



Available online at <http://scik.org>

J. Math. Comput. Sci. 10 (2020), No. 6, 2926-2942

<https://doi.org/10.28919/jmcs/5005>

ISSN: 1927-5307

## DEVELOPMENT AND ANALYSIS OF FIFTH STAGE INVERSE POLYNOMIAL SCHEME FOR THE SOLUTION OF STIFF LINEAR AND NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS

S.E. FADUGBA<sup>1</sup>, V.J. SHAALINI<sup>2,\*</sup>, A.A. IBRAHIM<sup>3</sup>

<sup>1</sup>Department of Mathematics, Ekiti State University, Ado Ekiti, 360001, Nigeria

<sup>2</sup>Department of Mathematics, Bishop Heber College, Trichy, India

<sup>3</sup>Department of Mathematical Sciences, Baze University, Abuja, Nigeria

Copyright © 2020 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

**Abstract:** This paper presents the development and analysis of fifth stage inverse polynomial scheme (ISM5) for the solution of Initial Value Problems (IVPs) in Ordinary Differential Equations (ODEs) of stiff types. The properties of the scheme were investigated and analyzed. The new scheme is tested on three numerical examples and the results were compared with the Runge-Kutta method of order five (RK5) in the context of the analytical solution. The results generated via ISM5 agreed with the analytical solution and outperformed RK5. Hence, ISM5 is found to be accurate, efficient and a powerful method for obtaining the numerical solution of stiff ODEs.

**Keywords:** analytical solution; fifth stage; inverse polynomial; linearity, nonlinearity; stiff ordinary differential equation.

**2010 AMS Subject Classification:** 34A12, 65L05, 65L20, 65L70.

---

\*Corresponding author

E-mail address: [shaalini.vinci@gmail.com](mailto:shaalini.vinci@gmail.com)

Received September 8, 2020

## 1. INTRODUCTION

Within the past two decades, many numerical methods have been developed to generate numerical solutions to IVPs of ODEs. Such problems often arise in different areas of engineering and applied sciences such as celestial mechanics, quantum mechanics, electrodynamics, theoretical physics and chemistry, and electronics.

Many mathematical models in ODEs are ‘stiff’ in nature. The step size is taken to be extremely small while solving in stiff differential equations. Also, many problems may be stiff in some intervals and non-stiff in others. Therefore, we need an efficient technique to be suitable for stiff and non-stiff problems. Some remarkable works are done by [1], [2], [3], [4], [5] and [6], just to mention a few. There are many one-step numerical techniques that have been developed for the solution of first order ODEs by means of interpolating functions. These works can be referred in [7 - 13].

In this paper, fifth order one-step method which is based on inverse polynomial interpolating function that can effectively handle stiff IVPs in ODEs is proposed. The organization of the paper is as follows: In Section 2, the derivation of the new scheme is presented. In Section 3, the properties of the scheme such as local truncation error, order of convergence, consistency and stability were investigated and analysed. Finally, the numerical examples were solved to demonstrate the accuracy, efficiency and effectiveness of the scheme in Section 4. Also, the concluding remarks are presented.

## 2. DERIVATION OF FIFTH STAGE INVERSE POLYNOMIAL SCHEME

The derivation of the fifth stage inverse polynomial scheme for the solution of IVP of the form

$$y' = f(x, y), \quad y(a) = y_0, x \in [a, b], y \in R \quad (1)$$

Consider the inverse polynomial of  $k$ -stage given by

$$y_{n+1} = y_n \left[ \sum_{j=0}^k a_j x_n^j \right]^{-1} \quad (2)$$

where the parameters  $a_j$ 's are to be determined from nonlinear equations that will be generated

by the fifth order inverse polynomial,  $k = 5$

$$y_{n+1} = y_n \left[ \sum_{j=0}^5 a_j x_n^j \right]^{-1} \quad (3)$$

$$= y_n [a_0 x_n^0 + a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5]^{-1} \quad (4)$$

Setting  $a_0 = 1$ , (4) becomes

$$y_{n+1} = y_n [1 + (a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5)]^{-1} \quad (5)$$

Taking the binomial expansion of (5), we get

$$\begin{aligned} y_{n+1} = & y_n [1 + (-1)(a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5) \\ & + \frac{(-1)(-2)}{2!} (a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5)^2 \\ & + \frac{(-1)(-2)(-3)}{3!} (a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5)^3 \\ & + \frac{(-1)(-2)(-3)(-4)}{4!} (a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5)^4 \\ & + \frac{(-1)(-2)(-3)(-4)(-5)}{5!} (a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5)^5 ] \end{aligned} \quad (6)$$

$$\begin{aligned} y_{n+1} = & y_n [1 - (a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5) + (a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5)^2 \\ & - (a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5)^3 + (a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5)^4 \\ & + (a_1 x_n + a_2 x_n^2 + a_3 x_n^3 + a_4 x_n^4 + a_5 x_n^5)^5 ] \end{aligned} \quad (7)$$

Expressing the LHS of (5) in terms of Taylor's series expansion, we have

$$y(x_n + h) = y_n + h y'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y'''_n + \frac{h^4}{4!} y^{iv}_n + \frac{h^5}{5!} y^v_n \quad (8)$$

Substituting (8) and (7) into (5), we have

$$\begin{aligned} y_n + h y'_n + \frac{h^2}{2!} y''_n + \frac{h^3}{3!} y'''_n + \frac{h^4}{4!} y^{iv}_n + \frac{h^5}{5!} y^v_n = & y_n - a_1 x_n y_n + (a_1^2 - a_2) x_n^2 y_n \\ & + (-a_1^3 + 2a_1 a_2 - a_3) x_n^3 y_n + (a_1^4 - 3a_1^2 a_2 + 2a_1 a_3 + a_2^2 - a_4) x_n^4 y_n \\ & + (-a_1^5 + 4a_1^3 a_2 - 3a_1^2 a_3 - 3a_1 a_2^2 + 2a_1 a_4 + 2a_2 a_3 - a_5) x_n^5 y_n \end{aligned} \quad (9)$$

Equating (9) term by term, we obtain the values of  $a_1, a_2, a_3, a_4$  and  $a_5$  as follows

$$a_1 = \frac{-hy'_n}{x_n y_n} \tag{10}$$

$$a_2 = \frac{2h^2(y'_n)^2 - h^2 y_n y''_n}{2x_n^2 y_n^2} \tag{11}$$

$$a_3 = \left[ \frac{-h^3 y_n^2 y'''_n - 6h^3 (y'_n)^3 + 6h^3 y_n y'_n y''_n}{6x_n^3 y_n^3} \right] \tag{12}$$

$$a_4 = \frac{24h^4 (y'_n)^4 - 36h^4 y_n (y'_n)^2 y''_n + 8h^4 y_n^2 y'_n y'''_n - h^4 y_n^3 y_n^{iv} + 6h^4 (y_n)^2 (y''_n)^2}{24x_n^4 y_n^4} \tag{13}$$

$$a_5 = \frac{1}{120x_n^5 y_n^5} [-120h^5 (y'_n)^5 + 240h^5 y_n (y'_n)^3 y''_n - 180h^5 y_n^2 (y'_n)^2 y_n''' - 90h^5 y_n^2 (y'_n)(y''_n)^2 + 10h^5 y_n^3 (y'_n) y_n^{iv} + 20h^5 y_n^3 y_n'' y_n''' - h^5 y_n^{(5)}] \tag{14}$$

Substituting (10), (11), (12), (13), (14) into (5), one obtains

$$y_{n+1} = y_n \left[ 1 - \left[ \frac{hy'_n}{x_n y_n} \right] x_n + \left[ \frac{2h^2 (y'_n)^2 - h^2 y_n y''_n}{2x_n^2 y_n^2} \right] x_n^2 + \left[ \frac{-h^3 y_n^2 y'''_n - 6h^3 (y'_n)^3 + 6h^3 y_n y'_n y''_n}{6x_n^3 y_n^3} \right] x_n^3 + \frac{1}{24x_n^4 y_n^4} \left[ \frac{24h^4 (y'_n)^4 - 36h^4 y_n (y'_n)^2 y''_n + 8h^4 y_n^2 y'_n y'''_n - h^4 y_n^3 y_n^{iv} + 6h^4 (y_n)^2 (y''_n)^2}{24x_n^4 y_n^4} \right] x_n^4 + \frac{1}{120x_n^5 y_n^5} [-120h^5 (y'_n)^5 + 240h^5 y_n (y'_n)^3 y''_n - 180h^5 y_n^2 (y'_n)^2 y_n''' - 90h^5 y_n^2 (y'_n)(y''_n)^2 + 10h^5 y_n^3 (y'_n) y_n^{iv} + 20h^5 y_n^3 y_n'' y_n''' - h^5 y_n^{(5)}] x_n^5 \right]^{-1} \tag{15}$$

Simplifying (15) further, yields

$$y_{n+1} = 120y_n^6 \left[ \begin{aligned} &120y_n^5 - 120hy'_n y_n^4 + 60y_n^3 (2h^2 (y'_n)^2 - h^2 y_n y''_n) \\ &+ 20y_n^2 (-h^3 y_n^2 (y'_n)''' - 6h^3 (y'_n)^3 + 6h^3 y_n y'_n y''_n) \\ &+ 5y_n (24h^4 (y'_n)^4 - 36h^4 y_n (y'_n)^2 y''_n + 8h^4 y_n^2 y'_n y'''_n - h^4 y_n^3 y_n^{iv} + 6h^4 (y_n)^2 (y''_n)^2) \\ &- 120h^5 (y'_n)^5 + 240h^5 y_n (y'_n)^3 y''_n - 180h^5 y_n^2 (y'_n)^2 y_n''' - 90h^5 y_n^2 (y'_n)(y''_n)^2 \\ &+ 10h^5 y_n^3 (y'_n) y_n^{(4)} + 20h^5 y_n^3 y_n'' y_n''' - h^5 y_n^{(5)} \end{aligned} \right]^{-1} \tag{16}$$

$$y_{n+1} = 120y_n^6 \left[ \begin{array}{l} 120y_n^5 - 120hy'_n y_n^4 + 120h^2 y_n^3 (y'_n)^2 - 60h^2 y_n^4 y_n'' - 20h^3 y_n^4 (y'_n)''' \\ -120h^3 y_n^2 (y'_n)^3 + 120h^3 y_n^3 y'_n y_n'' \\ +120h^4 y_n (y'_n)^4 - 180h^4 y_n^2 (y'_n)^2 y_n'' + 40h^4 y_n^3 y'_n y_n''' \\ -5h^4 y_n^4 y_n^{iv} + 30h^4 (y_n)^3 (y_n'')^2 \\ -120h^5 (y'_n)^5 + 240h^5 y_n (y'_n)^3 y_n'' - 180h^5 y_n^2 (y'_n)^2 y_n'' \\ -90h^5 y_n^2 (y'_n)(y_n'')^2 + 10h^5 y_n^3 (y'_n) y_n^{iv} + 20h^5 y_n^3 y_n'' y_n''' - h^5 y_n^v \end{array} \right]^{-1} \quad (17)$$

Equation (17) is the new fifth stage inverse polynomial scheme.

### 3. ANALYSIS OF PROPERTIES OF FIFTH STAGE INVERSE POLYNOMIAL SCHEME

In this section, we analyze the basic properties of the developed fifth inverse polynomial scheme, such as the consistency, stability and convergence.

#### 3.1 Consistency Property of the Scheme

A numerical scheme with an increment function  $\phi(x_n, y_n; h)$  is said to be consistent with (1) if

$$\phi(x_n, y_n; 0) = f(x_n, y_n) \quad (18)$$

From the general linear one-step method, we have that

$$y_{n+1} - y_n = h\phi(x_n, y_n; h) \quad (19)$$

This implies that

$$\frac{y_{n+1} - y_n}{h} = \phi(x_n, y_n; h) \quad (20)$$

Now substitute the new scheme in (15) into (20), we get

$$y_{n+1} - y_n = 120y_n^6 \left[ \begin{array}{l} 120y_n^5 - 120hy'_n y_n^4 + 120h^2 y_n^3 (y'_n)^2 - 60h^2 y_n^4 y_n'' - 20h^3 y_n^4 (y'_n)''' \\ -120h^3 y_n^2 (y'_n)^3 + 120h^3 y_n^3 y'_n y_n'' \\ +120h^4 y_n (y'_n)^4 - 180h^4 y_n^2 (y'_n)^2 y_n'' + 40h^4 y_n^3 y'_n y_n''' \\ -5h^4 y_n^4 y_n^{iv} + 30h^4 (y_n)^3 (y_n'')^2 \\ -120h^5 (y'_n)^5 + 240h^5 y_n (y'_n)^3 y_n'' - 180h^5 y_n^2 (y'_n)^2 y_n'' \\ -90h^5 y_n^2 (y'_n)(y_n'')^2 + 10h^5 y_n^3 (y'_n) y_n^{iv} + 20h^5 y_n^3 y_n'' y_n''' - h^5 y_n^v \end{array} \right]^{-1} - y_n \quad (21)$$

Simplifying (21) further yields

## FIFTH STAGE INVERSE POLYNOMIAL SCHEME

$$\frac{y_{n+1} - y_n}{h} = \frac{\begin{aligned} &120y'_n y_n^5 - h \left[ 120y_n^4 (y'_n)^2 + 60y_n^5 y_n'' \right] \\ &+ h^2 \left[ 20y_n^5 (y'_n)''' + 120y_n^3 (y'_n)^3 - 120y_n^4 y'_n y_n'' \right] \\ &+ h^3 \left[ \begin{aligned} &-120y_n^2 (y'_n)^4 + 180y_n^3 (y'_n)^2 y_n'' \\ &-40y_n^4 y'_n y_n''' + 5y_n^5 y_n^{iv} - 30(y_n)^4 (y_n'')^2 \end{aligned} \right] \\ &+ h^4 \left[ \begin{aligned} &120y_n (y'_n)^5 - 240y_n^2 (y'_n)^3 y_n'' + 180y_n^3 (y'_n)^2 y_n''' \\ &+ 90y_n^3 (y'_n) (y_n'')^2 - 10y_n^4 (y'_n) y_n^{iv} - 20y_n^4 y_n'' y_n''' + y_n^v \end{aligned} \right] \end{aligned}}{\begin{aligned} &120y_n^5 - 120hy'_n y_n^4 + 60h^2 \left[ 2y_n^3 (y'_n)^2 - y_n^4 y_n'' \right] \\ &- 20h^3 \left[ y_n^4 (y'_n)''' + 6y_n^2 (y'_n)^3 - 6y_n^3 y'_n y_n'' \right] \\ &+ 5h^4 \left[ \begin{aligned} &24y_n (y'_n)^4 - 36y_n^2 (y'_n)^2 y_n'' \\ &+ 8y_n^3 y'_n y_n''' - y_n^4 y_n^{iv} + 6(y_n)^3 (y_n'')^2 \end{aligned} \right] \\ &+ h^5 \left[ \begin{aligned} &-120(y'_n)^5 + 240y_n (y'_n)^3 y_n'' - 180y_n^2 (y'_n)^2 y_n''' \\ &- 90y_n^2 (y'_n) (y_n'')^2 + 10y_n^3 (y'_n) y_n^{iv} + 20y_n^3 y_n'' y_n''' - y_n^v \end{aligned} \right] \end{aligned}} \quad (22)$$

Therefore,

$$\varphi(x_n, y_n; h) = \frac{\begin{aligned} &120y'_n y_n^5 - h \left[ 120y_n^4 (y'_n)^2 + 60y_n^5 y_n'' \right] \\ &+ h^2 \left[ 20y_n^5 (y'_n)''' + 120y_n^3 (y'_n)^3 - 120y_n^4 y'_n y_n'' \right] \\ &+ h^3 \left[ \begin{aligned} &-120y_n^2 (y'_n)^4 + 180y_n^3 (y'_n)^2 y_n'' \\ &-40y_n^4 y'_n y_n''' + 5y_n^5 y_n^{iv} - 30(y_n)^4 (y_n'')^2 \end{aligned} \right] \\ &+ h^4 \left[ \begin{aligned} &120y_n (y'_n)^5 - 240y_n^2 (y'_n)^3 y_n'' + 180y_n^3 (y'_n)^2 y_n''' \\ &+ 90y_n^3 (y'_n) (y_n'')^2 - 10y_n^4 (y'_n) y_n^{iv} - 20y_n^4 y_n'' y_n''' + y_n^v \end{aligned} \right] \end{aligned}}{\begin{aligned} &120y_n^5 - 120hy'_n y_n^4 + 60h^2 \left[ 2y_n^3 (y'_n)^2 - y_n^4 y_n'' \right] \\ &- 20h^3 \left[ y_n^4 (y'_n)''' + 6y_n^2 (y'_n)^3 - 6y_n^3 y'_n y_n'' \right] \\ &+ 5h^4 \left[ \begin{aligned} &24y_n (y'_n)^4 - 36y_n^2 (y'_n)^2 y_n'' \\ &+ 8y_n^3 y'_n y_n''' - y_n^4 y_n^{iv} + 6(y_n)^3 (y_n'')^2 \end{aligned} \right] \\ &+ h^5 \left[ \begin{aligned} &-120(y'_n)^5 + 240y_n (y'_n)^3 y_n'' - 180y_n^2 (y'_n)^2 y_n''' \\ &- 90y_n^2 (y'_n) (y_n'')^2 + 10y_n^3 (y'_n) y_n^{iv} + 20y_n^3 y_n'' y_n''' - y_n^v \end{aligned} \right] \end{aligned}} \quad (23)$$

Taking the limit as  $h$  approaches zero, (23) becomes

$$\varphi(x_n, y_n; 0) = \left[ \frac{120y'_n y_n^5}{120y_n^5} \right] \quad (24)$$

Hence,

$$\varphi(x_n, y_n; 0) = y'_n = f(x_n, y_n) \quad (25)$$

This shows the consistency property of the scheme. Also it is clearly seen from (9) that the scheme is of order five. The local truncation error of the scheme is obtained as

$$T_{n+1} = O(h^6) \quad (26)$$

### 3.2 Stability Property of the Scheme

The new one step scheme is said to be stable if for any initial error  $e_0$ , there exist a constant  $N$  and  $h > 0$  such that when the general one-step scheme is applied to IVP with  $h \in (0, h_0)$ , the ultimate error  $e_n$  satisfies the following inequalities

$$e_n \leq Ne_0, 0 < N < 1 \quad (27)$$

For  $k=5$ , one obtains

$$y_{n+h} = y_{n+h-1} \left[ \sum_{j=0}^5 a_j x_{n+h}^j \right]^{-1} \quad (28)$$

The theoretical solution  $y(x)$  is given by

$$y(x_{n+h}) = y(x_{n+h-1}) \left[ \sum_{j=0}^5 a_j x_{n+h}^j \right]^{-1} + T_{n+h} \quad (29)$$

Subtracting (28) from (29), yields

$$y(x_{n+h}) - y_{n+h} = y(x_{n+h-1}) \left[ \sum_{j=0}^5 a_j x_{n+h}^j \right]^{-1} - y_{n+h-1} \left[ \sum_{j=0}^5 a_j x_{n+h}^j \right]^{-1} + T_{n+h} \quad (30)$$

$$e_{n+h} = e_{n+h-1} \left[ \sum_{j=0}^5 a_j x_{n+h}^j \right]^{-1} + T_{n+h} \quad (31)$$

Taking the norm of both sides of (31), yields

$$\|e_{n+h}\| = \left\| e_{n+h-1} \left[ \sum_{j=0}^5 a_j x_{n+h}^j \right]^{-1} \right\| + \|T_{n+h}\| \quad (32)$$

Assuming that

$$P = \sum_{j=0}^5 a_j x_{n+h}^j \quad (33)$$

Then

$$\left\| \left[ \sum_{j=0}^5 a_j x_{n+h}^j \right]^{-1} \right\| = \left[ \sum_{j=0}^5 a_j x_{n+h}^j \right]^{-1} \quad (34)$$

$$\left[ \sum_{j=0}^5 a_j x_{n+h}^j \right]^{-1} = P^{-1} = N \quad (35)$$

$$\|e_{n+h}\| \leq N \|e_{n+h-1}\| + \|T_{n+h}\| \quad (36)$$

Let  $E_{n+h} = \sup(e_{n+h})$  and  $T = \sup(T_{n+h})$ . Similarly,  $E_{n+h-1} = \sup(e_{n+h-1})$  with  $n \in (0, \infty)$ . We

then have

$$E_{n+h} \leq N E_{n+h-1} + T \quad (37)$$

For  $h = 1$ ,

$$E_{n+1} \leq N E_n + T \quad (38)$$

For  $h = 2$ ,

$$\begin{aligned} E_{n+2} &\leq N E_{n+1} + T \\ &\leq N(N E_n + T) + T \\ &\leq N^2 E_n + N T + T \end{aligned} \quad (39)$$

Continuing this way, the general form is obtained as

$$E_{n+h} \leq N^k E_n + \sum_{r=0}^{k-1} N^r T \quad (40)$$

$$\|e_{n+h}\| < E_{n+h} < N^k E_n + \sum_{r=0}^{k-1} N^r T \quad (41)$$

Since  $N < 1$  and as  $n \rightarrow \infty$ ,  $E_{n+h} \rightarrow 0$  and this shows that the new scheme is stable and convergent.



### 3.3 Convergence Property of the Scheme

For any numerical scheme to be convergent, it must be consistent and stable [14]. Since these conditions have been achieved previously, we conclude that the new scheme is convergent. Also, it has fifth order convergent.

## 4. NUMERICAL EXAMPLES

The performance of the new scheme is tested on the following stiff IVPs in ODEs:

### Example 1

Consider the following system of stiff ordinary differential equations with Stiffness Ratio;

SR=1000 given by

$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 998 & 1998 \\ -999 & -1999 \end{pmatrix} \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} \quad (42)$$

with initial conditions as follows

$$\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (43)$$

whose exact solutions are given by

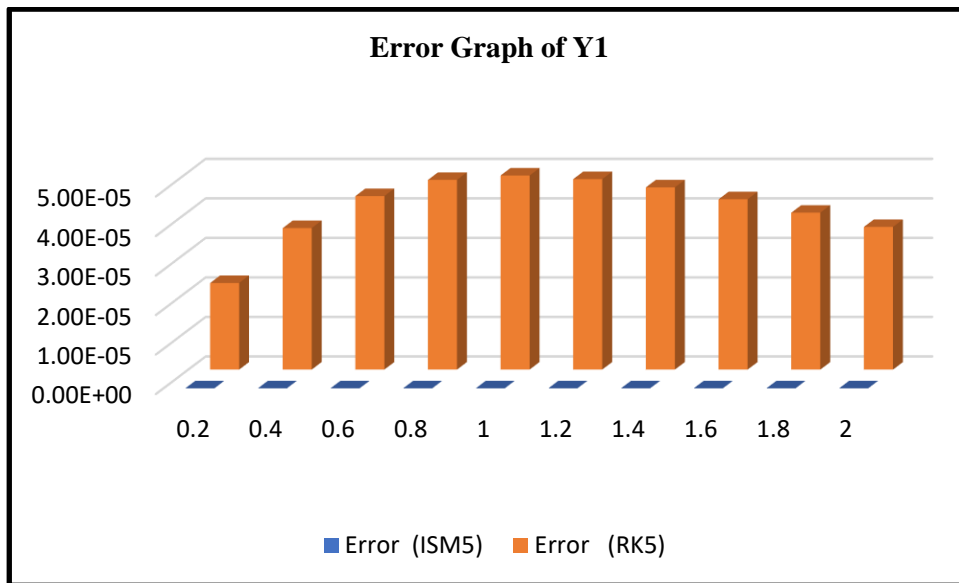
$$\begin{aligned} y_1(t) &= 4 \exp(-t) - 3 \exp(-1000t), \\ y_2(t) &= -2 \exp(-t) + 3 \exp(-1000t). \end{aligned} \quad (44)$$

The results generated via the new scheme (ISM5), Runge-Kutta method of order five (RK5) and the analytical solution are displayed in the Tables 1 and 2.

FIFTH STAGE INVERSE POLYNOMIAL SCHEME

**Table 1: The comparative results analyses of ISM5, RK5 and analytical solution**

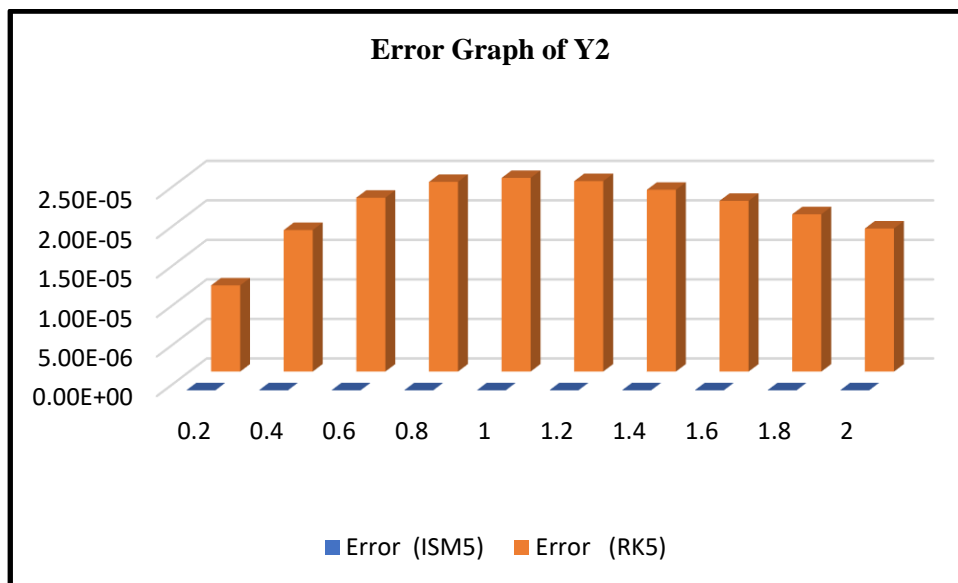
Time (t)	Y1(ISM5)	Y1 (RK5)	Analytical Solution
0.2	3.274923012311923	3.274901157720647	3.274923012311927
0.4	2.681280184142582	2.681244398210025	2.681280184142556
0.6	2.195246544376139	2.195202595957511	2.195246544376105
0.8	1.797315856468904	1.797267880733162	1.797315856468885
1.0	1.471517764685790	1.471468665836800	1.471517764685768
1.2	1.204776847648827	1.204728609324655	1.204776847648834
1.4	0.986387855766441	0.986341779354132	0.986387855766469
1.6	0.807586071978629	0.807542958778775	0.807586071978674
1.8	0.661195552886353	0.661155842653446	0.661195552886404
2.0	0.541341132946454	0.541305008634886	0.541341132946510



**Figure 1: The plot of the error generated via ISM5 and RK5 by using Table 1**

**Table 2: The comparative results analyses of ISM5, RK5 and analytical solution**

Time (t)	Y2(ISM5)	Y2 (RK5)	Analytical Solution
0.2	-1.637461506155961	-1.637450578860323	-1.637461506155963
0.4	-1.340640092071291	-1.340622199105012	-1.340640092071278
0.6	-1.097623272188069	-1.097601297978756	-1.097623272188052
0.8	-0.898657928234452	-0.898633940366581	-0.898657928234443
1.0	-0.735758882342895	-0.735734332918400	-0.735758882342884
1.2	-0.602388423824414	-0.602364304662328	-0.602388423824417
1.4	-0.493193927883220	-0.493170889677066	-0.493193927883234
1.6	-0.403793035989315	-0.403771479389387	-0.403793035989337
1.8	-0.330597776443177	-0.330577921326723	-0.330597776443202
2.0	-0.270670566473227	-0.270652504317443	-0.270670566473255

**Figure 2: The plot of the error generated via ISM5 and RK5 by using Table 2**

**Example 2**

Consider the stiff system of two non-linear differential equations given by

$$\left. \begin{aligned} y_1'(t) &= -1002y_1(t) + 1000y_2^2, y_1(0) = 1 \\ y_2'(t) &= y_1(t) - y_2(1 + y_2), y_2(0) = 1 \end{aligned} \right\} \quad (45)$$

As the independent variable 't' does not appear explicitly in (45), it is an autonomous system.

The exact solution of the system is given by

$$y_1(t) = \exp(-2t), y_2(t) = \exp(-t) \quad (46)$$

For this problem, the Jacobian at  $t = 0$  is

$$J = \begin{bmatrix} -1002 & 2000 \\ 1 & -3 \end{bmatrix} \quad (47)$$

whose eigenvalues are  $\lambda = [-1, -1004]$  and Stiffness Ratio; SR = 1004. Hence it is classified as stiff at  $t = 0$ . Further, this problem is super stable since there is at least one eigenvalue with a large negative real part. The results generated via ISM5, RK5 and the analytical solutions are displayed in the Tables 3 and 4.

**Table 3: The comparative results analyses of ISM5, RK5 and analytical solution**

Time (t)	Y1(ISM5)	Y1 (RK5)	Analytical Solution
1.0	0.135335283236616	0.135326223229009	0.135335283236613
2.0	0.018315638888735	0.018313190038883	0.018315638888738
3.0	0.002478752176667	0.002478255344673	0.002478752176667
4.0	0.000335462627903	0.000335373006275	0.000335462627903
5.0	0.000045399929763	0.000045384771944	0.000045399929763
6.0	0.000006144212353	0.000006141751089	0.000006144212353
7.0	0.000000831528719	0.000000831140156	0.000000831528719
8.0	0.000000112535175	0.000000112475082	0.000000112535175
9.0	0.000000015229980	0.000000015220831	0.000000015229980
10.0	0.000000002061154	0.000000002059778	0.000000002061154

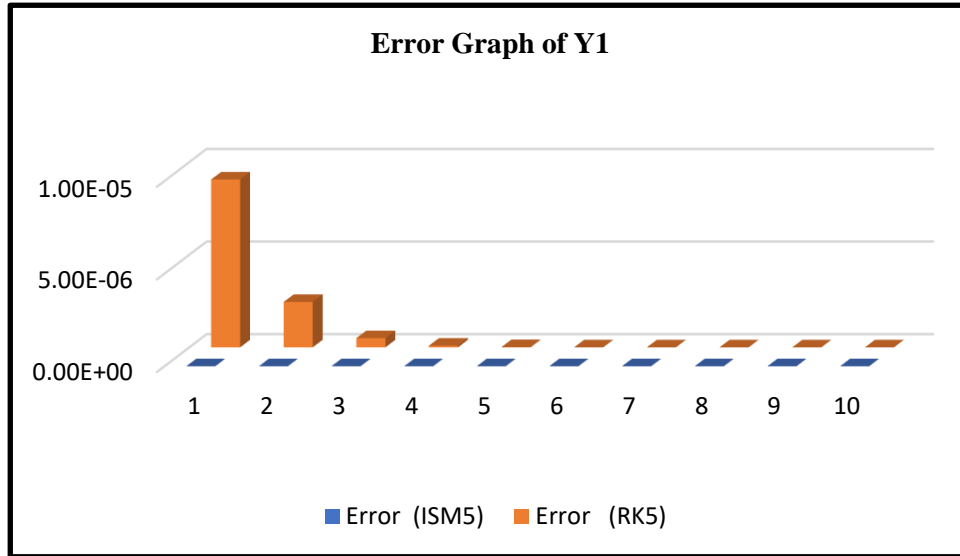
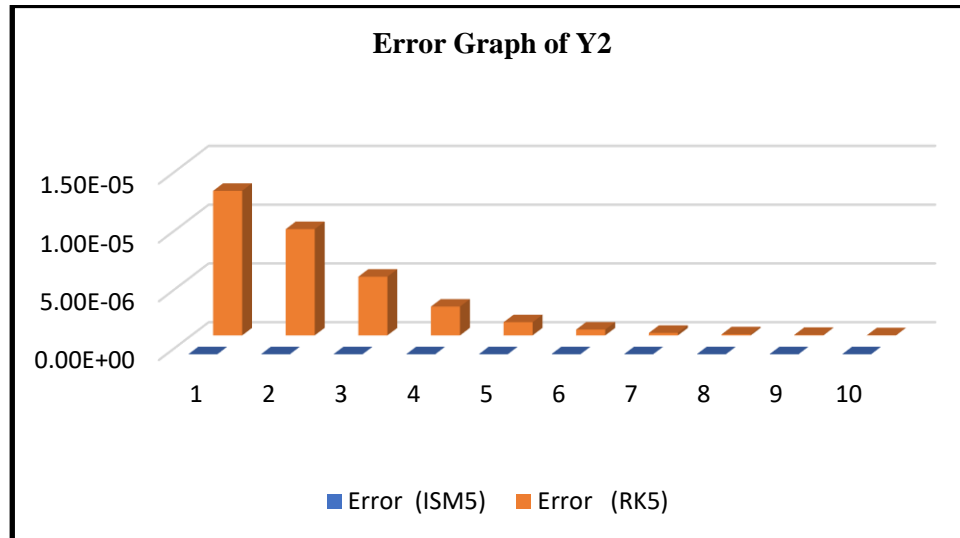


Figure 3: The plot of the error generated via ISM5 and RK5 by using Table 3

Table 4: The comparative results analyses of ISM5, RK5 and analytical solution

Time (t)	Y2(ISM5)	Y2 (RK5)	Analytical Solution
1.0	0.367879441171446	0.367867150974789	0.367879441171442
2.0	0.135335283236615	0.135326244358821	0.135335283236628
3.0	0.049787068367865	0.049782081775069	0.049787068367875
4.0	0.018315638888735	0.018313193324493	0.018315638888740
5.0	0.006737946999086	0.006736822531885	0.006737946999085
6.0	0.002478752176667	0.002478255815037	0.002478752176666
7.0	0.000911881965555	0.000911668944343	0.000911881965554
8.0	0.000335462627903	0.000335373071291	0.000335462627902
9.0	0.000123409804087	0.000123372741441	0.000123409804087
10.0	0.000045399929763	0.000045384780812	0.000045399929763

## FIFTH STAGE INVERSE POLYNOMIAL SCHEME



**Figure 4:** The plot of the errors generated via ISM5 and RK5 by using Table 4

### Example 3

Consider the IVP of the form

$$y'(t) = -100y(t) + 99\exp(2t), y(0) = 0, t \in [0, 0.5] \quad (48)$$

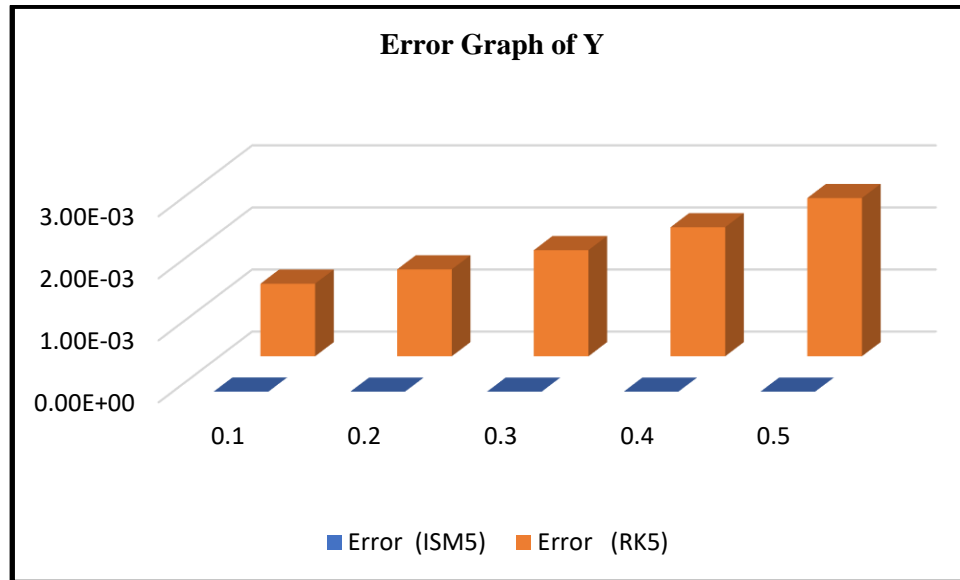
The exact solution is obtained

$$y(t) = \frac{33}{34}(\exp(2t) - \exp(-100t)) \quad (49)$$

The results generated via ISM5, RK5 and the analytical solutions are displayed in the Tables 5.

**Table 5:** The comparative results analyses of ISM5, RK5 and analytical solution

Time (t)	Y(ISM5)	Y(RK5)	Analytical Solution
0.1	1.185436514288875	1.186607569525299	1.185435082988336
0.2	1.447947498778944	1.449344632360295	1.447947498651290
0.3	1.768527070966936	1.770233531789713	1.768527070967168
0.4	2.160083842360042	2.162168118315579	2.160083842360337
0.5	2.638332362915774	2.640878103316632	2.638332362916133



**Figure 5: The plot of the errors generated via ISM5 and RK5 by using Table 5**

## 5. CONCLUDING REMARKS

In this paper, a new fifth stage inverse polynomial scheme has been developed for the solution of stiff IVPs in ODEs. The properties of the scheme were also analyzed. The performance of the scheme is examined on three numerical examples. The numerical results of new inverse polynomial one-step scheme and RK method of order five are displayed in Tables 1-5 by taking the step size  $h = 0.001$  and their corresponding error graphs are given in Figures 1-5. From Figures 1-5, it is evident that the absolute errors generated via the new fifth order inverse polynomial one-step scheme are lesser than that of the fifth order RK5 scheme. This proves that the new inverse polynomial one-step scheme has an edge over RK5 scheme in terms of accuracy. Hence, it is concluded that the proposed scheme is consistent, highly stable, convergent and very much applicable for solving stiff linear and non-linear stiff IVPs in ODEs. The methodology can further be extended for the solution of stiff delay differential equations.

## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

**REFERENCES**

- [1] C.E. Abhulimen, An exponential fitting predictor-corrector formula for stiff systems of ordinary differential equations, *Int. J. Comput. Appl. Math.* 4 (2) (2009), 115-126.
- [2] K. Burrage, Efficient block predictor-corrector methods with a small number of corrections, *J. Comput. Appl. Math.* 45 (1993), 139-150.
- [3] G. Dahlquist, On accuracy and unconditional stability of linear multistep methods for second order differential equations, *BIT Numer. Math.* 18 (1978), 133-136.
- [4] S.O. Fatunla, Block methods for second order ODEs, *Int. J. Comput. Math.* 41(1991), 55-63.
- [5] P. Onumanyi, U.W. Sirisena, S.N. Jator Continuous finite difference approximations for solving differential equations, *Int. J. Comput. Math.* 72 (1999), 15-27.
- [6] D. Sarafyan, New algorithms for the continuous approximate solution of ordinary differential equations and the upgrading of the order of the process, *Comput. Math. Appl.* 20 (1990), 77-100.
- [7] S.O. Fatunla, A new algorithm for numerical solution of ordinary differential equations, *Comput. Math. Appl.* 2 (1976), 247-253.
- [8] R.B. Ogunrinde, On a new inverse polynomial numerical scheme for the solution of initial value problems in ordinary differential equations, *Int. Sch. Sci. Res. Innov.* 9 (2015), 134-137.
- [9] K.O. Okosun, kth order inverse polynomial methods for the integration of the ordinary differential equations of the singularities, Industrial Mathematics and Computer Department, Federal University of Technology, Akure, Nigeria, 2003.
- [10] S. Fadugba, O. Falodun, Development of a new one-step method scheme for the solution of initial value problem (IVP) in ordinary differential equations, *Int. J. Theor. Appl. Math.* 3 (2017), 58-63.
- [11] O.E. Abolarin, S.W. Akingbade, Derivation and application of fourth stage inverse polynomial scheme to initial value problems, *Int. J. Appl. Math.* 47 (2017), 99-104.
- [12] P. Kama, E.A. Ibijola, On a new one-step method for numerical solution of initial-value problems in ordinary differential equations, *Int. J. Comput. Math.* 77 (2000), 457-467.
- [13] S.E. Fadugba, Development of an improved numerical integration method via the transcendental function of exponential form, *J. Interdiscipl. Math.* (2020). <https://doi.org/10.1080/09720502.2020.1747196>.



- [14] S.O. Fatunla, Numerical methods for IVPs in ODEs, Academic Press, USA, 1988.