ there exists $u \in X$ such that

$$\lim_{n\to\infty} p_b^r(x_n, u) = \lim_{n,m\to\infty} p_b^r(x_n, x_m) = p_b^r(u, u) = 0 \tag{7}$$

Finally we show that $u$ is a fixed point of $T$.

$$\begin{aligned} p_b^r(u,Tu) &\leq s[p_b^r(u,x_n)+p_b^r(x_n,x_{n+1})+p_b^r(x_{n+1},Tu)-p_b^r(x_n,x_n) \\ &\quad -p_b^r(x_{n+1},x_{n+1})]+\frac{1-s}{2}[p_b^r(u,u)+p_b^r(Tu,Tu)] \\ &\leq s[p_b^r(u,x_n)+p_b^r(x_n,x_{n+1})+p_b^r(Tx_n,Tu)] \\ &\leq s[p_b^r(u,x_n)+p_b^r(x_n,x_{n+1})+\lambda p_b^r(u,Tu)+\lambda p_b^r(x_n,Tx_n)] \end{aligned}$$

Taking Limit we have $p_b^r(u,Tu)=0$. Hence $Tu=u$. The uniqueness of the fixed point $u$ follows from the contraction principle. □

## 4. Conclusion

There are some mappings which fails to form a metric for assuming nonzero values in its diagonal of domain or not satisfying triangular inequality. Motivated by the study of S. G. Matthews, I. A. Bakhtin, S. Shukla for these types of mappings an attempt have been made to generalize both the concept of partial metric spaces and rectangular *b*-metric spaces and introduced the concept of rectangular partial *b*-metric spaces. A connection with rectangular *b*-metric spaces have been pointed out. Moreover analog to Cantor intersection theorem, Banach and Kannan fixed point theorem have been studied in the defined spaces.

### Conflict of Interests

The author(s) declare that there is no conflict of interests.

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