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RESOLVENT EQUATION APPROACH CONNECTED WITH $H(.,.)$-φ-η-MIXED MONOTONE MAPPING WITH AN APPLICATION

SANJEEV GUPTA[1,*], MANOJ SINGH[2]

[1]Department of Mathematics, GLA University, Mathura-281406, India

[2]Department of Mathematics, College of Science, Jazan University, Jazan-45142, Saudi Arabia

Abstract. The objective of this work is to use an application of $H(.,.)$-φ-η-mixed monotone mappings [13] via resolvent equation technique to solve the set-valued variational-like inclusions in semi-inner product spaces. We aim to establish an equivalence between the set-valued variational-like inclusion problem and fixed point problem. A relationship also obtain between the set-valued variational-like inclusion problem and the resolvent equation problem. This equivalent formulation suggests an idea to construct an iterative algorithm to find a solution of the resolvent equation problem.

Keywords: $H(.,.)$-φ-η-mixed monotone; resolvent equation; iterative algorithm; set-valued variational-like inclusion.

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1. Introduction

Variational Inequality theory is very important due to its large application in various problem e.g. partial differential equation and optimization problems, see [3]. Therefore it have

*Corresponding author
E-mail address: guptasanmp@gmail.com
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been developed and generalized in numerous directions. Variational inclusions is a natural generalization of variational inequalities. Monotonicity have a very crucial role in the study of variational inclusions. Therefore researchers introduced and studied many types of monotonicity e.g. maximal monotone mapping, relaxed monotone mapping, H-monotone mapping, A-monotone mapping etc., and discussed the solvability of different variational inclusion problems with the help of underlying different monotone mappings, see [4, 5],[8]-[11],[16],[25, 26],[28]-[31],[34, 35, 38]. The resolvent operator technique which is the generalized form of projection technique, is very efficient tool to solve variational inclusions and their generalizations. The resolvent equation is also a very significant approach. The resolvent operator equations technique is utilized to expand significant and feasible numerical approaches to find a the solution of many variational inequalities (inclusions) and linked optimization problems, see [1, 2].

Many heuristics generalized the monotonicity such as (H,η)-monotone, (A,η)-monotone, (A,η)-maximal relaxed monotone etc. They introduced and studied different variational inclusions problems involving these monotone mapping in Hilbert spaces (Benach spaces), see [9, 10, 25, 28, 29, 34].

"Recently, Sahu et al. [30] proved the existence of solutions for a class of nonlinear implicit variational inclusion problems in semi-inner product spaces, which is more general than the results studied in [31]. Moreover, they constructed an iterative algorithm for approximating the solution for the class of implicit variational inclusion problems involving A-monotone and H-monotone operators by using the generalized resolvent operator technique. It is remarked that they discussed the existence and convergence analysis by relaxing the condition of monotonicity on the set-valued map considered", [4].

Very recently Luo and Huang [26], introduced and studied (H,φ)-η-monotone mapping in Banach spaces which provides a unifying framework for various classes of monotone mapping. Most recently, Bhat and Zahoor [4, 5], introduced and studied (H,ϕ)-η-monotone mapping in semi-inner product space and discussed the convergence analysis of proposed iterative schemes for some classes of variational inclusion through generalized resolvent operator. For the applications point of view of discussed operators in variational inequalities and variational inclusion, see [8]-[11],[14, 15],[17]-[26],[28]-[35],[37, 39].

The proposed work is impelled by the noble research works discussed above. First, we establish a relation between the set-valued variational-like inclusions and fixed point problem and also obtain a equivalance between the set-valued variational-like inclusion amd the resolvent operator equation involving $H(.,.)$-φ-η-mixed monotone mapping [13]. These equivalant fixed point problem and the resolvent equation problem formulation suggest us an idea to develop an iterative algorithm. We also make an attempt to find the existence of solution of set valued variational inclusion involving nonlinear operators in 2-uniformly smooth Banach space. The obtained results are quite similar to above discussed research work but we utilize distinguished notion and approach to solve variational inclusion problems in 2-uniformly smooth Banach space. Our work is the extension and refinement of the existing results, see [1, 2, 4, 5, 17, 24, 26, 39].

Definition 1.1. [27, 30] *Let us consider the vector space Y over the field F of real or complex numbers. A functional $[.,.] : Y \times Y \to F$ is called a semi inner product if*
(i) $[u^1 + u^2, v^1] = [u^1, v^1] + [u^2, v^1],\ \forall u^1, u^2, v^1 \in Y$
(ii) $[\alpha u^1, v^1] = \alpha[u^1, v^1],\ \forall \alpha \in F,\ u^1, v^1 \in Y$
(iii) $[u^1, u^1] \geq 0,\ for\ u^1 \neq 0$
(iv) $|[u^1, v^1]|^2 \leq [u^1, u^1][v^1, v^1],\ \forall u^1, v^1 \in Y$
The pair $(Y, [.,.])$ is called a semi-inner product space.

"We observed that $\|u^1\| = [u^1, u^1]^{1/2}$ is a norm and we can say a semi-inner product space is a normed linear space with the norm. Every normed linear space can be made into a semi-inner product space in infinitely many different ways. Giles [12] had shown that if the underlying space Y is a uniformly convex smooth Banach space then it is possible to define a semi-inner product uniquely" [4].

Remark 1.2. "This unique semi-inner product has the following nice properties:
(i) $[u^1, v^1] = 0$ iff v^1 is orthogonal to u^1, that is iff $\|v^1\| \leq \|v^1 + \alpha u^1\|$, for all scalars α.
(ii) Generalized Riesz representation theorem: If f is a continuous linear functional on Y then there is a unique vector $v^1 \in Y$ such that $f(u^1) = [u^1, v^1]$, for all $u^1 \in Y$.
(iii) The semi-inner product is continuous, that is for each $u^1, v^1 \in Y$, we have $Re[v^1, u^1 + \alpha v^1] \to Re[v^1, u^1]$ as $\alpha \to 0$", [4].

Since the sequence space $l^p, p > 1$ and the function space $L^p, p > 1$ are uniformly convex smooth Banach spaces, we can define a semi-inner product on these spaces, uniquely.

Definition 1.3. [30] *The real sequence space l^p for $1 < p < 1$ is a semi-inner product space with the semi-inner product defined by*

$$[v,w] = \frac{1}{\|w\|_p^{p-2}} \sum_j v_j w_j |w|^{p-2},\ v,w \in l^p.$$

Definition 1.4. [12, 30] *The real Banach space $L^p(Y,\mu)$ for $1 < p < 1$ is a semi-inner product space with the semi-inner product defined by*

$$[g,h] = \frac{1}{\|h\|_p^{p-2}} \int_Y g(u)|h(u)|^{p-1} sgn(h(u)) d\mu,\ v,w \in L^p.$$

Definition 1.5. [30, 36] *The Y be a Banach space, then*

(i) *modulus of smoothness of Y defined as*

$$\rho_Y(s) \ = \ \sup\left\{\frac{\|u^1+v^1\|+\|u^1-v^1\|}{2} - 1 :\ \|u^1\| \le 1,\ \|v^1\| \le s\right\}.$$

(ii) *Y be uniformly smooth if* $\lim_{s\to 0} \rho_Y(s)/s \ = \ 0$

(iii) *Y be p-uniformly smooth for $p > 1$, if there exists $c > 0$ such that $\rho_Y(s) \le cs^p$.*

(iv) *Y be 2-uniformly smooth if there exists $c > 0$ such that $\rho_Y(s) \le cs^2$.*

Lemma 1.6. [30, 36] *Let $p > 1$ be a real number and Y be a smooth Banach space. Then the following statements are equivalent:*

(i) *Y is 2-uniformly smooth.*

(ii) *There is a constant $k > 0$ such that for every $v^1, w^1 \in Y$, the following inequality holds*

$$\|v^1+w^1\|^2 \le \|v^1\|^2 + 2\langle w^1, f_{v^1}\rangle + k\|w^1\|^2, \tag{1.1}$$

where $f_{v^1} \in J(v^1)$ and $J(v^1) = \{v^{1*} \in Y^* : [v^1, v^{1*}] = \|v^1\|^2$ and $\|v^{1*}\| = \|v^1\|\}$ is the normalized duality mapping.

Remark 1.7. *" Every normed linear space Y is a semi-inner product space (see* [27]*). Infact, by Hahn-Banach theorem, for each $v^1 \in Y$, there exists at least one functional $f_{v^1} \in Y^*$ such that*

$\langle v^1, f_{v^1}\rangle = \|v^1\|^2$. *Given any such mapping* $f : Y \to Y^*$, *we can verify that* $[w^1, v^1] = \langle w^1, f_{v^1}\rangle$ *defines a semi-inner product. Hence we can write the inequality (2.1) as*

$$\|v^1 + w^1\|^2 \leq \|v^1\|^2 + 2[w^1, f_{v^1}] + s\|w^1\|^2. \tag{1.2}$$

The constant s is known as constant of smoothness of Y, is chosen with best possible minimum value”, [30].

Example 1.8. *“The function space* L^p *is* 2*-uniformly smooth for* $p \geq 2$ *and it is p-uniformly smooth for* $1 < p < 2$. *If* $2 \leq p < \infty$, *then we have for all* $v^1, w^1 \in L^p$,

$$\|v^1 + w^1\|^2 \leq \|v^1\|^2 + 2[w^1, f_{v^1}] + (p-1)\|w^1\|^2.$$

where the constant of smoothness is $p - 1$*”,* [30].

2. Preliminaries

Let Y be a 2-uniformly smooth Banach space. Its norm and topological dual space is given by $\|.\|$ and Y^*, respectively. The semi-inner product $[.,.]$ signify the dual pair among Y and Y^*. In order to proceed the next, we recall some basic concepts, which will be needed in the subsequent sections.

Definition 2.1. [26, 30] *Let Y be real 2-uniformly smooth Banach space. Let single-valued mappings* $H,\ \eta : Y \times Y \to Y$, *and* $Q, R : Y \to Y$, *then*

(i) Q *is* (r, η)*-strongly monotone if there* $\exists$ *constant* $r > 0$ *such that*

$$[Q(u) - Q(u'), \eta(u,u')] \ \geq\ r\ \|u - u'\|^2,\ \forall u,\ u' \in Y;$$

(ii) Q *is* (s, η)*-relaxed monotone if there* $\exists$ *constant* $s > 0$ *such that*

$$[Q(u) - Q(u'), \eta(u,u')] \ \geq\ -s\ \|u - u'\|^2,\ \forall u,\ u' \in Y;$$

(iii) Q *is* α*-expansive if there* $\exists$ *constant* $\alpha > 0$

$$\|Q(u) - Q(u')\| \ \geq\ \alpha\ \|u - u'\|,\ \forall u,\ u' \in Y;$$

(iv) $H(Q,.)$ *is* (μ, η)*-cocoercive in regards R if there* $\exists$ *constant* $\mu > 0$ *such that*

$$[H(Qu,x) - H(Qu',x),\ \eta(u,u')] \ \geq\ \mu\ \|Qu - Qu'\|^2,\ \forall x,\ u,\ u' \in Y;$$

(v) $H(.,R)$ *is* (γ,η)*-relaxed monotone in regards R if there* $\exists$ *constant* $\gamma > 0$ *such that*

$$[H(x,Ru) - H(x,Ru'), \eta(u,u')] \geq -\gamma \|u-u'\|^2, \ \forall x,\ u,\ u' \in Y;$$

(vi) $H(Q,.)$ *is* κ_1*-Lipschitz continuous in regards Q if there* $\exists$ *constant* κ_1 *such that*

$$\|H(Qu,x) - H(Qu',x)\| \leq \kappa_1 \|u-u'\|, \ \forall x,\ u,u' \in Y;$$

(vii) $H(.,R)$ *is* κ_2*-Lipschitz continuous in regards R if there* $\exists$ *constant* κ_2 *such that*

$$\|H(x,Ru) - H(x,Ru')\| \leq \kappa_2 \|u-u'\|, \ \forall x,\ u,\ u' \in Y;$$

(viii) η *is be* τ*-Lipschitz continuous if there* $\exists$ *constant* $\tau > 0$ *such that*

$$\|\eta(u,u')\| \leq \tau \|u-u'\|, \ \forall u,\ u' \in Y.$$

"Let $M : Y \multimap Y$ be a set-valued mapping, then graph of M is given by graph$(M) = \{(v,w) : w \in M(v)\}$. The domain of M is given by

$$\text{Dom}(M) = \{v \in Y : \exists w \in Y : (v,w) \in M\}.$$

The Range of (M) is given by

$$\text{Range}(M) = \{w \in Y : \exists V \in Y : (v,w) \in M\}.$$

The inverse of (M) is given by

$$M^{-1} = \{(w,v) : (v,w) \in M\}.$$

For any two set-valued mappings N and M, and any real number β, we define

$$N + M = \{(v, w+w') : (v,w) \in N, (v,w') \in M\},$$

$$\beta M = \{(v, \beta w) : (v,w,) \in M\}.$$

For a mapping A and a set-valued map $M : Y \multimap Y$, we define $A + M = \{(v, w+w') : Av = w, (v,w') \in M\}$", [4].

Definition 2.2. [26, 30] *A set-valued mapping* $M : Y \multimap Y$ *is said to be*

(i) (n,η)*-strongly monotone if* $\exists$ *a constant* $n > 0$ *such that*

$$[v^* - w^*, \eta(v,w)] \geq n \|v-w\|^2, \ \forall v,w \in Y,\ v^* \in M(v),\ w^* \in M(w);$$

(ii) (m,η)*-relaxed monotone if* $\exists$ *a constant* $m>0$ *such that*

$$[v^*-w^*,\eta(v,w)]\ \geq\ -m\ \|v-w\|^2,\ \forall v,w\in Y,\ v^*\in M(v),\ w^*\in M(w).$$

Definition 2.3. Let $P,\eta: Y\times Y\to Y$ be the mappings and $M: Y\times Y\multimap Y$ be the multi-valued mapping. Then

(i) *M is* (n,η)*-strongly monotone if* $\exists$ *a constant* $n>0$ *such that*

$$[v^*-w^*,\eta(v,w)]\ \geq\ n\ \|v-w\|^2,\ \forall v,w\in Y,\ v^*\in M(v,t),\ w^*\in M(w,t),\ \text{for each fixed t} \in \text{Y};$$

(ii) *M is* (m,η)*-relaxed monotone if* $\exists$ *a constant* $m>0$ *such that*

$$[v^*-w^*,\eta(v,w)]\ \geq\ -m\ \|v-w\|^2,\ \forall v,w\in Y,\ v^*\in M(v,t),\ w^*\in M(w,t),\ \text{for each fixed t} \in \text{Y};$$

(iii) *P is* (ν,η)*-relaxed monotone in regards first component if* $\exists$ *a constant* $\nu>0$ *such that*

$$[P(v,u^*)-P(w,u^*),\eta(v,w)]\ \geq\ -\nu\|v-w\|^2,\ \forall v,w,u^*\in Y;$$

(iv) $P(.,.)$ *is* ε_1*-Lipschitz continuous in regards first component if* $\exists$ *a constant* $\varepsilon_1>0$ *such that*

$$\|P(v,u^*)-P(w,u^*)\|\ \leq\ \varepsilon_1\ \|v-w\|,\ \forall v,w,u^*\in Y;$$

(v) $P(.,.)$ *is* ε_2*-Lipschitz continuous in regards second component if* $\exists$ *a constant* $\varepsilon_2>0$ *such that*

$$\|P(u^*,v)-P(u^*,w)\|\ \leq\ \varepsilon_2\ \|v-w\|,\ \forall v,w,u^*\in Y.$$

Definition 2.4. [7] *The Hausdorff metric* $D(.,.)$ *on* $CB(Y)$*, is defined by*

$$D(A,B)=\max\left\{\sup_{u\in A}\inf_{v\in B}d(u,v),\sup_{v\in B}\inf_{u\in A}d(u,v)\right\},\ A,B\in CB(Y),$$

where $d(.,.)$ *is the induced metric on* Y *and* $CB(Y)$ *denotes the family of all nonempty closed and bounded subsets of* X*.*

Definition 2.5. [7] *A multi-valued mapping* $S: Y\multimap CB(Y)$ *is called D-Lipschitz continuous with constant* $\lambda_S>0$*, if*

$$D(Sv,Sw)\leq\ \lambda_S\ \|v-w\|,\ \forall v,\ w\in Y.$$

3. $H(.,.)$-ϕ-η-Mixed Monotone Mappings

First, we give some definitions and important theorems associates with $H(.,.)$-φ-η-mixed monotone mapping.

Let Y be 2-uniformly smooth Banach space. Assume that $\eta, H : Y \times Y \to Y$, and $\varphi, Q, R : Y \to Y$ be single-valued mappings and $M : Y \times Y \multimap Y$ be a multi-valued mapping.

Definition 3.1. [13] Let $H(.,.)$ is (μ, η)-cocoercive in regards Q with non-negative constant μ and (γ, η)-relaxed monotone in regards R with non-negative constant γ, then M is called $H(.,.)$-φ-η-*mixed monotone* in regards Q and R if

(i) for each fixed t, $\varphi o M(.,t)$ is (m, η)-relaxed monotone in regards first argument;

(ii) $(H(.,.) + \lambda \varphi o M(.,t))(Y) = Y$, $\lambda > 0$.

Remark 3.2. *If $H(.,.) = H$ and $\varphi o M$ is η-monotone. Then $H(.,.)$-φ-η-mixed monotone reduces to (H, φ)-η-monotone mapping, see* [4]. *In addition, if $\varphi o M$ is monotone. Then $H(.,.)$-φ-η-mixed monotone reduces to (H, φ)-monotone mapping, see* [26]. *If $\varphi(v) = v$, $\forall\, v \in Y$ and $\lambda > 0$. Then $H(.,.)$-φ-η-mixed monotone reduces to $(H(.,.), \eta)$-monotone mapping, see* [37].

Let us consider the following

Assumption $\mathbf{M_1}$: Let H is (μ, η)-cocoercive in regards Q with non-negative constant μ and (γ, η)-relaxed monotone in regards R with non-negative constant γ with $\mu > \gamma$.

Assumption $\mathbf{M_2}$: Let Q is α-expansive.

Assumption $\mathbf{M_3}$: Let η is τ-Lipschitz continuous.

Assumption $\mathbf{M_4}$: Let M is $H(.,.)$-φ-η-mixed monotone mapping in regards Q and R for each fixed $t \in Y$.

Theorem 3.3. [13] *Let assumptions M_1, M_2 and M_4 hold good with $\ell = \mu\alpha^2 - \gamma > m\lambda$, then $(H(Q,R) + \lambda \varphi o M(.,t))^{-1}$ is single-valued.*

Definition 3.4. [13] Let assumptions M_1, M_2 and M_4 hold good with $\ell = \mu\alpha^2 - \gamma > m\lambda$ then the *resolvent operator* $R^{H(.,.)-\eta}_{M(.,t),\varphi} : Y \to Y$ is given as

$$R^{H(.,.)-\eta}_{M(.,t),\varphi}(u) = (H(Q,R) + \lambda \varphi o M(.,t))^{-1}(u), \ \forall\, u \in Y. \tag{3.1}$$

The next attempt is to prove the Lipschitz continuity of the resolvent operator defined by (3.1).

Theorem 3.5. [13] *Let assumptions M_1-M_4 hold good with $\ell = \mu\alpha^2 - \gamma > m\lambda$ and η is τ-Lipschitz then $R^{H(.,.)-\eta}_{M(.,t),\varphi} : Y \to Y$ is $\frac{\tau}{\ell - m\lambda}$-Lipschitz continuous, that is,*

$$\|R^{H(.,.)-\eta}_{M(.,t),\varphi}(y) - R^{H(.,.)-\eta}_{M(.,t),\varphi}(z)\| \leq \frac{\tau}{\ell - m\lambda}\|y - z\|, \ \forall\, y, z \in Y, \text{ and fixed } t \in Y.$$

4. Formulation of the Problem and Existence of Solution

Now we make an attempt to show that $H(.,.)$-φ-η-mixed monotone mapping under acceptable assumptions can be used as a powerful tool to solve variational inclusion problems.

Let Y be 2-uniformly smooth Banach space. Let $S, T, G : Y \multimap CB(Y)$ be the multi-valued mappings, and let $Q, R, \varphi : Y \to Y$, $P : Y \times Y \to Y$ and η, $H : Y \times Y \to Y$ be single-valued mappings. Suppose that multi-valued mapping $M : Y \times Y \multimap Y$ be a $H(.,.)$-φ-η-mixed monotone mapping in regards Q, R. We consider the following generalized set-valued variational like inclusion problem to find $u \in Y$, $v \in S(u)$, $w \in T(u)$ and $t \in G(u)$ such that

$$0 \in P(v, w) + M(u, t). \tag{4.1}$$

If Y is real Hilbert space and $M(., t)$ is maximal monotone operator, then the similar problem to (4.1) studied by Huang et al. [16].

Lemma 4.1. *Let us consider the mapping $\varphi : Y \to Y$ such that $\varphi(v + w) = \varphi(v) + \varphi(w)$ and $Ker(\varphi) = \{0\}$, where $Ker(\varphi) = \{v \in Y : \ \varphi(v) = 0\}$. If (u, v, w, t), where $u \in Y$, $v \in S(u)$, $w \in T(u)$ and $t \in G(u)$ is a solution of problem (4.1) if and only if (u, v, w, t) satisfies the following relation:*

$$u \ = \ R^{H(.,.)-\eta}_{M(.,t),\varphi}\ [H(Qu, Ru) - \lambda\,\varphi o P(v, w)]. \tag{4.2}$$

The resolvent equation corresponding to generalized set-valued variational-like inclusion problem (4.1).

$$\varphi o P(v, w) + \lambda^{-1} J^{H(.,.)-\eta}_{M(.,t),\varphi}(x) = 0. \tag{4.3}$$

where $\lambda > 0$,

$$J^{H(.,.)-\eta}_{M(.,t),\varphi}(x) = \left[I - H(Q(R^{H(.,.)-\eta}_{M(.,t),\varphi}(x)), R(R^{H(.,.)-\eta}_{M(.,t),\varphi}(x)))\right],$$

I is the identity mapping and $H(Q,R)\left[R^{H(.,.)-\eta}_{M(.,t),\varphi}(x)\right] = H\Big(Q(R^{H(.,.)-\eta}_{M(.,t),\varphi}(x)), R(R^{H(.,.)-\eta}_{M(.,t),\varphi}(x))\Big)$.
Now, we show that the problem (4.1) is equivalent to the resolvent equation problem (4.3).

Lemma 4.2. *If (u,v,w,t) with $u \in Y$, $v \in S(u)$, $w \in T(u)$ and $t \in G(u)$ is a solution of problem (4.1) if and only if the resolvent equation problem (4.3) has a solution (x,u,v,w,t) with $x,u \in Y$, $v \in S(u)$, $w \in T(u)$ and $t \in G(u)$, where*

$$u = R^{H(.,.)-\eta}_{M(.,t),\varphi}(x), \tag{4.4}$$

and $x = H(Qu,Ru) - \lambda\varphi oP(v,w)$.

Proof: Let (u,v,w,t) be a solution of problem (4.1), and from Lemma 4.1 Using the fact that

$$J^{H(.,.)-\eta}_{M(.,t),\varphi} = \left[I - H\Big(Q(R^{H(.,.)-\eta}_{M(.,t),\varphi}), R(R^{H(.,.)-\eta}_{M(.,t)),\varphi})\Big)\right],$$

$$\begin{aligned}
J^{H(.,.)-\eta}_{M(.,t),\varphi}(x) &= J^{H(.,.)-\eta}_{M(.,t),\varphi}\left[H(Qu,Ru) - \lambda\varphi oP(v,w)\right]\\
&= \left[I - H\Big(Q(R^{H(.,.)-\eta}_{M(.,t),\varphi}), R(R^{H(.,.)-\eta}_{M(.,t)),\varphi})\Big)\right]\left[H(Qu,Ru) - \lambda\varphi oP(v,w)\right]\\
&= \left[H(Qu,Ru) - \lambda\varphi oP(v,w)\right] - H\Big(Q(R^{H(.,.)-\eta}_{M(.,t),\varphi}), R(R^{H(.,.)-\eta}_{M(.,t)),\varphi})\Big)\Big]\Big(H(Qu,Ru) - \lambda\varphi oP(v,w)\Big)\\
&= \left[H(Qu,Ru) - \lambda\varphi oP(v,w)\right]\\
&\quad - H\Big(Q(R^{H(.,.)-\eta}_{M(.,t),\varphi})(H(Qu,Ru) - \lambda\varphi oP(v,w)), R(R^{H(.,.)-\eta}_{M(.,t)),\varphi})(H(Qu,Ru) - \lambda\varphi oP(v,w))\Big)\\
&= \left[H(Qu,Ru) - \lambda\varphi oP(v,w)\right] - H(Qu,Ru)\\
&= -\lambda\varphi oP(v,w)
\end{aligned}$$

This implies that

$$\varphi oP(v,w) + \lambda^{-1} J^{H(.,.)-\eta}_{M(.,t),\varphi}(x) = 0.$$

Conversely, let (x,u,v,w,t) is a solution of resolvent equation problem (4.3), then

$$\begin{gathered}
J^{H(.,.)-\eta}_{M(.,t),\varphi}(x) = -\lambda\varphi oP(v,w)\\
\left[I - H\Big(Q(R^{H(.,.)-\eta}_{M(.,t),\varphi}), R(R^{H(.,.)-\eta}_{M(.,t)),\varphi})\Big)\right](x) = -\lambda\varphi oP(v,w)\\
x - H(Qu,Ru) = -\lambda\varphi oP(v,w).
\end{gathered}$$

This implies that

$$x = H(Qu,Ru) - \lambda\varphi oP(v,w).$$

Hence (u,v,w,t) is a solution of variational inclusion problem (4.1).

Lemma 4.1 and Lemma 4.2 are very crucial from the numerical point of view. They permit us to suggest the following iterative scheme for finding the approximate solution of (4.3).

Algorithm 4.3. For any given (x_0,u_0,v_0,w_0,t_0), we can choose $x_0,u_0 \in Y$, $v_0 \in S(u_0)$, $w_0 \in T(u_0)$, $t_0 \in G(u_0)$ and $0 < \varepsilon < 1$ such that sequences $\{x_k\}$, $\{u_k\}$, $\{v_k\}$, $\{w_k\}$ and $\{t_k\}$ satisfy

$$\begin{cases} u_k = R^{H(.,.)-\eta}_{M^k(.,t),\varphi}(x_k), \\ v_k \in S(u_k),\ \|v_k - v_{k+1}\| \le D(S(u_k),S(u_{k+1})) + \varepsilon^{k+1}\|u_k - u_{k+1}\|, \\ w_k \in T(u_k), \|w_k - w_{k+1}\| \le D(T(u_k),T(u_{k+1})) + \varepsilon^{k+1}\|u_k - u_{k+1}\|, \\ t_k \in G(u_k),\ \|t_k - t_{k+1}\| \le D(G(u_k),G(u_{k+1})) + \varepsilon^{k+1}\|u_k - u_{k+1}\|, \\ x_{k+1} = H(Qu_k,Ru_k) - \lambda\varphi oP(v_k,w_k), \end{cases}$$

where $\lambda > 0$, $k \ge 0$, and $D(.,.)$ is the Hausdorff metric on $\mathrm{CB}(Y)$.

Next, we find the convergence of the iterative algorithm for the resolvent equation problem (4.3) corresponding generalized set-valued variational inclusion problem (4.1).

Theorem 4.4. *Let us consider the problem (4.1) with assumptions M_1-M_4 and $\varphi : Y \to Y$ be a single-valued mapping with $\varphi(v+w) = \varphi(v)+\varphi(w)$ and $Ker(\varphi) = \{0\}$. Let assume that*

(i) *S,T and G are λ_S,λ_T and λ_G continuous, respectively;*

(ii) *$H(Q,R)$ is κ_1, κ_2-Lipschitz continuous in regards Q and R, respectively;*

(iii) *φoP is (ν,η)-relaxed monotone in regards first component;*

(iv) *φoP is ε_1, ε_2-Lipschitz continuous in regards first and second component, respectively;*

(v) $0 < \sqrt{\left\{(\kappa_1+\kappa_2)^2 + 2\nu\lambda\lambda_S^2 - 2\varepsilon_1\lambda\lambda_S\Big((\kappa_1+\kappa_2)+\tau\lambda_S\Big) + \varepsilon_1^2\lambda^2\lambda_S^2\right\}} < \frac{(1-\xi\lambda_G)(\ell-m\lambda)}{\tau} - \varepsilon_2\lambda\lambda_T$;

(vi) $\|R^{H(.,.)-\varphi-\eta}_{M^k(.,t_k)}(u) - R^{H(.,.)-\varphi-\eta}_{M^{k-1}(.,t_{k-1})}(u)\| \le \xi\|t_k - t_{k-1}\|,\ \forall\, t,t^* \in Y, \xi > 0;$

Then the iterative sequences $\{x_k\},\{u_k\},\{v_k\},\{w_k\}$, and $\{t_k\}$ generated by Algorithm 4.3 converges strongly to the unique solution (x,u,v,w,t) of the resolvent equation problem (4.3).

Proof. Using Algorithms 4.3 and $\lambda_S, \lambda_T, \lambda_G$-$D$ Lipschitz continuity of S, T and G, we have

$$\|v_k - v_{k-1}\| \leq D(S(u_k), S(u_{k-1})) + \varepsilon^k \|u_k - u_{k-1}\| \leq \{\lambda_S + \varepsilon^k\}\|u_k - u_{k-1}\|, \tag{4.5}$$

$$\|w_k - w_{k-1}\| \leq D(T((u_k), T(u_{k-1})) + \varepsilon^k \|u_k - u_{k-1}\| \leq \{\lambda_T + \varepsilon^k\}\|u_k - u_{k-1}\|, \tag{4.6}$$

$$\|t_k - t_{k-1}\| \leq D(G((u_k), G(u_{k-1})) + \varepsilon^k \|u_k - u_{k-1}\| \leq \{\lambda_G + \varepsilon^k\}\|u_k - u_{k-1}\|, \tag{4.7}$$

where $k = 1,\ 2,\ .....$

Now, we compute

$$\begin{aligned}\|x_{k+1} - x_k\| &= \|H(Qu_k, Ru_k) - H(Qu_{k-1}, Ru_{k-1}) - \lambda(\varphi oP(v_k, w_k) - \varphi oP(v_{k-1}, w_{k-1}))\| \\ &\leq \|H(Qu_k, Ru_k) - H(Qu_{k-1}, Ru_{k-1}) - \lambda(\varphi oP(v_k, w_k) - \varphi oP(v_{k-1}, w_k))\| \\ &\quad + \lambda\|\varphi oP(v_{k-1}, w_k) - \varphi oP(v_{k-1}, w_{k-1}))\|.\end{aligned} \tag{4.8}$$

$$\begin{aligned}&\|H(Qu_k, Ru_k) - H(Qu_{k-1}, Ru_{k-1}) - \lambda(\varphi oP(v_k, w_k) - \varphi oP(v_{k-1}, w_k))\|^2 \\ &\leq \|H(Qu_k, Ru_k) - H(Qu_{k-1}, Ru_{k-1})\|^2 \\ &\quad -2\lambda[\varphi oP(v_k, w_k) - \varphi oP(v_{k-1}, w_k), \eta(v_k, v_{k-1})] \\ &\quad +2\lambda\|\varphi oP(v_k, w_k) - \varphi oP(v_{k-1}, w_k)\| \\ &\quad \times\Big\{\|H(Qu_k, Ru_k) - H(Qu_{k-1}, Ru_{k-1})\| + \|\eta(v_n, v_{n-1})\|\Big\} \\ &\quad +\lambda^2\|\varphi oP(v_k, w_k) - \varphi oP(v_{k-1}, w_k)\|^2.\end{aligned} \tag{4.9}$$

Since $H(Q,R)$ is κ_1, κ_2-Lipschitz continuous in regards Q, R, respectively, We have

$$\|H(Qu_k, Ru_k) - H(Qu_{k-1}, Ru_{k-1})\|^2 \leq (\kappa_1 + \kappa_2)^2 \|u_k - u_{k-1}\|^2. \tag{4.10}$$

Since φoP is (ν, η)-relaxed monotone, then we have

$$[\varphi oP(v_k, w_k) - \varphi oP(v_{k-1}, w_k), \eta(v_k, v_{k-1})] \geq -\nu\|v_k - v_{k-1}\|^2 \geq -\nu\{\lambda_S + \varepsilon^k\}^2 \|u_k - u_{k-1}\|.^2 \tag{4.11}$$

As $\varphi oP(.,.)$ is $\varepsilon_1,\varepsilon_2$-Lipschitz continuous in the first, second arguments, respectively and using (4.5),(4.6), we have

$$\|\varphi oP(v_k,w_k)-\varphi oP(v_{k-1},w_k)\| \leq \varepsilon_1\|v_k-v_{k-1}\| \leq \varepsilon_1\{\lambda_S+\varepsilon^k\}\|u_k-u_{k-1}\|, \tag{4.12}$$

$$\|\varphi oP(v_{k-1},w_k)-\varphi oP(v_{k-1},w_{k-1})\| \leq \varepsilon_2\|w_k-w_{k-1}\| \leq \ \varepsilon_2\{\lambda_T+\varepsilon^k\}\|u_k-u_{k-1}\|. \tag{4.13}$$

By using M-3 and (4.10)-(4.13) in (4.9), we have

$$\begin{aligned}
&\|H(Qu_k,Ru_k)-H(Qu_{k-1},Ru_{k-1})-(\varphi oP(v_k,w_k)-\varphi oP(v_{k-1},w_k))\|^2 \\
&\leq \Big\{(\kappa_1+\kappa_2)^2+2\nu\lambda\{\lambda_S+\varepsilon^k\}^2-2\varepsilon_1\lambda\{\lambda_S+\varepsilon^k\}\Big\{(\kappa_1+\kappa_2)+\tau\{\lambda_S+\varepsilon^k\}\Big\}+\varepsilon_1^2\lambda^2\{\lambda_S+\varepsilon^k\}^2\Big\} \\
&\qquad\qquad \times\|u_k-u_{k-1}\|^2,
\end{aligned}$$

$$\begin{aligned}
&\|H(Qu_k,Ru_k)-H(Qu_{k-1},Ru_{k-1})-(\varphi oP(v_k,w_k)-\varphi oP(v_{k-1},w_k))\| \\
&\leq \sqrt{\Big[(\kappa_1+\kappa_2)^2+2\nu\lambda\{\lambda_S+\varepsilon^k\}^2-2\varepsilon_1\lambda\{\lambda_S+\varepsilon^k\}\Big\{(\kappa_1+\kappa_2)+\tau\{\lambda_S+\varepsilon^k\}\Big\}+\varepsilon_1^2\lambda^2\{\lambda_S+\varepsilon^k\}^2\Big]} \\
&\qquad\qquad \times\|u_k-u_{k-1}\|.
\end{aligned} \tag{4.14}$$

Using (4.14) in (4.8), we get

$$\begin{aligned}
&\|x_{k+1}-x_k\| = \|H(Qu_k,Ru_k)-H(Qu_{k-1},Ru_{k-1})-(\varphi oP(v_k,w_k)-\varphi oP(v_{k-1},w_k))\| \\
&\leq \Big[\sqrt{\Big[(\kappa_1+\kappa_2)^2+2\nu\lambda\{\lambda_S+\varepsilon^k\}^2-2\varepsilon_1\lambda\{\lambda_S+\varepsilon^k\}\Big\{(\kappa_1+\kappa_2)+\tau\{\lambda_S+\varepsilon^k\}\Big\}+\varepsilon_1^2\lambda^2\{\lambda_S+\varepsilon^k\}^2\Big]} \\
&\qquad\qquad +\varepsilon_2\lambda\{\lambda_T+\varepsilon^k\}\Big]\times\|u_k-u_{k-1}\|.
\end{aligned} \tag{4.15}$$

By Lipschitz continuity of resolvent operator and condition (vi),(4.7), we have

$$\begin{aligned}
\|u_k-u_{k-1}\| &\leq \|R^{H(.,.)-\eta}_{M^k(.,t_k),\varphi}(x_k)-R^{H(.,.)-\eta}_{M^k(.,t_{k-1}),\varphi}(x_{k-1})\| \\
&\leq \|R^{H(.,.)-\eta}_{M^k(.,t_k),\varphi}(x_k)-R^{H(.,.)-\eta}_{M^k(.,t_k),\varphi}(x_{k-1})\| \\
&\quad +\|R^{H(.,.)-\eta}_{M^k(.,t_k),\varphi}(x_{k-1})-R^{H(.,.)-\eta}_{M^{k-1}(.,t_{k-1}),\varphi}(x_{k-1})\| \\
&\leq \frac{\tau}{\ell-m\lambda}\|x_k-x_{k-1}\|+\xi\|t_k-t_{k-1}\| \\
&\leq \frac{\tau}{\ell-m\lambda}\|x_k-x_{k-1}\|+\xi\{\lambda_G+\varepsilon^k\}\|u_k-u_{k-1}\|
\end{aligned}$$

$$\|u_k - u_{k-1}\| \leq \frac{\tau}{(\ell - m\lambda)(1-\xi(\lambda_G + \varepsilon^k))}\|x_k - x_{k-1}\|. \tag{4.16}$$

Using (4.16), in (4.15), equation (4.15) becomes

$$\|x_{k+1} - x_k\| \leq \Theta(\varepsilon^k)\|x_k - x_{k-1}\|, \text{ where} \tag{4.17}$$

$$\Theta(\varepsilon^k) =$$

$$\frac{\tau\sqrt{\left\{(\kappa_1+\kappa_2)^2 + 2\nu\lambda\{\lambda_S+\varepsilon^k\}^2 - 2\varepsilon_1\lambda\{\lambda_S+\varepsilon^k\}\Big((\kappa_1+\kappa_2)+\tau\{\lambda_S+\varepsilon^k\}\Big)+\varepsilon_1^2\lambda^2\{\lambda_S+\varepsilon^k\}^2\right\}}+\tau\varepsilon_2\lambda\{\lambda_T+\varepsilon^k\}}{(\ell-m\lambda)(1-\xi(\lambda_G+\varepsilon^k))}.$$

Since $0 < \varepsilon < 1$, this implies that $\Theta(\varepsilon^k) \to \Theta$ as $k \to \infty$, where

$$\Theta = \frac{\tau\left[\sqrt{\left\{(\kappa_1+\kappa_2)^2 + 2\nu\lambda\lambda_S^2 - 2\varepsilon_1\lambda\lambda_S\Big((\kappa_1+\kappa_2)+\tau\lambda_S\Big)+\varepsilon_1^2\lambda^2\lambda_S^2\right\}}+\varepsilon_2\lambda\lambda_T\right]}{(\ell-m\lambda)(1-\xi\lambda_G)}.$$

It is given that $\Theta < 1$, then $\{x_k\}$ is a Cauchy sequence in Banach space Y, then $x_k \to x$ as $k \to \infty$. From (4.16), $\{u_k\}$ is also Cauchy sequence in Banach space Y, then there exist u such that $u_k \to u$.

From equation (4.5)-(4.7) and Algorithm 4.3, the sequences $\{v_k\}$, $\{w_k\}$ and $\{t_k\}$ are also Cauchy sequences in Y. Thus, there exist v, w and t such that $v_k \to v$, $w_k \to w$ and $t_k \to t$ as $k \to \infty$. Next we will prove that $v \in S(u)$. Since $v_k \in S(u_k)$, then

$$\begin{aligned} d(v,\, S(u)) &\leq \|v - v_k\| + d(v_k,\, S(u)) \\ &\leq \|v - v_k\| + D(S(u_k),\, S(u)) \\ &\leq \|v - v_k\| + \lambda_S\,\|u_k - u\| \to 0, \text{ as } k \to \infty, \end{aligned}$$

which gives $d(v,\, S(u)) = 0$. Due to $S(u) \in CB(Y)$, we have $v \in S(u)$. In the same manner, we easily show that $w \in T(u)$ and $t \in G(u)$.

By the continuity of $R^{H(.,.)-\eta}_{M(.,t),\varphi}$, Q, R, S, T G, φoP, η and M and Algorithms 4.3, we know that u, v, w and t satisfy

$$\begin{aligned} x_{k+1} &= [H(Qu_k, Ru_k) - \lambda\varphi oP(v_k, w_k)], \\ \to x &= [H(Qu, Ru) - \lambda\varphi oP(v, w)] \; as\; k \to \infty. \end{aligned}$$

$R^{H(.,.,)-\eta}_{M(.,t),\varphi}(x_k) = u_k \to u = R^{H(.,.,)-\eta}_{M(.,t),\varphi}(x)$, as $k \to \infty$. Using the Lemma 4.2 and above equation we have

$$\varphi o P(v,w) + \lambda^{-1}(x - H(Q(R^{H(.,.)-\eta}_{M(.,t),\varphi}(x)), R(R^{H(.,.)-\eta}_{M(.,t),\varphi}(x))) = 0, \tag{4.18}$$

$$\varphi o P(v,w) + \lambda^{-1} J^{H(.,.)-\eta}_{M(.,t),\varphi}(x) = 0. \tag{4.19}$$

Hence (x,u,v,w,t) is a solution of the problem (4.3).

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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