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ON A v-ANALOGUE OF THE GAMMA FUNCTION AND SOME ASSOCIATED INEQUALITIES

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Abstract. In this paper, we introduce a new one-parameter deformation of the classical Gamma function, which we call a v-analogue. We also establish some properties generalizing those satisfied by the classical Gamma function. In addition, we establish some inequalities involving this new function.

Keywords: Gamma function; v-analogue; v-polygamma functions; inequality.

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1. Introduction

The classical Euler's Gamma function is defined for $s \in \mathbb{R}^+$ as

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt. \tag{1}$$

It was first introduced into the mathematical literature by Leonhard Euler in 1730 to extend the factorial function to non negative integers. It was later studied by other mathematicians

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including Friederick Gauss and Karl Weierstrass who gave their respective definitions as

$$\Gamma(s) = \lim_{n\to\infty}\left[\frac{n!n^s}{s(s+1)(s+2)\cdots(s+n)}\right],\ s\in\mathbb{R}^+, \tag{2}$$

$$\frac{1}{\Gamma(s)} = se^{\gamma}\prod_{n=1}^{\infty}\left(1+\frac{s}{n}\right)e^{-\frac{s}{n}},\ s\in\mathbb{R}^+, \tag{3}$$

where γ is the Euler-Mascheroni constant which is defined as

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right).$$

Euler also defined the Gamma function in terms of its infinite product for $s\in\mathbb{R}^+$ as

$$\Gamma(s) = s^{-1}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{s}\left(1+\frac{s}{n}\right)^{-1}. \tag{4}$$

For more information on the Gamma function and its properties, see [1, 2, 4] and the related references.

The digamma and polygamma functions are closely related to the Gamma function. The digamma function is the logarithmic derivative of the Gamma function. That is, $\psi(s) = \frac{d}{ds}\ln\Gamma(s)$. It has the following series and integral representations:

$$\psi(s) = -\gamma - \frac{1}{s} + \sum_{n=1}^{\infty}\frac{s}{n(n+s)} = -\gamma + \int_0^{\infty}\frac{e^{-t}-e^{-st}}{1-e^{-t}}dt, \tag{5}$$

where $s\in\mathbb{R}^+$ and γ is the Euler-Mascheroni constant. The polygamma function $\psi^m(s)$ is the m^{th} derivative of the digamma function. That is, $\psi^m(s) = \frac{d^m}{ds^m}\psi(s)$, $m\in\mathbb{N}$. It also has the following series and integral representations:

$$\psi^m(s) = (-1)^{m+1}m!\sum_{n=1}^{\infty}\frac{1}{(s+n)^{m+1}} = (-1)^{m+1}\int_0^{\infty}\frac{t^m}{1-e^{-t}}e^{-st}dt, \tag{6}$$

for $s\in\mathbb{R}^+$ and $m\in\mathbb{N}$. For more information on digamma and polygamma functions, see [2, 6, 7, 8].

The generalization of the Gamma function has attracted much attention from many researchers. For example, Euler and Jackson introduced the p and q-analogues of the Gamma function respectively and established some properties of the said functions (see [3, 5, 12] and the references therein). Chaudry and Zubair [9] established another p-analogue of the Gamma function via its integral representation. Díaz and Pariguan [10] later introduced the k- analogue of the Gamma function by generalizing the Pochhammer k-symbol. Diaz and Teruel [11], Krasniqi and Merovci [13] and Nantomah, Prempeh and Twum [14] introduced a two-parameter deformation of the Gamma function. The authors respectively established the (q,k), (p,q) and (p,k)-analogues of the Gamma function with applications to inequalities.

In this paper, we introduce a new one-parameter deformation of the *integral*, *limit* and *product* representations of the Gamma function. We also establish some properties generalizing those satisfied by the classical Gamma function. In addition, we establish some inequalities involving this new function. We present our results in the following section.

2. PRELIMINARIES

In this section, we present some definitions and Lemmas which are well known in the literature and which play a key role in the proofs of our main results.

Definition 2.1. A function $f: I \to (0,\infty)$ is said to be logarithmically convex if $\ln f$ is convex on I. That is,

$$lnf(\alpha x+\beta y) \leq \alpha \ln f(x) + \beta \ln f(y), \tag{7}$$

or equivalently

$$f(\alpha x+\beta y) \leq (f(x))^{\alpha} (f(y))^{\beta}, \tag{8}$$

for each $x,y \in I$ and $\alpha, \beta \in (0,1)$ such that $\alpha+\beta = 1$.

Lemma 2.2. *Let $A,B \geq 0$ and $k \geq 1$. Then the inequality $A^k + B^k \leq (A+B)^k$ holds.*

Lemma 2.3. [15]

Let $p>1$ and $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$. Let f and g be continuous functions on $[a,b]$. Then Hölder's inequality for integrals is given by

$$\int_a^b f(t)g(t)dt \leq \left(\int_a^b f(t)^p dt\right)^{\frac{1}{p}} \left(\int_a^b g(t)^q dt\right)^{\frac{1}{q}}. \tag{9}$$

Lemma 2.4. [15]

Let $p>1$ and f and g be continuous functions on $[a,b]$. Then Minkowski's inequality for integrals is given as

$$\left(\int\limits_a^b |f(t)+g(t)|^p dt\right)^{\frac{1}{p}} \leq \left(\int\limits_a^b |f(t)|^p dt\right)^{\frac{1}{p}} + \left(\int\limits_a^b |g(t)|^p dt\right)^{\frac{1}{p}}. \tag{10}$$

3. Main Results

Definition 3.1. Let $s,v\in\mathbb{R}^+$. Then the v-analogue (also called v-deformation or v-generalization) of the Gamma function is defined as

$$\Gamma_v(s) = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{s}{v}-1} e^{-t}dt. \tag{11}$$

Note that when $v=1$, we have $\Gamma_v(s)=\Gamma(s)$.

From the relation (11), we can easily show that

$$\Gamma_v(s+v) = sv^{-2}\Gamma_v(s), \tag{12}$$

$$\Gamma_v(s) = v^{1-\frac{s}{v}}\Gamma\left(\frac{s}{v}\right), \tag{13}$$

$$\Gamma_v(v) = 1. \tag{14}$$

Proposition 3.2. *Let $s,v\in\mathbb{R}^+$. Then the v-analogue of the Gamma function satisfies the relation:*

$$\Gamma_v(s) = \lim_{n\to\infty}\frac{n!\left(\frac{n}{v}\right)^{\left(\frac{s}{v}\right)} v^{n+1}}{s(s+v)(s+2v)\cdots(s+nv)}. \tag{15}$$

Proof. Using the fact that $\left(1-\frac{t}{n}\right)^n$ converges to e^{-t} as $n\to\infty$, we write (11) as

$$\Gamma_v(s)=\lim_{n\to\infty}\int_0^n\left(\frac{t}{v}\right)^{\frac{s}{v}-1}\left(1+\frac{t}{n}\right)^n dt.$$

Repeated integration by parts yields

$$\begin{aligned}
I_n &= \int_0^n\left(\frac{t}{v}\right)^{\frac{s}{v}-1}\left(1+\frac{t}{n}\right)^n dt\\
&= \left[\frac{v^2}{s}\left(\frac{t}{v}\right)^{\frac{s}{v}}\left(1-\frac{t}{n}\right)\right]_0^n+\frac{nv^2}{ns}\int_0^n\left(\frac{t}{v}\right)^{\frac{s}{v}}\left(1+\frac{t}{n}\right)^{n-1}dt\\
&= \frac{nv^2}{ns}\int_0^n\left(\frac{t}{v}\right)^{\frac{s}{v}}\left(1+\frac{t}{n}\right)^{n-1}dt\\
&\;\vdots\\
&= \frac{nv^2}{ns}\frac{(n-1)v^2}{n(s+v)}\frac{(n-2)v^2}{n(s+2v)}\cdots\frac{v^2}{(s+(n-1)v)}\int_0^n\left(\frac{t}{v}\right)^{\frac{s}{v}+n-1}dt\\
&= \frac{n!\left(\frac{n}{v}\right)^{\frac{s}{v}}v^{n+1}}{s(s+v)(s+2v)\cdots(s+nv)}.
\end{aligned}$$

By taking limit on both sides, we obtain

$$\Gamma_v(s)=\lim_{n\to\infty}I_n=\lim_{n\to\infty}\frac{n!\left(\frac{n}{v}\right)^{\frac{s}{v}}v^{n+1}}{s(s+v)(s+2v)\cdots(s+nv)},$$

which completes the proof.

Proposition 3.3. *Let $s,v\in\mathbb{R}^+$. Then the v-analogue of the Gamma function can be expressed as*

$$\frac{1}{\Gamma_v(s)}=v^{\frac{s}{v}-1}se^{\frac{\gamma s}{v}}\prod_{k=1}^{\infty}\left(1+\frac{s}{kv}\right)e^{-\frac{s}{kv}}, \tag{16}$$

where γ is the Euler-Mascheroni constant.

Proof. By the relation (15), we write

$$\Gamma(s)=\lim_{n\to\infty}\left(\frac{n}{v}\right)^{\frac{s}{v}}\frac{v}{s}\left(\frac{v}{s+v}\right)\left(\frac{2v}{s+2v}\right)\cdots\left(\frac{nv}{s+nv}\right).$$

This implies that

$$\begin{aligned}
\frac{1}{\Gamma(s)} &= \lim_{n\to\infty}\left(\frac{v}{n}\right)^{\frac{s}{v}}\frac{s}{v}\left(1+\frac{s}{v}\right)\left(1+\frac{s}{2v}\right)\cdots\left(1+\frac{s}{nv}\right)\\
&= \lim_{n\to\infty}\left(\frac{v}{n}\right)^{\frac{s}{v}}\frac{s}{v}\prod_{k=1}^{n}\left(1+\frac{s}{kv}\right).
\end{aligned}$$

To evaluate the above limit, we introduce a convergent factor $e^{-\frac{s}{kv}}$. This gives

$$\begin{aligned}\frac{1}{\Gamma(s)} &= \lim_{n\to\infty}\left(\frac{v}{n}\right)^{\frac{s}{v}}\frac{s}{v}e^{\frac{s}{v}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)}\prod_{k=1}^{n}\left(1+\frac{s}{kv}\right)e^{-\frac{s}{kv}}\\ &= \lim_{n\to\infty}e^{\frac{s}{v}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-\ln n\right)}v^{\frac{s}{v}}\frac{s}{v}\prod_{k=1}^{n}\left(1+\frac{s}{kv}\right)e^{-\frac{s}{kv}}\\ &= v^{\frac{s}{v}-1}se^{\frac{\gamma s}{v}}\prod_{k=1}^{\infty}\left(1+\frac{s}{kv}\right)e^{-\frac{s}{kv}}.\end{aligned}$$

This completes the proof.

Proposition 3.4. *Let $s,v\in\mathbb{R}^+$. Then the v-analogue of the Gamma function satisfies the relation:*

$$\Gamma_v(s) = v^{-\frac{s}{v}}s^{-1}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{\frac{s}{v}}\left(1+\frac{s}{nv}\right)^{-1}. \tag{17}$$

Proof. By replacing s with $\frac{s}{v}$ in (4) and using relation (13), we obtain

$$\begin{aligned}\Gamma_v(s) &= v^{\left(1-\frac{s}{v}\right)}\left[\frac{v}{s}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{\frac{s}{v}}\left(1+\frac{s}{nv}\right)^{-1}\right]\\ &= v^{-\frac{s}{v}}s^{-1}\prod_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{\frac{s}{v}}\left(1+\frac{s}{nv}\right)^{-1},\end{aligned}$$

which completes the proof.

Remark 3.5. By letting $v=1$ in Propositions (3.2), (3.3) and (3.4) we obtain (2), (3) and (4) as special cases.

Proposition 3.6. *Let $s,v\in\mathbb{R}^+$. Then the v-digamma function, ψ_v has the series representation*

$$\psi_v(s) = -\frac{(\ln v+\gamma)}{v}-\frac{1}{s}+\sum_{n=1}^{\infty}\left[\frac{1}{nv}-\frac{1}{s+nv}\right]. \tag{18}$$

Proof. Taking log on both sides of (16) gives

$$\ln\Gamma_v(s) = -\ln v^{\left(\frac{s}{v}-1\right)}-\ln s-\frac{\gamma s}{v}-\sum_{n=1}^{\infty}\ln\left(1+\frac{s}{nv}\right)+\sum_{n=1}^{\infty}\frac{s}{nv}.$$

By differentiating both sides with respect to s, we obtain

$$\begin{aligned}\psi_v(s) &= -\frac{\ln v}{v}-\frac{1}{s}-\frac{\gamma}{v}-\sum_{n=1}^{\infty}\frac{1}{nv+s}+\sum_{n=1}^{\infty}\frac{1}{nv}\\ &= -\frac{(\ln v+\gamma)}{v}-\frac{1}{s}+\sum_{n=1}^{\infty}\left[\frac{1}{nv}-\frac{1}{s+nv}\right].\end{aligned}$$

This completes the proof.

Proposition 3.7. *Let $s, v \in \mathbb{R}^+$. Then the v-digamma function, ψ_v has the integral representation*

$$\psi_v(s) = -\left(\frac{\ln v + \gamma}{v}\right) + \int_0^\infty \frac{e^{-vt} - e^{-st}}{1 - e^{-vt}} dt, \tag{19}$$

where γ is the Euler-Mascheroni constant.

Proof. By relation (18), we have

$$\begin{aligned}
\psi_v(s) &= -\left(\frac{\ln v + \gamma}{v}\right) - \frac{1}{s} + \sum_{n=1}^{\infty}\left(\frac{1}{nv} - \frac{1}{s + nv}\right) \\
&= -\left(\frac{\ln v + \gamma}{v}\right) - \int_0^\infty e^{-st} dt + \sum_{n=1}^{\infty}\int_0^\infty \left(e^{-nvt} - e^{-(s+nv)t}\right) dt \\
&= -\left(\frac{\ln v + \gamma}{v}\right) - \int_0^\infty e^{-st} dt + \sum_{n=1}^{\infty}\int_0^\infty e^{-nvt}\left(1 - e^{-st}\right) dt \\
&= -\left(\frac{\ln v + \gamma}{v}\right) - \int_0^\infty e^{-st} dt + \int_0^\infty \left(1 - e^{-st}\right)\sum_{n=1}^{\infty} e^{-nvt} dt \\
&= -\left(\frac{\ln v + \gamma}{v}\right) - \int_0^\infty e^{-st} dt + \int_0^\infty \left(1 - e^{-st}\right)\left(\frac{e^{-vt}}{1 - e^{-vt}}\right) dt \\
&= -\left(\frac{\ln v + \gamma}{v}\right) + \int_0^\infty \frac{e^{-vt} - e^{-st}}{1 - e^{-vt}} dt,
\end{aligned}$$

which completes the proof.

Proposition 3.8. *Let $s, v \in \mathbb{R}^+$ and $m \in \mathbb{N}$. Then the v-polygamma function, $\psi_v^{(m)}$ has the series representation*

$$\psi_v^{(m)}(s) = (-1)^{m+1}\, m! \sum_{n=0}^{\infty} \frac{1}{(s + nv)^{m+1}}. \tag{20}$$

Proof. By differentiating (18) successively, we have

$$\psi_v^{(1)}(s) = \sum_{n=0}^{\infty} \frac{1}{(s+nv)^2},\ \psi_v^{(2)}(s) = -\sum_{n=0}^{\infty} \frac{2}{(s+nv)^3},\ \psi_v^{(3)}(s) = \sum_{n=0}^{\infty} \frac{6}{(s+nv)^4} \cdots$$

The m^{th} order derivative yields

$$\psi_v^{(m)}(s) = (-1)^{m+1}\, m! \sum_{n=1}^{\infty} \frac{1}{(s + nv)^{m+1}},$$

which completes the proof.

Proposition 3.9. *Let $s,v\in\mathbb{R}^+$ and $m\in\mathbb{N}$. Then the v-polygamma function, $\psi_v^{(m)}$ has the integral representation*

$$\psi_v^{(m)}(s)=(-1)^{m+1}\int_0^\infty \frac{t^m e^{-st}}{1-e^{-vt}}dt. \tag{21}$$

Proof. Recall relation (20):

$$\psi_v^m(s)=(-1)^{m+1}\sum_{n=0}^\infty \frac{m!}{(s+nv)^{m+1}}.$$

We introduce the inverse Laplace transform of the summand to get

$$\begin{aligned}\psi_v^m(s)&=(-1)^{m+1}\sum_{n=0}^\infty\int_0^\infty t^m e^{-(s+nv)t}dt\\ &=(-1)^{m+1}\sum_{n=0}^\infty \frac{m!}{(s+nv)^{m+1}}\\ &=(-1)^{m+1}\sum_{n=0}^\infty\int_0^\infty t^m e^{-(s+nv)t}dt\\ &=(-1)^{m+1}\int_0^\infty t^m\sum_{n=0}^\infty e^{-(s+nv)t}dt\\ &=(-1)^{m+1}\int_0^\infty \frac{t^m e^{-st}}{1-e^{-vt}}dt.\end{aligned}$$

This completes the proof.

Theorem 3.10. *Let $r,s,v\in\mathbb{R}^+$, $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. Then the inequality*

$$\Gamma_v\left(\frac{r}{p}+\frac{s}{q}\right)\leq[\Gamma_v(r)]^{\frac{1}{p}}[\Gamma_v(s)]^{\frac{1}{q}} \tag{22}$$

holds.

Proof. By replacing s with $\frac{r}{p}+\frac{s}{q}$ in (11), we obtain

$$
\begin{aligned}
\Gamma_v\left(\frac{r}{p}+\frac{s}{q}\right) &= \int\limits_0^\infty \left(\frac{t}{v}\right)^{\frac{r}{pv}+\frac{s}{qv}-1} e^{-t}dt \\
&= \int\limits_0^\infty \left(\frac{t}{v}\right)^{\frac{r}{pv}+\frac{s}{qv}-\frac{1}{p}+\frac{1}{q}} e^{-t\left(\frac{1}{p}+\frac{1}{q}\right)}dt \\
&= \int\limits_0^\infty \left(\frac{t}{v}\right)^{\frac{1}{p}\left(\frac{r}{v}-1\right)} e^{-\frac{t}{p}} \left(\frac{t}{v}\right)^{\frac{1}{q}\left(\frac{s}{v}-1\right]} e^{-\frac{t}{q}}dt.
\end{aligned}
$$

By applying Hölder's inequality for integrals, we have

$$
\begin{aligned}
\Gamma_v\left(\frac{r}{p}+\frac{s}{q}\right) &\leq \left[\int\limits_0^\infty \left(\left(\frac{t}{v}\right)^{\frac{1}{p}\left(\frac{r}{v}-1\right)} e^{-\frac{t}{p}}\right)^p dt\right]^{\frac{1}{p}} \left[\int\limits_0^\infty \left(\left(\frac{t}{v}\right)^{\frac{1}{q}\left(\frac{s}{v}-1\right)} e^{-\frac{t}{q}}\right)^q dt\right]^{\frac{1}{q}} \\
&= \left[\int\limits_0^\infty \left(\frac{t}{v}\right)^{\left(\frac{r}{v}-1\right]} e^{-t}dt\right]^{\frac{1}{p}} \left[\int\limits_0^\infty \left(\frac{t}{v}\right)^{\left(\frac{s}{v}-1\right)} e^{-t}dt\right]^{\frac{1}{q}} \\
&= \left[\Gamma_v(r)\right]^{\frac{1}{p}} \left[\Gamma_v(s)\right]^{\frac{1}{q}}.
\end{aligned}
$$

This completes the proof.

Definition 3.11. The n^{th} derivative of the v-analogue of the gamma function is defined for $s,v \in \mathbb{R}^+$ as

(23) $$\Gamma_v^{(n)}(s) = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{s}{v}-1} \left[\ln\left(\frac{t}{v}\right)\right]^n \frac{e^{-t}}{v^n}dt.$$

Theorem 3.12. *Let $r,s,v \in \mathbb{R}^+$, $k \geq 1$, and $m,n \in \{2h : h \in \mathbb{N}\}$. Then the following inequality holds.*

(24) $$\left[\Gamma_v^{(m)}(r)+\Gamma_v^{(n)}(s)\right]^{\frac{1}{k}} \leq \left[\Gamma_v^{(m)}(r)\right]^{\frac{1}{k}} + \left[\Gamma_v^{(n)}(s)\right]^{\frac{1}{k}}.$$

Proof. From the relation (23), we have

$$\left[\Gamma_v^{(m)}(r)+\Gamma_v^{(n)}(s)\right]^{\frac{1}{k}}$$

$$=\left[\int_0^\infty\left(\frac{t}{v}\right)^{\frac{r}{v}-1}\left[\ln\left(\frac{t}{v}\right)\right]^m\frac{e^{-t}}{v^m}dt+\int_0^\infty\left(\frac{t}{v}\right)^{\frac{s}{v}-1}e^{-t}\left[\ln\left(\frac{t}{v}\right)\right]^n\frac{e^{-t}}{v^n}dt\right]^{\frac{1}{k}}$$

$$=\left[\left[\int_0^\infty\left(\frac{t}{v}\right)^{\frac{1}{k}\left(\frac{r}{v}-1\right)}\left[\ln\left(\frac{t}{v}\right)\right]^{\frac{m}{k}}\frac{e^{-\frac{t}{k}}}{v^{\frac{m}{k}}}\right]^k+\left[\int_0^\infty\left(\frac{t}{v}\right)^{\frac{1}{k}\left(\frac{s}{v}-1\right)}\left[\ln\left(\frac{t}{v}\right)\right]^{\frac{n}{k}}\frac{e^{-\frac{t}{k}}}{v^{\frac{n}{k}}}\right]^k dt\right]^{\frac{1}{k}}.$$

By applying Lemma 4.3 to the above expression, we obtain

$$\left[\Gamma_v^{(m)}(r)+\Gamma_v^{(n)}(s)\right]^{\frac{1}{k}}\leq\left[\left[\int_0^\infty\left(\frac{t}{v}\right)^{\frac{1}{k}\left(\frac{r}{v}-1\right)}\left[\ln\left(\frac{t}{v}\right)\right]^{\frac{m}{k}}\frac{e^{-\frac{t}{k}}}{v^{\frac{m}{k}}}+\int_0^\infty\left(\frac{t}{v}\right)^{\frac{1}{k}\left(\frac{s}{v}-1\right)}\left[\ln\left(\frac{t}{v}\right)\right]^{\frac{n}{k}}\frac{e^{-\frac{t}{k}}}{v^{\frac{n}{k}}}\right]^k dt\right]^{\frac{1}{k}}.$$

Finally, we apply Minkowski's inequality to get

$$\left[\Gamma_v^{(m)}(r)+\Gamma_v^{(n)}(s)\right]^{\frac{1}{k}}\leq\left[\int_0^\infty\left(\frac{t}{v}\right)^{\left(\frac{r}{v}-1\right)}\left[\ln\left(\frac{t}{v}\right)\right]^m\frac{e^{-t}}{v^m}dt\right]^{\frac{1}{k}}+\left[\int_0^\infty\left(\frac{t}{v}\right)^{\left(\frac{s}{v}-1\right)}\left[\ln\left(\frac{t}{v}\right)\right]^n\frac{e^{-t}}{v^n}dt\right]^{\frac{1}{k}}$$
$$=\left[\Gamma_v^{(m)}(r)\right]^{\frac{1}{k}}+\left[\Gamma_v^{(n)}(s)\right]^{\frac{1}{k}}.$$

This completes the proof.

Remark 3.13. A similar results was proved for the (p,k)-gamma function, see Theorem 2.3 of [14].

Theorem 3.14. *Let $r,s\in\mathbb{R}^+$, $m,n\in\mathbb{N}$ and $a>1$ such that $\frac{m}{a}+\frac{n}{b}\in\mathbb{N}$. Then the inequality*

$$\left|\psi_v^{\left(\frac{m}{a}+\frac{n}{b}\right)}\left(\frac{r}{a}+\frac{s}{b}\right)\right|\leq\left|\psi_v^{(m)}(r)\right|^{\frac{1}{a}}+\left|\psi_v^{(n)}(s)\right|^{\frac{1}{b}}\tag{25}$$

holds.

Proof. By using the integral representation (21), we obtain

$$\begin{aligned}
\left|\psi_v^{\left(\frac{m}{a}+\frac{n}{b}\right)}\left(\frac{r}{a}+\frac{s}{b}\right)\right| &= \int_0^\infty \frac{t^{\left(\frac{m}{a}+\frac{n}{b}\right)} e^{-\left(\frac{r}{a}+\frac{s}{b}\right)t}}{1-e^{-vt}} dt \\
&= \int_0^\infty \frac{t^{\left(\frac{m}{a}+\frac{n}{b}\right)} e^{-\left(\frac{r}{a}+\frac{s}{b}\right)t}}{(1-e^{-vt})^{\left(\frac{1}{a}+\frac{1}{b}\right)}} dt \\
&= \int_0^\infty \left(\frac{t^{\frac{m}{a}} e^{-\frac{r}{a}t}}{(1-e^{-vt})^{\frac{1}{a}}} \frac{t^{\frac{n}{b}} e^{-\frac{s}{b}t}}{(1-e^{-vt})^{\frac{1}{b}}}\right) dt \\
&= \int_0^\infty \left(\frac{t^m e^{-rt}}{1-e^{-vt}}\right)^{\frac{1}{a}} \left(\frac{t^n e^{-st}}{1-e^{-vt}}\right)^{\frac{1}{b}} dt,
\end{aligned}$$

and by applying Hölder's inequality, we obtain

$$\begin{aligned}
\int_0^\infty \left(\frac{t^m e^{-rt}}{1-e^{-vt}}\right)^{\frac{1}{a}} \left(\frac{t^n e^{-st}}{1-e^{-vt}}\right)^{\frac{1}{b}} dt &\leq \left[\int_0^\infty \frac{t^m e^{-rt}}{1-e^{-vt}} dt\right]^{\frac{1}{a}} \left[\int_0^\infty \frac{t^n e^{-st}}{1-e^{-vt}} dt\right]^{\frac{1}{b}} \\
&= \left|\psi_v^{(m)}(r)\right|^{\frac{1}{a}} \left|\psi_v^{(n)}(s)\right|^{\frac{1}{b}}.
\end{aligned}$$

This completes the proof.

Remark 3.15. A similar results was proved for the (p,k)-gamma function, see Theorem 2.5 of [14].

Theorem 3.16. *Let $m, p \in \mathbb{N}$ and $k > 0$. Then the following inequalities hold.*

(26) $$\left(e^{\psi_v^{(m)}(s)}\right)^2 \geq e^{\psi_v^{(m+1)}(s)} e^{\psi_v^{(m-1)}(s)}, \ (m \text{ is odd}),$$

(27) $$\left(e^{\psi_v^{(m)}(s)}\right)^2 \leq e^{\psi_v^{(m+1)}(s)} e^{\psi_v^{(m-1)}(s)}, \ (m \text{ is even}).$$

Proof. Using the integral representation (21), we obtain

$$\psi_v^{(m)}(s) - \tfrac{1}{2}\left(\psi_v^{(m+1)}(s) + \psi_v^{(m-1)}(s)\right)$$

$$= (-1)^{m+1}\int_0^\infty \frac{m!t^m e^{-st}}{1-e^{-vt}}dt - \frac{(-1)^{m+2}}{2}\int_0^\infty \frac{(m+1)!t^{m+1}e^{-st}}{1-e^{-vt}}dt - \frac{(-1)^m}{2}\int_0^\infty \frac{(m-1)!t^{m-1}e^{-st}}{1-e^{-vt}}dt$$

$$= \frac{(-1)^{m+1}}{2}\left[\int_0^\infty \frac{2m(m-1)!t^m e^{-st}}{1-e^{-vt}}dt + \int_0^\infty \frac{(m+1)m(m-1)!t^{m+1}e^{-st}}{1-e^{-vt}}dt + \int_0^\infty \frac{(m-1)!t^{m-1}e^{-st}}{1-e^{-vt}}dt\right]$$

$$= \frac{(-1)^{m+1}}{2}\left(2m + m(m+1) + 1\right)\int_0^\infty \frac{(m-1)!t^{m-1}e^{-st}}{1-e^{-vt}}dt$$

$$= \frac{(-1)^{m+1}}{2}\left(m^2 + 3m + 1\right)\int_0^\infty \frac{(m-1)!t^{m-1}e^{-st}}{1-e^{-vt}}dt$$

$$= \begin{cases} \geq 0, & m \text{ is odd} \\ \leq 0, & m \text{ is even} \end{cases}.$$

This implies that the inequalities

$$2\psi_v^{(m)}(s) \geq \psi_v^{(m+1)}(s) + \psi_v^{(m-1)}(s) \tag{28}$$

and

$$2\psi_v^{(m)}(s) \leq \psi_v^{(m+1)}(s) + \psi_v^{(m-1)}(s) \tag{29}$$

hold respectively for odd m and even m. By exponentiating both sides of the inequalities (28) and (29), we obtained the desired results.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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