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AN ADAPTIVE LEAST SQUARES MIXED FINITE ELEMENT METHOD FOR TWO DIMENSIONAL VISCOELASTIC PROBLEMS

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Abstract. A least squares mixed finite element (LSMFE) method for the numerical solution of two dimensional viscoelastic problems is analyzed and developed in this paper. *A posteriori* error estimator which is needed in the adaptive refinement algorithm is proposed. The local evaluation of the least squares functional serves as *a posteriori* error estimator. The posteriori errors are effectively estimated.

Keywords: adaptive method; least squares mixed finite element; two dimensional viscoelastic problems; least squares functional; *a posteriori* error; adaptive algorithm.

2000 AMS Subject Classification: 65N30, 35Q10

1. Introduction

A general theory of the least squares method has been developed by A K Aziz, R B Kellogg and A B Stephens in [1]. The most important advantage leads to a symmetric positive definite problem. In the least squares mixed finite element approach, a least

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squares residual minimization is introduced. This method has an advantage which is not subject to the LBB condition. The mixed finite element methods of least squares type have been the object of many studies recently (see, e.g. Stokes Equation[2], Elliptic Problem[3], Newtonian Fluid Flow Problem[4], Transmission Problems[5] *et al.*). The adaptive least squares mixed finite element method have been studied in recent several years (see, e.g. the linear elasticity[6]), but the research of adaptive method about two dimensional viscoelastic problems is not common.

Adaptive methods are now widely used in the scientific computation. In this paper, we are interested in the adaptive least squares mixed finite element method for two dimensional viscoelastic problems, two dimensional viscoelastic problems are fundamental partial differential equations. It occurs in various areas of applied mathematics and science. Our emphasis in this paper is on the performance of an adaptive refinement strategy based on the *a posteriori* error estimator inherent in the least squares formulation by the local evaluation of the functional. During the last $15-20$ years a big amount of work has been devoted to *a posteriori* error estimation problem, i.e., computing reliable bounds on the error of given numerical approximation to the solution of partial differential equations using only numerical solution and the given data. In order to be operating the *a posteriori* error estimator should be neither under nor overestimate the error. The *a posteriori* error is effectively estimated, and proved the convergence of the adaptive least squares mixed finite element method in this paper.

An outline of the paper is as follows. The least squares formulation of two dimensional viscoelastic problems is described in Section 2. It includes continuous and coercivity properties of the least squares variational formulation. Appropriate spaces for the finite element approximation and a generalization of the coercivity shown in Section 2 to the discrete form is discussed in Section 3. In Section 4, *a posteriori* error estimators which are needed in an adaptive refinement algorithm are composed with the least squares functional, and posteriori errors are effectively estimated. The adaptive algorithm is described in Section 5. Finally, we summarize our findings and present conclusions in Section 6. In this paper, we define C to be a generic positive constant.

2. A Least Squares Formulation of Two Dimensional Viscoelastic Problems

We start from the equations of two dimensional viscoelastic problems in the form:

$$\begin{cases} u_{tt} - \nabla \cdot (a(x,t)\nabla u_t + b(x,t)\nabla u) = 0, in\ \Omega \times (0,T) \\ u(x,t) = 0, on\ \partial\Omega \times (0,T) \\ u(x,0) = u_0(x),\ x \in \Omega \\ u_t(x,0) = u_1(x),\ x \in \Omega \end{cases} \tag{2.1}$$

where $\Omega \subset R^n$ is a bounded domain, with boundary $\partial\Omega$, $\forall T > 0$, $u_t = \frac{\partial u}{\partial t}$, $u_{tt} = \frac{\partial^2 u}{\partial t^2}$. $a(x,t), b(x,t)$ are bounded functions for $x \in \Omega, t \in (0,T)$, the functions $a(x,t), b(x,t)$ are assumed to be C^2. The inner product is denoted by $(\cdot,\cdot)_{0,\Omega}$. The description of viscoelastic problems are practical problems such as the heat conduction, the nuclear reaction dynamics, viscoelastic mechanics, biomechanics, the pressure on the porous media and so on.

We shall consider an adaptive least squares mixed finite element method for (2.1). Now we set $\nabla u = \sigma$, then, we have:

$$\begin{cases} u_{tt} - \nabla \cdot (a(x,t)\sigma_t + b(x,t)\sigma) = 0, in\ \Omega \times (0,T) \\ \nabla u - \sigma = 0, in\ \Omega \times (0,T) \\ u(x,t) = 0, on\ \partial\Omega \times (0,T) \\ u(x,0) = u_0(x),\ x \in \Omega \\ u_t(x,0) = u_1(x).\ x \in \Omega \end{cases} \tag{2.2}$$

we know, the first equation of (2.2) is equivalent to

$$u_{tt} - (a'(x,t)\sigma_t + a(x,t)\nabla\sigma_t + b'(x,t)\sigma + b(x,t)\nabla\sigma) = 0. \tag{2.3}$$

We introduce the Sobolev spaces:

$$H^1(\Omega) = \{p \in L^2(\Omega) : \nabla p \in L^2(\Omega)^2\},$$

$$H_0^m(\Omega) = \{v \in H^m(\Omega) : D^\alpha v|_{\partial\Omega} = 0, |\alpha| < m\}.$$

Let $U(\Omega) = H_0^1(\Omega) \cap C(\Omega)$, $Q(\Omega) = H^1(\Omega) \cap C(\Omega)$.

Now, let us define the least squares problem: find $(\sigma, u)\in Q(\Omega)\times U(\Omega)$ such that

$$J(\sigma, u) = \inf_{q\in Q(\Omega), v\in U(\Omega)} J(q, v), \tag{2.4}$$

where

$$\begin{aligned} J(q, v) = (&v_{tt} - (a'(x,t)q_t + a(x,t)\nabla q_t + b'(x,t)q + b(x,t)\nabla q), \\ &v_{tt} - (a'(x,t)q_t + a(x,t)\nabla q_t + b'(x,t)q + b(x,t)\nabla q))_{0,\Omega} \\ &+(q - \nabla v, q - \nabla v)_{0,\Omega}. \end{aligned} \tag{2.5}$$

We introduce the least squares functional:

$$\begin{aligned} \mathscr{F}(\sigma, u) = &\|u_{tt} - (a'(x,t)\sigma_t + a(x,t)\nabla\sigma_t + b'(x,t)\sigma + b(x,t)\nabla\sigma)\|_{0,\Omega}^2 \\ &+\|\nabla u - \sigma\|_{0,\Omega}^2. \end{aligned} \tag{2.6}$$

Taking variations in (2.4) with respect to q and v, the weak statement becomes: find $(\sigma, u)\in Q(\Omega)\times U(\Omega)$ such that

$$\mathscr{B}(\sigma, u; q, v) = 0, \ (\forall v \in U(\Omega), \forall q \in Q(\Omega))$$

where

$$\begin{aligned} \mathscr{B}(\sigma, u; q, v) = (&u_{tt} - (a'(x,t)\sigma_t + a(x,t)\nabla\sigma_t + b'(x,t)\sigma + b(x,t)\nabla\sigma), \\ &v_{tt} - (a'(x,t)q_t + a(x,t)\nabla q_t + b'(x,t)q + b(x,t)\nabla q))_{0,\Omega} \\ &+(\nabla u - \sigma, q - \nabla v)_{0,\Omega}. \end{aligned} \tag{2.7}$$

Theorem 2.1. The bilinear form $\mathscr{B}(\cdot,\cdot;\cdot,\cdot)$ is continuous and coercive. In other words, there exist positive constants α and β, such that

$$\begin{aligned} \mathscr{B}(\sigma, u; q, v)\le\beta(&\|u_{tt}\|_{0,\Omega}^2 + \|\sigma_t\|_{0,\Omega}^2 + \|\nabla\sigma_t\|_{0,\Omega}^2 + \|\sigma\|_{0,\Omega}^2 + \|\nabla\sigma\|_{0,\Omega}^2 \\ &+\|\nabla u\|_{0,\Omega}^2)^{\frac{1}{2}}(\|v_{tt}\|_{0,\Omega}^2 + \|q_t\|_{0,\Omega}^2 + \|\nabla q_t\|_{0,\Omega}^2 + \|q\|_{0,\Omega}^2 \\ &+\|\nabla q\|_{0,\Omega}^2 + \|\nabla v\|_{0,\Omega}^2)^{\frac{1}{2}}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} \mathscr{B}(q, v; q, v)\ge\alpha(&\|v_{tt}\|_{0,\Omega}^2 + \|q_t\|_{0,\Omega}^2 + \|\nabla q_t\|_{0,\Omega}^2 + \|q\|_{0,\Omega}^2 + \|\nabla q\|_{0,\Omega}^2 \\ &+\|\nabla v\|_{0,\Omega}^2), \end{aligned} \tag{2.9}$$

holds for all $(\sigma, u), (q, v)\in Q(\Omega)\times U(\Omega)$.

Proof: i) For the upper bound we have:

$$\begin{aligned}
\mathscr{B}(q,v;q,v) &= (v_{tt} - (a'(x,t)q_t + a(x,t)\nabla q_t + b'(x,t)q + b(x,t)\nabla q),\\
&\quad v_{tt} - (a'(x,t)q_t + a(x,t)\nabla q_t + b'(x,t)q + b(x,t)\nabla q))_{0,\Omega}\\
&\quad +(q-\nabla v, q-\nabla v)_{0,\Omega}\\
&= \|v_{tt} - (a'(x,t)q_t + a(x,t)\nabla q_t + b'(x,t)q + b(x,t)\nabla q)\|^2_{0,\Omega}\\
&\quad +\|q-\nabla v\|^2_{0,\Omega}\\
&\le C(\|v_{tt}\|^2_{0,\Omega} + \|q_t\|^2_{0,\Omega} + \|\nabla q_t\|^2_{0,\Omega} + \|q\|^2_{0,\Omega} + \|\nabla q\|^2_{0,\Omega} + \|\nabla v\|^2_{0,\Omega}).
\end{aligned}$$

Since the bilinear form is symmetric, this is sufficient for the upper bound in Theorem 2.1.

ii) For the lower bound.

$$\begin{aligned}
\mathscr{B}(q,v;q,v) &= (v_{tt} - (a'(x,t)q_t + a(x,t)\nabla q_t + b'(x,t)q + b(x,t)\nabla q),\\
&\quad v_{tt} - (a'(x,t)q_t + a(x,t)\nabla q_t + b'(x,t)q + b(x,t)\nabla q))_{0,\Omega}\\
&\quad +(q-\nabla v, q-\nabla v)_{0,\Omega}\\
&= (v_{tt}, v_{tt})_{0,\Omega} + (a'(x,t)q_t + a(x,t)\nabla q_t + b'(x,t)q + b(x,t)\nabla q,\\
&\quad a'(x,t)q_t + a(x,t)\nabla q_t + b'(x,t)q + b(x,t)\nabla q)_{0,\Omega}\\
&\quad -2(v_{tt}, a'(x,t)q_t + a(x,t)\nabla q_t + b'(x,t)q + b(x,t)\nabla q)_{0,\Omega}\\
&\quad +(q,q)_{0,\Omega} + (\nabla v, \nabla v)_{0,\Omega} - 2(q, \nabla v)_{0,\Omega}
\end{aligned}$$

$$
\begin{aligned}
= \ & (v_{tt}, v_{tt})_{0,\Omega} + (a'(x,t)q_t, a'(x,t)q_t)_{0,\Omega} + (a'(x,t)q_t, a(x,t)\nabla q_t)_{0,\Omega} \\
& +(a'(x,t)q_t, b'(x,t)q)_{0,\Omega} + (a'(x,t)q_t, b(x,t)\nabla q)_{0,\Omega} \\
& +(a(x,t)\nabla q_t, a'(x,t)q_t)_{0,\Omega} + (a(x,t)\nabla q_t, a(x,t)\nabla q_t)_{0,\Omega} \\
& +(a(x,t)\nabla q_t, b'(x,t)q)_{0,\Omega} + (a(x,t)\nabla q_t, b(x,t)\nabla q)_{0,\Omega} \\
& +(b'(x,t)q, a'(x,t)q_t)_{0,\Omega} + (b'(x,t)q, a(x,t)\nabla q_t)_{0,\Omega} \\
& +(b'(x,t)q, b'(x,t)q)_{0,\Omega} + (b'(x,t)q, b(x,t)\nabla q)_{0,\Omega} \\
& +(b(x,t)\nabla q, a'(x,t)q_t)_{0,\Omega} + (b(x,t)\nabla q, a(x,t)\nabla q_t)_{0,\Omega} \\
& +(b(x,t)\nabla q, b'(x,t)q)_{0,\Omega} + (b(x,t)\nabla q, b(x,t)\nabla q)_{0,\Omega} + (q,q)_{0,\Omega} \\
& +(\nabla v, \nabla v)_{0,\Omega} - 2(v_{tt}, a'(x,t)q_t)_{0,\Omega} - 2(v_{tt}, a(x,t)\nabla q_t)_{0,\Omega} \\
& -2(v_{tt}, b'(x,t)q)_{0,\Omega} - 2(v_{tt}, b(x,t)\nabla q)_{0,\Omega} - 2(q, \nabla v)_{0,\Omega} \\
= \ & (v_{tt}, v_{tt})_{0,\Omega} + (a'(x,t)q_t, a'(x,t)q_t)_{0,\Omega} + (a(x,t)\nabla q_t, a(x,t)\nabla q_t)_{0,\Omega} \\
& +(b'(x,t)q, b'(x,t)q)_{0,\Omega} + (b(x,t)\nabla q, b(x,t)\nabla q)_{0,\Omega} \\
& +2(a'(x,t)q_t, a(x,t)\nabla q_t)_{0,\Omega} + 2(a'(x,t)q_t, b'(x,t)q)_{0,\Omega} \\
& +2(a'(x,t)q_t, b(x,t)\nabla q)_{0,\Omega} + 2(a(x,t)\nabla q_t, b'(x,t)q)_{0,\Omega} \\
& +2(a(x,t)\nabla q_t, b(x,t)\nabla q)_{0,\Omega} + 2(b'(x,t)q, b(x,t)\nabla q)_{0,\Omega} + (q,q)_{0,\Omega} \\
& +(\nabla v, \nabla v)_{0,\Omega} - 2(v_{tt}, a'(x,t)q_t)_{0,\Omega} - 2(v_{tt}, a(x,t)\nabla q_t)_{0,\Omega} \\
& -2(v_{tt}, b'(x,t)q)_{0,\Omega} - 2(v_{tt}, b(x,t)\nabla q)_{0,\Omega} - 2(q, \nabla v)_{0,\Omega} \\
\geq \ & \|v_{tt}\|_{0,\Omega}^2 + \varepsilon\|q_t\|_{0,\Omega}^2 + \varepsilon\|\nabla q_t\|_{0,\Omega}^2 + (1+\varepsilon)\|q\|_{0,\Omega}^2 + \varepsilon\|\nabla q\|_{0,\Omega}^2 + \varepsilon\|\nabla v\|_{0,\Omega}^2 \\
& -C(\|q_t\|_{0,\Omega}^2 + \|\nabla q_t\|_{0,\Omega}^2 + \|q\|_{0,\Omega}^2 + \|\nabla q\|_{0,\Omega}^2) - (\delta_1+\delta_2+\delta_3+\delta_4)\|v_{tt}\|_{0,\Omega}^2 \\
& -(\frac{\|q_t\|_{0,\Omega}^2}{\delta_1} + \frac{\|\nabla q_t\|_{0,\Omega}^2}{\delta_2} + \frac{\|q\|_{0,\Omega}^2}{\delta_3} + \frac{\|\nabla q\|_{0,\Omega}^2}{\delta_4}) - \frac{\|q\|_{0,\Omega}^2}{\delta_5} - \delta_5\|\nabla v\|_{0,\Omega}^2 \\
\geq \ & \|v_{tt}\|_{0,\Omega}^2 + \varepsilon\|q_t\|_{0,\Omega}^2 + \varepsilon\|\nabla q_t\|_{0,\Omega}^2 + (1+\varepsilon)\|q\|_{0,\Omega}^2 + \varepsilon\|\nabla q\|_{0,\Omega}^2 + \varepsilon\|\nabla v\|_{0,\Omega}^2
\end{aligned}
$$

so we can select the positive constants $\varepsilon, \delta_1, \delta_2, \delta_3, \delta_4, \delta_5$ satisfying

$$
\begin{aligned}
(1-\delta_1-\delta_2-\delta_3-\delta_4) &> 0, \\
(C-3\varepsilon-\frac{1}{\delta_1}) &> 0, \\
(C-3\varepsilon-\frac{1}{\delta_2}) &> 0, \\
(1+C-3\varepsilon-\frac{1}{\delta_3}-\frac{1}{\delta_5}) &> 0, \\
(C-3\varepsilon-\frac{1}{\delta_4}) &> 0, \\
(1-\delta_5) &> 0,
\end{aligned}
$$

we have

$$
\begin{aligned}
\mathscr{B}(q,v;q,v) \geq\ & \alpha(\|v_{tt}\|^2_{0,\Omega}+\|q_t\|^2_{0,\Omega}+\|\nabla q_t\|^2_{0,\Omega}+\|q\|^2_{0,\Omega}+\|\nabla q\|^2_{0,\Omega} \\
& +\|\nabla v\|^2_{0,\Omega}).
\end{aligned}
$$

The proof of Theorem2.1 is therefore completed.

Theorem 2.2. The equations (2.2) has a unique solution in $Q(\Omega)\times U(\Omega)$.

Proof: From Theorem 2.1, the bilinear form $\mathscr{B}(\cdot,\cdot;\cdot,\cdot)$ is coercive and bounded on $Q(\Omega)\times U(\Omega)$. Then the result follows from Lax-Milgram theorem.

3. Finite Element Approximation

In principle, the LSMFE approach simply consists of minimizing (2.6) in finite dimensional subspaces $U_h(\Omega) \subset U(\Omega)$ and $Q_h(\Omega) \subset Q(\Omega)$. Suitable spaces are based on a triangulation $\mathcal{T}_h$ of Ω and consist of piecewise polynomials with sufficient continuity conditions.

Let $U_h(\Omega) = H^1_0(\Omega) \cap W^{1,\infty}(\Omega)$ and $Q_h(\Omega) \subset Q(\Omega)$, let $\mathcal{T}_h$ be a class qusi-uniform regular partition of Ω.

The least squares functional:

$$\begin{aligned}\mathscr{F}_h(\sigma, u) = \sum_{T\in\mathcal{T}_h} (&\|u_{tt} - (a'(x,t)\sigma_t + a(x,t)\nabla\sigma_t + b'(x,t)\sigma + b(x,t)\nabla\sigma)\|_{0,T}^2 \\ &+\|\nabla u - \sigma\|_{0,T}^2).\end{aligned} \tag{3.1}$$

Minimizing the functional (3.1) is equivalent to the following variational problem: find $\sigma_h \in Q_h$ and $u_h \in U_h$ such that

$$\mathscr{B}_h(\sigma_h, u_h; q, v) = 0, \tag{3.2}$$

holds for all $(q, v) \in Q_h(\Omega) \times U_h(\Omega)$.

The discrete bilinear form $\mathscr{B}_h(\cdot,\cdot;\cdot,\cdot)$ is defined as follows:

$$\begin{aligned}\mathscr{B}_h(\sigma, u; q, v) = \sum_{T\in\mathcal{T}_h} [&(u_{tt} - (a'(x,t)\sigma_t + a(x,t)\nabla\sigma_t + b'(x,t)\sigma \\ &+b(x,t)\nabla\sigma), v_{tt} - (a'(x,t)q_t + a(x,t)\nabla q_t + b'(x,t)q \\ &+b(x,t)\nabla q))_{0,T} + (\nabla u - \sigma, \nabla v - q)_{0,T}].\end{aligned} \tag{3.3}$$

Theorem 3.1. The bilinear $\mathscr{B}_h(\cdot,\cdot;\cdot,\cdot)$ is continuous and coercive, i.e., there exist positive constants α_h and β_h such that

$$\begin{aligned}\mathscr{B}_h(\sigma, u; q, v) \leq \beta_h (\sum_{T\in\mathcal{T}_h} (&\|u_{tt}\|_{0,T}^2 + \|\sigma_t\|_{0,T}^2 + \|\nabla\sigma_t\|_{0,T}^2 + \|\sigma\|_{0,T}^2 \\ &+\|\nabla\sigma\|_{0,T}^2 + \|\nabla u\|_{0,T}^2))^{\frac{1}{2}} (\sum_{T\in\mathcal{T}_h} (\|v_{tt}\|_{0,T}^2 + \|q_t\|_{0,T}^2 \\ &+\|\nabla q_t\|_{0,T}^2 + \|q\|_{0,T}^2 + \|\nabla q\|_{0,T}^2 + \|\nabla v\|_{0,T}^2))^{\frac{1}{2}},\end{aligned} \tag{3.4}$$

$$\begin{aligned}\mathscr{B}_h(q, v; q, v) \geq \alpha_h \sum_{T\in\mathcal{T}_h} (&\|v_{tt}\|_{0,T}^2 + \|q_t\|_{0,T}^2 + \|\nabla q_t\|_{0,T}^2 + \|q\|_{0,T}^2 \\ &+\|\nabla q\|_{0,T}^2 + \|\nabla v\|_{0,T}^2),\end{aligned} \tag{3.5}$$

which holds for all $(q, v) \in Q_h(\Omega) \times U_h(\Omega)$, $(\sigma, u) \in Q_h(\Omega) \times U_h(\Omega)$.

Proof: The theorem can be proved in a similar manner as in Theorem 2.1.

4. Posteriori Error Estimation

One of the main motivations for using least squares finite element approaches is the fact that the element-wise evaluation of the functional serves as an *a posteriori* error estimator.

A posteriori estimate attempt to provide quantitatively accurate measures of the discretization error through the so-called a posteriori error estimators which are derived by using the information obtained during the solution process. In recent years, the use of *a posteriori* error estimators has become an efficient tool for assessing and controlling computational errors in adaptive computations[9].

Now we define the least squares functional:

$$\begin{aligned}\mathscr{F}_h(\sigma_h, u_h) = \sum_{T\in\mathcal{T}_h} (\|u_{htt} - (a'(x,t)\sigma_{ht} + a(x,t)\nabla\sigma_{ht} + b'(x,t)\sigma_h \\ +b(x,t)\nabla\sigma_h)\|_{0,T}^2 + \|\nabla u_h - \sigma_h\|_{0,T}^2).\end{aligned} \tag{4.1}$$

We have

$$\begin{aligned}\mathscr{F}_h(\sigma-\sigma_h, u-u_h) &= \sum_{T\in\mathcal{T}_h} (\|u_{tt} - u_{htt} - (a'(x,t)(\sigma_t - \sigma_{ht}) \\ &\quad +a(x,t)\nabla(\sigma_t - \sigma_{ht}) + b'(x,t)(\sigma - \sigma_h) \\ &\quad +b(x,t)\nabla(\sigma - \sigma_h))\|_{0,T}^2 + \|\nabla(u-u_h) - (\sigma-\sigma_h)\|_{0,T}^2)).\end{aligned}$$

So we define the posteriori estimator as following:

$$\mathscr{F}_h(\sigma-\sigma_h, u-u_h) =: \sum_{T\in\mathcal{T}_h} \eta_T^2.$$

Theorem 4.1. The least squares functional constitutes an *a posteriori* error estimator. In other words, for

$$\begin{aligned}\eta_T^2 &= \|u_{tt} - u_{htt} - (a'(x,t)(\sigma_t - \sigma_{ht}) \\ &\quad +a(x,t)\nabla(\sigma_t - \sigma_{ht}) + b'(x,t)(\sigma - \sigma_h) \\ &\quad +b(x,t)\nabla(\sigma - \sigma_h))\|_{0,T}^2 + \|\nabla(u-u_h) - (\sigma-\sigma_h)\|_{0,T}^2,\end{aligned}$$

there exist positive constants α_T and β_T such that

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} \eta_T^2 &\leq \beta_T \sum_{T\in\mathcal{T}_h} (\|u_{tt} - u_{htt}\|_{0,T}^2 + \|\sigma_t - \sigma_{ht}\|_{0,T}^2 + \|\nabla(\sigma_t - \sigma_{ht})\|_{0,T}^2 \\ &\quad +\|\sigma - \sigma_h\|_{0,T}^2 + \|\nabla(\sigma - \sigma_h)\|_{0,T}^2 + \|\nabla(u - u_h)\|_{0,T}^2),\end{aligned}$$

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\eta_T^2 &\geq \alpha_T\sum_{T\in\mathcal{T}_h}(\|u_{tt}-u_{htt}\|_{0,T}^2+\|\sigma_t-\sigma_{ht}\|_{0,T}^2+\|\nabla(\sigma_t-\sigma_{ht})\|_{0,T}^2\\ &\quad+\|\sigma-\sigma_h\|_{0,T}^2+\|\nabla(\sigma-\sigma_h)\|_{0,T}^2+\|\nabla(u-u_h)\|_{0,T}^2).\end{aligned}$$

which holds for all $(\sigma_h, u_h)\in Q_h(\Omega)\times U_h(\Omega)$.

Proof: We know

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}\eta_T^2 &= \mathscr{F}_h(\sigma-\sigma_h,u-u_h)\\ &= \sum_{T\in\mathcal{T}_h}(\|u_{tt}-u_{htt}-(a'(x,t)(\sigma_t-\sigma_{ht})\\ &\quad+a(x,t)\nabla(\sigma_t-\sigma_{ht})+b'(x,t)(\sigma-\sigma_h)\\ &\quad+b(x,t)\nabla(\sigma-\sigma_h))\|_{0,T}^2+\|\nabla(u-u_h)-(\sigma-\sigma_h)\|_{0,T}^2))\\ &= \mathscr{B}_h(\sigma-\sigma_h,u-u_h;\sigma-\sigma_h,u-u_h).\end{aligned}$$

From Theorem 3.1, we have:

$$\begin{aligned}\mathscr{B}_h(\sigma-\sigma_h,u-u_h;\sigma-\sigma_h,u-u_h) &\leq \beta_h\sum_{T\in\mathcal{T}_h}(\|u_{tt}-u_{htt}\|_{0,T}^2+\|\sigma_t-\sigma_{ht}\|_{0,T}^2\\ &\quad+\|\nabla(\sigma_t-\sigma_{ht})\|_{0,T}^2+\|\sigma-\sigma_h\|_{0,T}^2\\ &\quad+\|\nabla(\sigma-\sigma_h)\|_{0,T}^2+\|\nabla(u-u_h)\|_{0,T}^2),\end{aligned}$$

$$\begin{aligned}\mathscr{B}_h(\sigma-\sigma_h,u-u_h;\sigma-\sigma_h,u-u_h) &\geq \alpha_h\sum_{T\in\mathcal{T}_h}(\|u_{tt}-u_{htt}\|_{0,T}^2+\|\sigma_t-\sigma_{ht}\|_{0,T}^2\\ &\quad+\|\nabla(\sigma_t-\sigma_{ht})\|_{0,T}^2+\|\sigma-\sigma_h\|_{0,T}^2\\ &\quad+\|\nabla(\sigma-\sigma_h)\|_{0,T}^2+\|\nabla(u-u_h)\|_{0,T}^2).\end{aligned}$$

The positive constants $\alpha_T = C\alpha_h$ and $\beta_T = C\beta_h$, this completes the proof.

Remark: The mesh is adapted and based on *a posteriori* error estimate of the two dimensional viscoelastic problems. Based on the computed *a posteriori* error estimator η_T, we use a mesh optimization procedure to compute the size of elements in the new mesh. Adaptive refinement strategies consist in refining those triangles with the largest values of η_T.

5. Adaptive Algorithm

An optimal strategy determines a refinement region $R \subset \Omega$ that minimizes the ratio

$$\frac{\textit{work to solve the new discrete problem}}{\textit{gain in accuracy}}.$$

We must somehow approximate both *work* and *gain*. A reasonable approximation to *work* is a linear function of the number of vertices in the finite element mesh, n_{old}, and the number of vertices that will be added due to refinement, n_{new}. The new vertices are added to the hierarchy of levels as the finest level. Hence, the *work* induced by these points is proportional to n_{new}. The *work* induced by the old vertices is proportional to n_{old}, since a multigrid cycle is used to solve the coarse level problem. Thus, the following approximation for work is used:

$$\textit{work to solve the new discrete problem} \; = \; an_{old} + bn_{new},$$

with suitable constants a and b. The *gain* in accuracy can be measured by calculating the ratio

$$\textit{gain in accuracy} \; = \; \frac{\mathscr{F}_h(\sigma_{h,old}, u_{h,old})}{\mathscr{F}_h(\sigma_{h,new}, u_{h,new})}.$$

Our adaptive algorithms is as follows:

1. Find the maximum of the *a posteriori* error estimates over all $T \in \mathcal{T}_h : \eta_{T,max}$.
2. Partition $\mathcal{T}_h$ into contour sets $C_i = \{T \mid \eta_T^2 \in (\frac{i-1}{N}\eta_{T,max}^2, \frac{i}{N}\eta_{T,max}^2)\}$, $i = 1, 2, \cdots, N$.
3. Calculate $w_i = \frac{work}{gain}$, for $i = 1, 2, \cdots, N$.
4. Find $i \in \{1, 2, \cdots, N\}$ for which w_i is minimal.
5. Refine contour sets $C_i, \cdots, C_N$.

Convergence of an Adaptive Algorithm

Suppose that σ_h and u_h are the best approximation to the solution σ and u of problem (2.6) on the current level $Q_h(\Omega)$ and $U_h(\Omega)$. Let $R \subset \Omega$ be a subregion in which further refinement is considered. Define the errors $e = \sigma - \sigma_h$ and $E = u - u_h$, define the set

$$U_R(\Omega) := \{\nu \in U(\Omega), \rho \in Q(\Omega) : \nu = 0, \rho = 0 \;\, on \;\, R^c \equiv \Omega - R\}.$$

Then, define

$$(l,\ L) := arg \inf_{\nu\in U_R(\Omega),\ \rho\in Q_R(\Omega)} \mathscr{B}(e+\rho, E+\nu; e+\rho, E+\nu).$$

Now, let

$$h = e - l, H = E - L,$$

and note that $h = e, H = E$ on R^c. Define h, H as follows:

$$(h,\ H) := arg \inf_{\rho=e\ on\ R^c,\ \nu=E\ on\ R^c} \mathscr{B}_R(\rho, \nu; \rho, \nu).$$

Theorem5.1 Given region $R \subset \Omega$, approximation $\sigma_h \in Q_h, u_h \in U_h$, and define ϵ by

$$\mathscr{B}_R(e, E; e, E) = (1-\epsilon)\mathscr{B}(e, E; e, E). \tag{5.1}$$

Assume that there exists $\gamma < 1 - \epsilon$ such that

$$\mathscr{B}_R(h, H; h, H) \le \gamma\mathscr{B}(h, H; h, H). \tag{5.2}$$

Then,

$$\mathscr{B}(h, H; h, H) \le \xi\mathscr{B}(e, E; e, E), \tag{5.3}$$

where $\xi = \frac{\epsilon}{1-\gamma} < 1$.

Proof: (5.1) and (5.2) imply that

$$\begin{aligned}
(1-\gamma)\mathscr{B}(h, H; h, H) &= \mathscr{B}(h, H; h, H) - \gamma\mathscr{B}(h, H; h, H) \\
&\le \mathscr{B}(h, H; h, H) - \mathscr{B}_R(h, H; h, H) \\
&= \mathscr{B}_{R^c}(e, E; e, E) \\
&= \mathscr{B}(e, E; e, E) - \mathscr{B}_R(e, E; e, E) \\
&= \mathscr{B}(e, E; e, E) - (1-\epsilon)\mathscr{B}(e, E; e, E) \\
&= \epsilon\mathscr{B}(e, E; e, E).
\end{aligned}$$

Hence,

$$\mathscr{B}(h, H; h, H) \le \frac{\epsilon}{1-\gamma}\mathscr{B}(e, E; e, E),$$

which completes the proof.

Assume that refinement by halving h reduces the local errors l, L in R by a factor of $1/4$:

$$\mathscr{B}(\sigma_{\frac{h}{2}}, u_{\frac{h}{2}}; \sigma_{\frac{h}{2}}, u_{\frac{h}{2}}) \leq \mathscr{B}_{R^c}(e, E; e, E) + \mathscr{B}_R(h, H; h, H) + \frac{1}{4}\mathscr{B}_R(l, L; l, L). \tag{5.4}$$

Consider the relations

$$\mathscr{B}_R(l, L; l, L) = \mathscr{B}_R(e, E; e, E) - \mathscr{B}_R(h, H; h, H), \tag{5.5}$$

and

$$\mathscr{B}_R(h, H; h, H) \leq \frac{\gamma}{1-\gamma}\mathscr{B}_{R^c}(e, E; e, E), \tag{5.6}$$

which follows from (5.2) and the fact that $h = e, H = E$ on R^c, and

$$\mathscr{B}_R(h, H; h, H) = \mathscr{B}_R(h, H; h, H) + \mathscr{B}_{R^c}(e, E; e, E). \tag{5.7}$$

Then, (5.4), (5.5) and (5.6) imply that

$$\begin{aligned}
\mathscr{B}(\sigma_{\frac{h}{2}}, u_{\frac{h}{2}}; \sigma_{\frac{h}{2}}, u_{\frac{h}{2}}) &\leq \mathscr{B}_{R^c}(e, E; e, E) + \mathscr{B}_R(h, H; h, H) \\
&\quad + \frac{1}{4}(\mathscr{B}_R(e, E; e, E) - \mathscr{B}_R(h, H; h, H)) \\
&= \mathscr{B}_{R^c}(e, E; e, E) + \frac{1}{4}\mathscr{B}_R(e, E; e, E) \\
&\quad + \frac{3}{4}\mathscr{B}_R(h, H; h, H) \\
&\leq (1 + \frac{3}{4}\frac{\gamma}{1-\gamma})\mathscr{B}_{R^c}(e, E; e, E) + \frac{1}{4}\mathscr{B}_R(e, E; e, E) \\
&= (1 + \frac{3}{4}\frac{\gamma}{1-\gamma})\epsilon\mathscr{B}(e, E; e, E) + \frac{1}{4}(1-\epsilon)\mathscr{B}(e, E; e, E) \\
&= (\frac{1}{4} + \frac{3}{4}\frac{\epsilon}{1-\gamma})\mathscr{B}(e, E; e, E).
\end{aligned}$$

Thus, we obtain a bound similar to (5.3), but now with $\xi = \frac{1}{4} + \frac{3}{4}\frac{\epsilon}{1-\gamma}$. This implies convergence for the case of one additional level of refinement when $\frac{\epsilon}{1-\gamma} < 1$.

6. Summary and Conclusions

We describe an adaptive least squares mixed finite element procedure for solving two dimensional viscoelastic problems in this paper, and the procedure uses a least squares mixed finite element formulation and adaptive refinement based on *a posteriori* error estimate. The methods were applied to study the continuous and coercivity of two dimensional viscoelastic problems.

In this paper, we applied relatively standard *a posteriori* error estimation techniques to adaptively solve two dimensional viscoelastic problems and described the adaptive algorithm.

References

[1] Aziz A K, Kellogg R B, Stephens A B. Least-squares Methods for Elliptic Systems. *Math.Comp.*, 44: 53-70 (1985).

[2] Gu Haiming, Yang Dangping, Sui Shulin, Liu Xinmin. Least-squares Mixed Finite Element Method for A Class of Stokes Equation. *Appl.Math.and Mech.*, 21(5): 557-566 (2000).

[3] Gu Hai-ming, Xu Xiu-ling. The Least-Squares Mixed Finite Element Methods for a Degenerate Elliptic Problem. *Math.Appl*, 15(1): 118-122 (2002).

[4] Zhiqiang Cai, Barry Lee, and Ping Wang. Least-Squares Methods for Incompressible Newtonian Fluid Flow: Linear Stationary Problems. *SIAM J.Numer Anal*, 42: 843-859 (2004).

[5] M.Maischak and E.P.Stephan. A Least Squares Coupling Method with Finite Elements and Boundary Elements for Transmission Problems. *Comput.Math.Appl.* 48: 995-1016 (2004).

[6] Haiming Gu and Danping Yang. Least-squares Mixed Finite Element Method for Sobolev Equations. *Indian J.Pure Appl.Math.*, 31(5): 505-517 (2000).

[7] M.-Y.Kim E.-J.Park J.Park. Mixed Finite Element Domain Decomposition for Nonlinear Parabolic Problems. *Comput.Math.Appl.*, 40: 1061-1070 (2000).

[8] Zhiqiang Cai, Johannes Korsawe, Gerhard Starke. An Adaptive Least Squares Mixed Finite Element Method for the Stress-Displacement Formulation of Linear Elasticity. *Numer Methods Partial Differential Eq*, 21: 132-148 (2005).

[9] Suh-Yuh Yang. Analysis of a least squares finite element method for the circular arch problem. *Appl.Math.Comput.*, 114: 263-278 (2000).

[10] Abdellatif Agouzal. A Posteriori Error Estimators for Nonconforming Approximation. *Numer Anal Model*, 5: 77-85 (2008).

[11] Markus Berndt, Thomas A. Manteuffel, and Stephen F. Mccormick. Local Error Estimates And Adaptive Refinement for First-order System Least Squares(FOSLS). *Electronic Transactions on Numerical Analysis*, 6: 35-43 (1997).

[12] Tao Tang, Jinchao Xu. Adaptive Computations: Theory and Algorithms. Science Press, Beijing, 2007.

[13] Z.D.Luo. Theoretical Bases for Mixed Finite Element Methods and Application (in Chinese). Science Press, Beijing, 2006.