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PBIB-DESIGNS ARISING FROM DIAMETRAL PATHS IN CUBIC GRAPHS

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Abstract. The Partially balanced incomplete block (PBIB)-designs and association schemes arising from different graph parameters is a well studied concept. In this paper, we determine the PBIB-designs with some association schemes through diametral paths in cubic graphs of order at most ten. The discussion of non-existence of some designs corresponding to diametral paths from certain graphs concludes the article.

Keywords: PBIB-designs; diametral paths; cubic graphs.

2010 AMS Subject Classification: 05C51, 53C22, 58E10.

1. Introduction

Design theory is a field of combinatorics with close ties to several other areas of mathematics including graph theory, group theory, number theory, the theory of finite fields, combinatorial matrix theory, and with a wide range of applications in information theory, statistics, computer science and engineering. The concept of an m-class association scheme was first introduced by Bose and Shimamoto [3]. Partially balanced incomplete block designs (PBIBD) were first studied by Bose and Nair, though the concept of association schemes [2], and they established

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the relation between PBIB-designs and strongly regular graphs. R. C. Bose [1] has shown that strongly regular graphs emerge from PBIB-designs with 2-association schemes.

Huilgol et al.[7], introduced a design called $(v,b,r,diam(G),k,\lambda_i)$-design arising from diametral paths in graphs, derived the governing results of these designs, if existed. The authors have constructed these diametral designs with 2-association schemes corresponding to the strongly regular graphs like $K_n \times K_n$, Square lattice graphs $L_2(n)$, Triangular graphs $T(n)$, the three Chang graphs, the Petersen graph, the Shrikhande graph, the Clebsch graph, the complement of Clebsch graph, the complement of Schlafli graph, the Hoffman-Singleton graph, complete bipartite graph etc. These serve as extensions for the collection of near impossible class of strongly regular graphs having 2-association schemes, next to Ionin and Shrikhande [6].

In this paper, we have extended the diametral designs introduced in[7], to cubic graphs of order at most 10. We establish a link between PBIB-design and graphs through the collection of all diametral paths in graphs. We also determine all cubic graphs on ten vertices in which the set of all diametral paths forms a PBIB-design.

2. PRELIMINARIES

Throughout this paper, $G = (V,E)$ stands for finite, connected, undirected graph with neither loops nor multiple edges. The terms not defined here are used in the sense of Buckley and Harary [4].

The distance between the two vertices x and y is defined as the length of a $x-y$ geodesic in G and is denoted by $d_G(x,y)$ or $d(x,y)$. For a vertex x in G, its *eccentricity*, $ecc(x)$ is defined as the distance to a farthest vertex from x in G. That is, $ecc(x) = max\{d(x,y) : \text{for all } y \in V(G)\}$. For a vertex x, a vertex y is an *eccentric vertex* if, $d(x,y) = ecc(x)$.
In a graph G, the *diameter* of G, denoted as $diam(G)$, is the maximum of the eccentricities of vertices in G, and the *radius* of G, denoted as $rad(G)$, is the minimum of the eccentricities of

G.

That is, $diam(G) = max\{ecc(x) : x \in V(G)\}$ and $rad(G) = min\{ecc(x) : x \in V(G)\}$.

For a vertex x in a connected graph G, let $d_i(x)$ be the number of vertices at distance i from x. The distance degree sequence of a vertex x is $dds(x) = (d_o(x), d_1(x), \ldots, d_{ecc(x)}(x))$.
The *distance degree sequence DDS(G)* of a graph G consists of the collection of sequences $dds(x)$ of its vertices, listed in numerical order. The *distance degree regular (DDR)* graph is a graph in which all vertices have the same distance degree sequence.

Definition 2.1. [9] *Given a set* $\{1, 2, \ldots, v\}$ *a relation satisfying the following conditions is said to be an association scheme with m classes.*

(1) Any two symbols α *and* β *are* i^{th} *associates for some i, with* $1 \leq i \leq m$ *and this relation of being* i^{th} *associates is symmetric.*
(2) The number of i^{th} *associates of each symbol is* n_i *.*
(3) If α *and* β *are two symbols which are* i^{th} *associates, then the number of symbols which are* j^{th} *associates of* α *and* k^{th} *associates of* β *is* p^i_{jk} *and is independent of the pair of* i^{th} *associates* α *and* β.

Definition 2.2. [9] *Consider a set of symbols* $V = \{1, 2, \ldots, v\}$ *and an association scheme with m classes on* V. *A partially balanced incomplete block design (PBIBD) is a collection of b subsets of* V *called blocks, each of them containing k symbols* $(k < v)$, *such that every symbol occurs in r blocks and two symbols* α *and* β *which are* i^{th} *associates occur together in* λ_i *blocks, the numbers* λ_i *being independent of the choice of the pair* α *and* β.
The numbers $v, b, r, k, \lambda_i (i = 1, 2, \ldots, d)$ *are called the parameters of the first kind.*

Definition 2.3. [7] *A* $(v, b, r, diam(G), k, \lambda_i)$*-design, called a diametral-design (in short) over a regular, self-centered graph* $G = (V, E)$ *of degree d, diameter* $diam(G)$, *is an ordered pair* $D = (V, B)$, *where* $V = V(G)$ *and B, the set of all diametral paths of G, called blocks, containing the vertices belonging to the diametral paths, such that two symbols* α *and* β *which are* i^{th}

associates occur together in λ_i blocks, the numbers λ_i being independent of the choice of the pair α and β.

Proposition 2.1. [7] *Let G be a regular, self centered graph of diameter $diam(G)$. Let $D = (V,B)$ be $(v,b,r,diam(G),k,\lambda_i)$-design, called the diametral-design, over G with regularity d. The number of i^{th} associates of each symbol is n_i. Then there exists an integer r called the repetition number of D such that any vertex $v' \in V$ is contained in exactly r blocks, such that the following two conditions hold:*

- $r \cdot diam(G) = \sum_{i=1}^{diam(G)} \lambda_i n_i$
- $vr = bk \quad or \quad r = \frac{b\cdot(diam(G)+1)}{v}$.

3. Cubic graphs of order at most 10

A cubic graph is a graph in which all vertices have degree three. There exist only one cubic graph of order four namely the complete graph K_4, 2 cubic graphs of order six, 5 cubic graphs of order eight, 21 cubic graphs of order ten and 85 cubic graphs of order 12 [5]. Here we give a list of those cubic graphs that give PBIB-designs from the diametral paths in them, and also we give the list of cubic graphs which do not give the PBIB-designs from diametral paths.

3.1. Cubic graphs with 6 vertices.

There exist 2 cubic graphs on six vertices [5]. Out of these two graphs only one graph namely G_2 forms a PBIB-design, whose blocks are diametral paths, which is clear from Table 1.

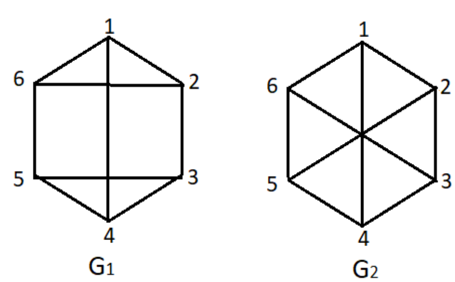

FIGURE 1. Cubic graphs with 6 vertices

TABLE 1. PBIB-Design Parameters

Graphs	$diam(G)$	$rad(G)$	k	b	r	λ_1	λ_2
G_1	2	2	3	12	6	2,4	2
G_2	2	2	3	18	9	4	3

Lemma 3.1. *The graph G_2 forms a PBIB-design with 2-association schemes.*

Proof. Proof follows from the Table 1. Since G_2 is a distance degree regular (DDR) graph with distance degree sequence (dds) of each vertex is $(1,3,2)$. We get the parameters of the second kind as distance degree sequence itself. And the association schemes will be 2 as G_2 is self-centered with diameter 2. Hence the parameters of second kind are given by $n_1 = 3$, $n_2 = 2$ with

$$P^1 = \begin{pmatrix} p^1_{11} & p^1_{12} \\ p^1_{21} & p^1_{22} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} p^2_{11} & p^2_{12} \\ p^2_{21} & p^2_{22} \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}. \qquad \square$$

3.2. Cubic graphs with 8 vertices.

There are 5 cubic graphs on eight vertices [5]. We observe that, $2 \leq diam(G) \leq 3$. Out of 5 graphs only one graph namely G_5 forms a PBIB-design, whose blocks are diametral paths, which is clear from the Table 2.

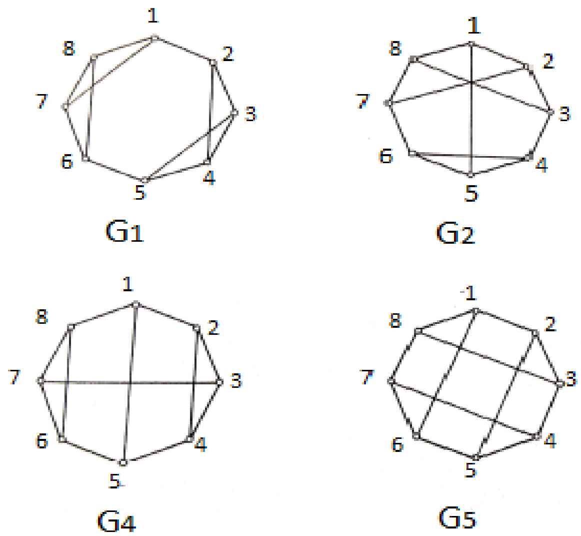

FIGURE 2. Cubic graphs with 8 vertices

TABLE 2. PBIB-design Parameters

Graphs	$diam(G)$	$rad(G)$	k	b	r	λ_1	λ_2	λ_3
G_1	3	3	4	16	6,10	..	..	..
G_2	2	2	3	21	6,9	..	..	..
G_3	2	2	3	24	9	4	1,2	..
G_4	3	2	4	8	2,4,5	..	..	..
G_5	3	3	4	24	12	6	4	6

Lemma 3.2. *The graph G_5 forms a PBIB-design with 3-association schemes.*

Proof. From the Table 2. we can conclude that graph G_5 form PBIB design, whose blocks are diametral paths. Since G_5 is a distance degree regular (DDR) graph with distance degree sequence (dds) of each vertex is $(1,3,3,1)$. We get the parameters of the second kind as distance degree sequence itself. And the association schemes will be 3 as G_5 is self-centered with diameter 3. Hence the parameter of second kind are given by $n_1 = 3$, $n_2 = 3$, $n_3 = 1$ with

$$P^1 = \begin{pmatrix} p^1_{11} & p^1_{12} & p^1_{13} \\ p^1_{21} & p^1_{22} & p^1_{23} \\ p^1_{31} & p^1_{32} & p^1_{33} \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} p^2_{11} & p^2_{12} & p^2_{13} \\ p^2_{21} & p^2_{22} & p^2_{23} \\ p^2_{31} & p^2_{32} & p^2_{33} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} p^3_{11} & p^3_{12} & p^3_{13} \\ p^3_{21} & p^3_{22} & p^3_{23} \\ p^3_{31} & p^3_{32} & p^3_{33} \end{pmatrix} = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad \square$$

3.3. Cubic graphs with 10 vertices.

There are 21 cubic graphs on ten vertices [5]. We observe that the $2 \leq diam(G) \leq 4$. Out of 21 graphs only one graph namely the Petersen graph forms a PBIB-design, whose blocks are diametral paths in Petersen graph.

The Petersen graph is a cubic graph with 10 vertices and 15 edges. It is a small graph that serves as an useful example and counterexample for many problems in graph theory. The Petersen graph is known to possess many important properties of graphs such as being cubic, distance transitive, distance regular, a Moore graph of diameter two, strongly regular graph and many more.

Next we prove the non-existence of a PBIB-designs for the remaining graphs from the collection.

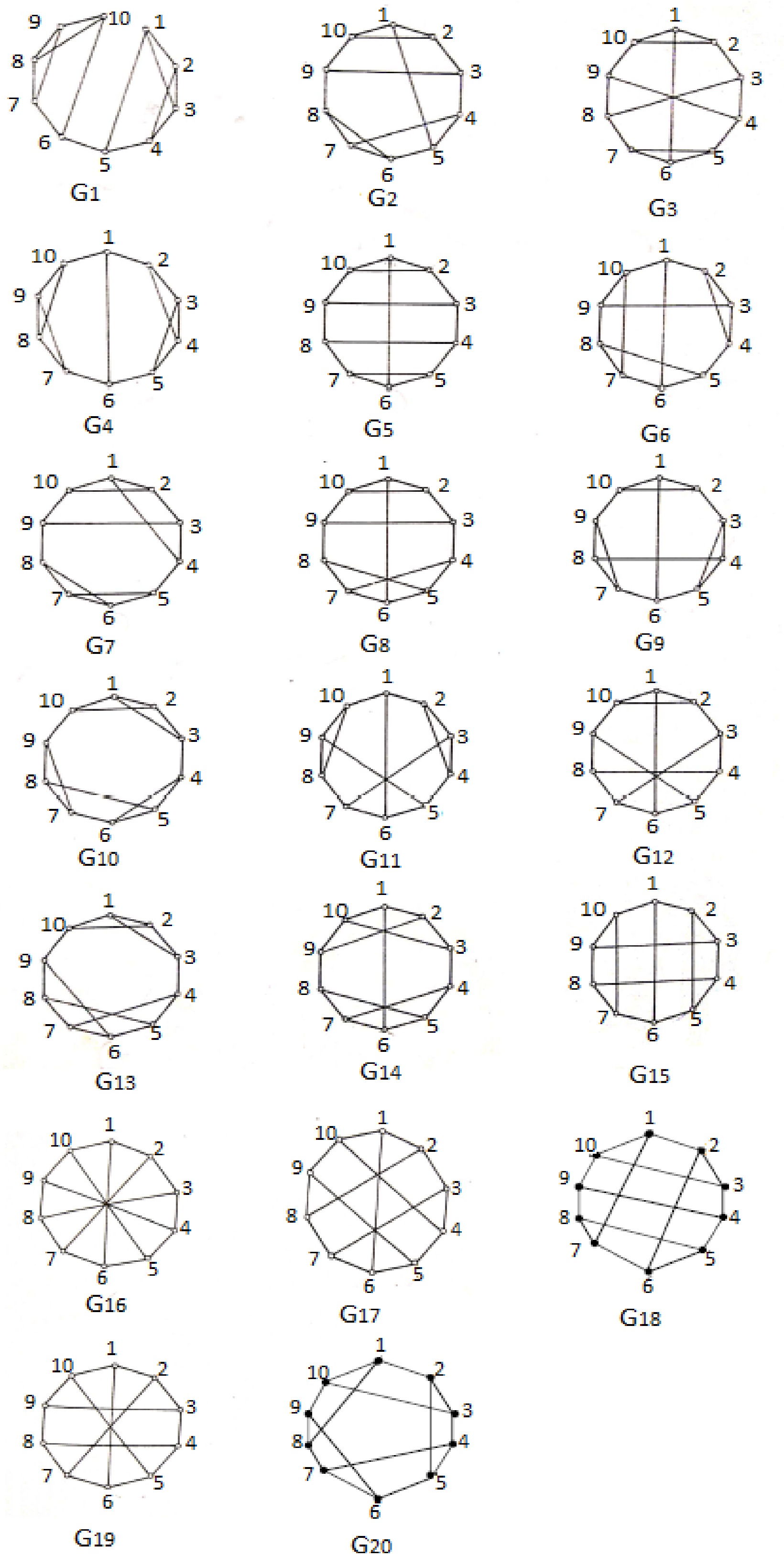

FIGURE 3. Cubic graphs with ten vertices.

TABLE 3. PBIB-design Parameters

Graphs	$diam(G)$	$rad(G)$	k	b	r	λ_1	λ_2	λ_3	λ_4	λ_5
G_1	5	3	6	16	6,8	..	..	..	..	..
G_2	3	3	4	20	5,8,9,10	..	..	..	..	..
G_3	3	3	4	20	7,12	..	..	..	..	..
G_4	4	2	5	8	4	2,4	2,4	2	2	..
G_5	3	3	4	26	9,10,12	..	..	..	..	..
G_6	3	3	4	24	6,10,12	..	..	..	..	..
G_7	4	3	5	8	2,4,8	..	..	..	..	..
G_8	3	3	4	24	8,9,12	..	..	..	..	..
G_9	3	2	4	18	6,8	..	..	..	..	..
G_{10}	3	3	4	18	5,6,10	..	..	..	..	..
G_{11}	3	2	4	12	4,6	..	..	..	..	..
G_{12}	3	2	4	20	4,6,9,10	..	..	..	..	..
G_{13}	3	3	4	22	6,7,8,12	..	..	..	..	..
G_{14}	3	3	4	36	12,16	..	..	..	..	..
G_{15}	3	3	4	30	12	6	2,4	3	..	..
G_{16}	3	3	4	30	16	9,6	4	4	..	..
G_{17}	3	2	4	14	2,5,8	..	..	..	..	..
G_{18}	3	3	4	20	12	6	2,4	3	..	..
G_{19}	3	3	4	20	5,9,12	..	..	..	..	..
G_{20}	3	3	4	40	6	9,6	4	4	..	..
G_{21}	2	2	3	30	9	4	1			

Theorem 3.1. *There do not exist PBIB-designs from the graphs G_1 to G_{20}.*

Proof. Observe that for the graphs G_1 to G_3, G_5 to G_{14}, G_{17} and G_{19} the value of r is not unique. For the graphs G_4, G_{15}, G_{16}, G_{18}, G_{20} the value of λ_i is not unique. Hence there do not exist diametral-designs from the graphs G_1 to G_{20}. □

Theorem 3.2. *The collection of all diametral paths in the Petersen graph forms a PBIB-design with* 2*-association schemes and parameters* $v = 10$*,* $b = 30$*,* $k = 3$*,* $r = 9$*,* $\lambda_1 = 4$ *and* $\lambda_2 = 1$.

Proof. Let the Petersen graph be labeled as $1, 2, \ldots, 10$, which is a strongly regular graph with the parameters $(10, 3, 0, 1)$ as shown in Figure 4.

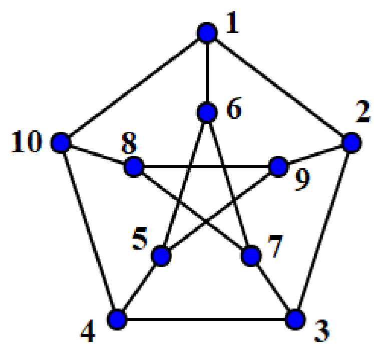

FIGURE 4. Petersen graph(G_{21})

The diametral paths of Petersen graph are given by,

{1,6,5}	{1,10,4}	{2,1,6}	{2,3,7}	{3,4,10}	{1,2,3}	{1,6,7}	{2,9,8}	{2,1,10}	{3,2,9}
{1,10,8}	{1,2,9}	{2,3,4}	{2,9,5}	{3,7,6}	{3,4,5}	{4,5,9}	{6,7,8}	{7,8,10}	{9,8,10}
{3,7,8}	{7,8,9}	{6,5,4}	{7,6,5}	{9,5,6}	{5,9,8}	{7,3,4}	{4,10,8}	{5,4,10}	{10,1,6}

The Petersen graph has exactly thirty diametral paths, that is, $b = 30$ and $k = 3$. Using the relation $vr = bk$, we get $r = 9$.

Thus, we have a PBIB-design with the parameters $(10, 30, 9, 3, 4, 1)$, where $v = 10$, $b = 30$, $r = 9$, $k = 3$, $\lambda_1 = 4$, $\lambda_2 = 1$ and whose blocks are the diametral paths in the graph.

Since the Petersen graph is distance degree regular graph with distance degree sequence of each vertex is $(1, 3, 9)$. Hence the parameters of second kind are given by, $n_1 = 3$ and $n_2 = 9$, with

$$P^1 = \begin{pmatrix} p^1_{11} & p^1_{12} \\ p^1_{21} & p^1_{22} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 2 & 4 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} p^2_{11} & p^2_{12} \\ p^2_{21} & p^2_{22} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}.$$

□

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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