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ON SEMIREAL HYPERRINGS AND REAL HYPERRINGS

DONGMING HUANG*, JINPING LI

School of Science, Hainan University, Haikou 570228, China

Abstract. The main aim of this paper is to develop the study of hyperrings in real algebra. By introducing the notions of real hyperrings and semireal hyperrings, we obtain some necessary and sufficient conditions for real (semireal) hyperrings. Also, we characterize the minimal prime hyperideals in hyperrings.

Keywords: minimal prime hyperideals; real hyperrings; semireal hyperrings.

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1. Introduction

In 1934, F.Marty [3] introduced the concept of hyperoperation at the Eighth Congress of Scandinavian Mathematicians, and then defined an algebraic hyperstructure called hypergroup. Hyperoperation is a generalization of binary operation. Specifically, for a non-empty set H, $(a,b) \in H \times H$ no longer maps to an element but a non-empty subset of H under a hyperoperation. In 1956, M.Krasner [4](or [5]) introduced a kind of hyperrings, called Krasner hyperring. Krasner hyperrings have two operations, where addition is a hyperoperation, and multiplication is an operation. This type of hyperring is widely studied [7,8].

*Corresponding author

E-mail address: huangdm35@126.com

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In 2006, M.Marshall [6] introduced the concept of multirings and established partial Artin-Schreier theory in this category. The concept of multirings is similar to the concept of Krasner hyperrings, and the only difference between them is about the distributive property. In view of the importance of the study on the reality of rings(fields) in real algebraic geometry [9,10], we introduce the concept of realities in hyperring category and obtain some necessary and sufficient conditions for real(semireal) hyperrings.

Throughout this paper, given two sets A and B, the relative complement of A in B, denoted by $B\setminus A$, is the set that contains exactly those elements belonging to B but not to A.

2. HYPERRINGS

In this section we review some basic definitions and properties of hyperrings which will be used in the sequel.

Definition 2.1. A Krasner hyperring $(R,+,\cdot,0,1)$ is an nonempty set R equipped with a hyperoperation $+$ and a usual operation $\cdot$ which satisfy the following conditions:

(1)$(R,+,0)$ is a canonical hypergroup,that is,

(i)$\forall x,y,z\in R, x+(y+z)=(x+y)+z$,that is,$+$ is associative,

(ii)$\forall x,y\in R, x+y=y+x$,that is,$+$is commutative,

(iii)there exists $0\in R$ such that $0+x=\{x\},\forall x\in R$,

(iv)for every $x\in R$ there exists a unique element $y\in R$ such that $0\in x+y$,we denote $y=-x$ and call it the inverse of x,

(v)$z\in x+y$ implies $y\in -x+z$ and $x\in z-y$;

(2)$(R,\cdot,1)$is a commutative monoid having zero as a bilaterally absorbing element,i.e.,$\forall x\in R, x\cdot 0=0\cdot x=0$;

(3)$\cdot$ is distributive with respect with +,i.e.,$\forall x,y,z\in R, x(y+z)=xy+xz,(x+y)z=xz+yz$.

By convention,we write $0+x=x$ instead of $0+x=\{x\}$.

Throughout the paper,all hyperrings are assumed to be Krasner hyperrings unless otherwise stated.

Proposition 2.2. *Let R be a hyperring,$\forall x,y,z,w\in R$,* *(1)* 0 *is unique;*

(2) $-(-x) = x$*;*

(3) $-(x+y) = -x-y$*,where*$-A = \{-a|a \in A\}$ *for a subset* A *in* R*;*

(4) $x(-y) = (-x)y = -xy$*;*

(5) $(-x)(-y) = xy$*;*

(6) $0 \in x-y$ *implies* $x = y$*;*

(7) $(x+y)(z+w) \subseteq xz+xw+yz+yw$*.*

Remark 2.3. In general,the doubly distributive property is not hold for hypering,i.e., the (7) in the above Proposition cannot be rewritten as $(x+y)(z+w) = xz+xw+yz+yw$.

Example 2.4. ([8])Let $R = \{0,1,a\}$ and $A = \{0,a\}$. Define a hyperoperation $+$ and a operation $\cdot$ on R as in the following tables.

$+$	0	1	a
0	0	1	a
1	1	A	1
a	a	1	0

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	0

Then R is a hyperring.Notice that$(1+1)(1+1) = AA = \{0\}$,and $1\cdot 1+1\cdot 1+1\cdot 1+1\cdot 1 = 1+1+1+1 = A+A = \{0,a\}$.It's clear that they are not equal,so the doubly distributive property does not hold in general.

Lemma 2.5. *(Proposition 3.12,* [8]*)Let* R *be a hyperring. For* $a,b \in R$ *,we have* $(a+b)^n \subseteq \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$.

Definition 2.6. Let R be a hyperring. A nonempty subset I of R is a hyperideal if:

(1) $\forall x,y \in I, x-y \subseteq I$;

(2) $\forall x \in I, r \in R, xr \in I$.

Definition 2.7. Let R be a hyperring. A hyperideal $I \subsetneq R$ is prime if for all $x,y \in R$ with $xy \in I$, we have $x \in I$ or $y \in I$.

Definition 2.8. Let R be a hyperring. A prime hyperideal I of R is called minimal if there is no prime hyperideal of R which is properly contained in I.

Definition 2.9. A nonempty subset S of hyperring R is called multiplicatively closed if S is closed under multiplication and does not contain zero element.A multiplicatively closed set S in R is called maximal if it is not properly contained in any other multiplicatively closed set in R.

Lemma 2.10. *Let R be a hyperring.*

(1)If P is a prime hyperideal of R,then $R\setminus P$ is a multiplicatively closed set of R;

(2)If I is a hyperideal of R such that $R\setminus I$ is multiplicatively closed set, then I is a prime hyperideal of R;

(3)Any multiplicatively closed set of R is contained in a maximal multiplicatively closed set.

Proof. The proof is similar to the classical case(Lemma2.2,[2]). □

Proposition 2.11. *Let R be a hyperring and S be a maximal multiplicatively closed set of R.For each $r\in R\setminus S$, there exists an element $s\in S$ such that rs is nilpotent.*

Proof. The proof is similar to the classical case(Theorem 2.3,[2]). □

Theorem 2.12. *A subset S of hyperring R is a maximal multiplicatively closed set if and only if $R\setminus S$ is a minimal prime hyperideal of R.*

Proof. Similar to the classical case(Theorem 2.6,[2]), it is enough to show that $p-q\subseteq R\setminus S$ for $p,q\in R\setminus S$ when we prove the "only if" part. Let $p,q\in R\setminus S$. By Proposition 2.11, there exist $p',q'\in S$, such that $(pp')^n=0$ and $(qq')^m=0$ for positive integers m,n. Assume that $p-q\nsubseteq R\setminus S$. Then there exists $t\in p-q$ such that $t\in S$. Since $t^{m+n}\in S,(p')^{m+n}\in S$ and $(q')^{m+n}\in S$, we have $(tp'q')^{m+n}\in S$. On the other hand, by Lemma 2.5, one has

$$(tp'q')^{m+n}\in[(p-q)p'q']^{m+n}\subseteq\sum_{i=0}^{m+n}\binom{m+n}{i}(-1)^{m+n-i}p^i(p')^{m+n}q^{m+n-i}(q')^{m+n}=\{0\}.$$

This means that $0\in S$, which is impossible. □

Corollary 2.13. *A prime hyperideal P of hyperring R is minimal if and only if for each $x\in P$, there exists an element $r\in R\setminus P$ such that rx is nilpotent.*

Proof. The "only if" part follows from Proposition 2.11 and Theorem 2.12 . Now we prove the "if" part. Assume, instead, that P is not minimal. Then there exist a prime hyperideal Q such that $Q \subsetneq P$. Hence, there exists $x \in P \setminus Q$. By assumption, there exists an element $r \in R \setminus P$ such that rx is nilpotent. This means that we have $(rx)^n = 0 \in Q$ for some positive integer n. Hence, $rx \in Q$, since Q is prime. It follows that $r \in Q$ or $x \in Q$. This is impossible. □

3. Semireal Hyperrings and Real Hyperrings

In this section, we introduce two notions of reality for hyperrings.

Let R be a hyperring. Then we obtain two multiplicatively closed subsets of R as follows:

$$\sum R^2 := \{\sum_{i=1}^{n} a_i^2 | n \text{ is a positive integer, and } a_i \in R \text{ for } i = 1, \cdots, n.\};$$

$$1 + \sum R^2 := \{1 + t | t \in \sum R^2\}.$$

Definition 3.1. A hyperring R is called a semireal hyperring if $0 \notin 1 + \sum R^2$.

Definition 3.2. A hyperring R is called a real hyperring if R satisfies the following property: for any positive integer n,if $0 \in \sum_{i=1}^{n} x_i^2$,then $x_1 = x_2 = \cdots = x_n = 0$.

In [1], B.Davvaz and A.Salasi constructed a factor hyperring R/I on a hyperring R and a normal hyperideal I of R. In [7], J. Jun shown that the condition "I is normal" is redundant. Let R be a hyperring and I a hyperideal of R. Let $\sim$ be a relation of R as follows:

$$x \sim y \Leftrightarrow x + I = y + I \text{ or equivalently } x \sim y \Leftrightarrow (x - y) \cap I \neq \varnothing.$$

Then $\sim$ is an equivalence relation. Hence, we can obtain the quotient class $R/I = \{[x] | x \in R\}$ of R by $\sim$. By defining a hyperoperation $\oplus$:

$$[a] \oplus [b] = (a + I) \oplus (b + I) := \{c + I | c \in a + b\}$$

and an operation $\odot$:

$$[a] \odot [b] := a \cdot b + I$$

on R/I, the quotient hyperrings R/I is a hyperring. For specific details, please refer to Reference [7].

Definition 3.3. Let R be a hyperring. A proper hyperideal I of R is called semireal(resp. real) if the quotient hyperring R/I is semireal(resp. real).

It's clear that a hyperring R(resp. hyperideal I) is real implies that R(resp. I) is semireal.

Proposition 3.4. *Let R be a hyperring and I be a hyperideal of R.Then*

(1)I is semireal if and only if $I \cap (1+\sum R^2) = \varnothing$*;*

(2)I is real if and only if for any positive integer n*,*$I \cap \sum_{i=1}^{n} x_i^2 \neq \varnothing$ *implies* $x_i \in I$ *for each* $i = 1,2,\cdots,n$*.*

Proof. This proposition follows from the definitions of real(semireal) hyperideals and of quotient hyperrings. In more detail,

(1) I is semireal iff $[0] \notin [1] \oplus \sum(R/I)^2$ iff $\forall t \in 1+\sum R^2, [0] \neq [t]$ iff $\forall t \in 1+\sum R^2, \{t\} \cap I = \varnothing$ iff $I \cap (1+\sum R^2) = \varnothing$;

(2) I is real iff for any positive integer n, $[0] \in [x_1]^2 \oplus \cdots \oplus [x_n]^2 = [\sum_{i=1}^{n} x_i^2]$ implies $[x_i] = [0]$ for each $i = 1,2,\cdots,n$ iff for any positive integer n,$I \cap \sum_{i=1}^{n} x_i^2 \neq \varnothing$ implies $x_i \in I$ for each $i = 1,2,\cdots,n$. □

Theorem 3.5. *Let R be a hyperring. Then the following statements are equivalent:*

(1)R is semireal;

(2)R has a semireal hyperideal;

(3)R has a semireal prime hyperideal;

(4)R has a real prime hyperideal.

Proof. $(1) \Rightarrow (2)$. If R is semireal, then (0) is a semireal hyperideal. Thus (2) holds;

$(2) \Rightarrow (3)$. Assume R has a semireal hyperideal I. Set $M = 1+\sum R^2$. Set

$$\Pi = \{P | P \text{ is a hyperideal of } R,\, I \subseteq P \text{ and } P \cap M = \varnothing\}.$$

The set Π is ordered by set inclusion,then by Zorn's Lemma, Π has a maximal element say Q. We shall show that Q is a prime hyperideal of R. If there exist $a,b \notin Q$ but $ab \in Q$, then the hyperideals $Q+(a)$ and $Q+(b)$ strictly contain Q, and so do not belong to Π. Hence $(Q+(a)) \cap M \neq \varnothing$ and $(Q+(b)) \cap M \neq \varnothing$. Therefore, there exist $m_1,m_2 \in M, q_1,q_2 \in Q, r_1,r_2 \in R$ such that $m_1 \in q_1 + r_1 a$ and $m_2 \in q_2 + r_2 b$. It follows that $r_1 a \in m_1 - q_1$ and $r_2 b \in m_2 - q_2$. By multiplying these two, one obtains

$$(r_1 r_2)ab \in (m_1 - q_1)(m_2 - q_2) \subseteq m_1 m_2 - m_1 q_2 - m_2 q_1 + q_1 q_2.$$

Hence, there exists $q \in -m_1 q_2 - m_2 q_1 + q_1 q_2 \subseteq Q$ such that $(r_1 r_2)ab \in m_1 m_2 + q$. It follows that $m_1 m_2 \in (r_1 r_2)ab - q \subseteq Q$. This implies that $M \cap Q \neq \varnothing$ since $m_1 m_2 \in M$, a contradiction.

$(3) \Rightarrow (4)$. Assume R has a semireal prime hyperideal I. Let Π be the set of hyperideals J of R such that $I \subseteq J$ and $J \cap (1+\sum R^2) = \varnothing$. Then $\Pi \neq \varnothing$ since we have $I \in \Pi$. By Zorn's Lemma, Π has a Maximal element Q. We shall show that Q is a real prime hyperideal of R. By the proof of above, Q is a prime hyperideal.Therefore, all that remains is to prove that Q is real. Assume, instead, that Q is not real. Then there exists a relation $(a_1^2+\cdots+a_n^2)\cap Q \neq \varnothing$,where $a_1,\cdots,a_n \in R$, and without loss of generality, $a_1 \notin Q$. Since $Q+(a_1)$ strictly contains Q, and so does not belong to Π. Hence, we have $(Q+(a_1))\cap(1+\sum R^2) = \varnothing$, and, thus, there exist $r,s \in R, q \in Q, t \in \sum R^2$, such that $r \in q+sa_1$ and $r \in 1+t$. It follows that $sa_1 \in r-q$ and, thus we have $s^2a_1^2 \in q^2-rq-rq+r^2$. By adding $s^2(a_2^2+\cdots+a_n^2)$ on two sides, one obtains

$$s^2(a_1^2+\cdots+a_n^2) \subseteq q^2-rq-rq+r^2+s^2a_2^2+\cdots+s^2a_n^2.$$

Since $(a_1^2+\cdots+a_n^2)\cap Q \neq \varnothing$, there exists $q_1 \in Q$,such that $q_1 \in s^2(a_1^2+\cdots+a_n^2)$. Furthermore, notice that $q^2-rq-rq \subseteq Q$ and $r^2+s^2a_2^2+\cdots+s^2a_n^2 \subseteq (1+t)^2+s^2a_2^2+\cdots+s^2a_n^2 \subseteq 1+t+t+t^2+s^2a_2^2+\cdots+s^2a_n^2 \subseteq 1+\sum R^2$. Hence, there exist $q_2 \in Q, w \in 1+\sum R^2$, such that $q_1 \in q_2+w$. It follows that $w \in q_1-q_2$. This implies that $Q\cap(1+\sum R^2) \neq \varnothing$, a contradiction.

$(4) \Rightarrow (1)$. Assume R has a real prime hyperideal P. If $0 \in 1+\sum R^2$, then we have $0 \in 1+\sum_{i=1}^n r_i^2$, where $r_i \in R, i = 1,\cdots,n$. This implies that $0 \in P\cap(1^2+r_1^2+\cdots r_n^2) \neq \varnothing$. By the reality of P, we have $1 \in P$, a contradiction. □

Corollary 3.6. *For any semireal hyperideal I of hyperring R, there exists a maximal semireal hyperideal containing I.Moreover, any maximal semireal hyperideal is prime.*

Corollary 3.7. *Any maximal semireal hyperideal of hyperring R is a maximal real hyperideal of R, that is, the maximal semireal hyperideals coincide with the maximal real hyperideals in a hyperring.*

Lemma 3.8. *(Lemma 3.17,* [7]*)Let R be a hyperring and I be a hyperideal of R. Then the radical of I*

$$\sqrt{I}=\{r \in R \mid r^n \in I \text{ for some positive integer } n\}$$

is a hyperideal.

Lemma 3.9. *(Lemma 3.19,* [7]*)Let R be a hyperring and I be a hyperideal of R. Then*

$$\sqrt{I} = \bigcap_{P \in V(I)} P,$$

where $V(I)$is the set of prime hyperideals of R containing I.

The nilradical of a hyperring R, denoted $nil(R)$, is the set of all nilpotent elements in R, i.e., $nil(R) = \{r \in R | r^n = 0 \text{ for some positive integer } n\}$. By Lemma 3.8 and Lemma 3.9 , $nil(R)$ is a hyperideal of R, and $nil(R)$ is the intersection of all prime hyperideals of R. A hyperring R is called reduced if its nilradical $nil(R)$ is zero.

Theorem 3.10. *Let R be a hyperring. Then the following statements are equivalent:*

(1)R is real;

(2)R is reduced and all minimal prime hyperideals of R are real;

(3)The intersection of all real prime hyperideals of R is equal to 0*.*

Proof. $(1) \Rightarrow (2)$. Let R is a real hyperring. It's clear that R is reduced. Let P be any minimal prime hyperideal. If $x \in (x_1^2 + \cdots + x_n^2) \cap P$, then there exist $r \in R \setminus P$, such that rx is nilpotent by Corollary 2.13. Hence, $rx = 0$. It follows that $0 = r^2x \in r^2x_1^2 + \cdots + r^2x_n^2$. By the reality of R, we have $rx_i = 0 \in P, i = 1, \cdots, n$. Thus $x_i \in P, i = 1, \cdots, n$. So P is real.

$(2) \Rightarrow (3)$. If R is reduced and all minimal prime hyperideals of R are real. Let X, X_m and X_r be the set of prime hyperideals, minimal prime hyerideals and real prime hyperideals of R respectively. Then by Lemma 3.9 ,we have

$$0 = Nil(R) = \bigcap_{P \in X} P = \bigcap_{P \in X_m} P = \bigcap_{P \in X_r} P.$$

$(3) \Rightarrow (1)$. If $0 \in \sum_{i=1}^n x_i^2$, then $\sum_{i=1}^n x_i^2 \cap P \neq \varnothing$ for any real hyperideal P. Hence $x_i \in P, i = 1, \cdots, n$. This implies that $x_i \in \bigcap_{P \in X_r} P$, where X_r is the set of real prime hyperideals of R. By assumption, we have $x_i = 0$ for $i = 1, \cdots, n$. □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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