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ON PRODUCT GRAPHS

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Abstract. In this paper we study product graphs and obtain some results. We also indicate the scope of its applications in a variety of fields.

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1. Introduction

All graphs considered in this paper are finite, simple and undirected.

Graphs and in particular graph products arise in a variety of different contexts, from computer science to theoretical biology, computational engineering or in studies on social networks. In the last few years graph products became again a very flourishing topic in graph theory. The revival of interest seems to be mostly due to the algorithmic point of view. In particular, algorithms for decomposing a graph with respect to a given product and for isometrically embedding a graph into a (Cartesian) product of graphs were proposed [3, 12, 14, 18, 45, 46]. Furthermore, retracts of graph products, the reconstruction of products and some other properties of products were investigated [11, 26, 27, 31, 32, 36].

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Finding a (prime factor) decomposition of a given graph with respect to a graph product is one of the basic problems in studying graph products from the algorithmic point of view. Among the four most interesting graph products (the lexicographic, the direct, the Cartesian and the strong product) the Cartesian product [3, 12, 46] and the strong product [14] are known to have polynomial algorithms for finding prime factor decompositions of connected graphs. An overview of complexity results for other products can be found in [14]. Because of Feigenbaum and Schaffer's polynomial result, it seems to be of vital interest to study those parameters of strong products of graphs whose determination is in general NP-complete.

2. Preliminaries

For any graph G, we let $V(G)$ be the vertex set of G, $E(G)$ the edge set G, and $l(G) =\mid V(G) \mid$ the number of vertices in G. Two graphs G and H with the same number of vertices are said to be isomorphic, denoted $G \cong H$, if there exists a bijection from $V(G)$ to $V(H)$ that preserves adjacency. Such a bijection is called an isomorphism from G to H. In the case when G and H are identical, this bijection is called an automorphism of G. The collection of all automorphisms of G, denoted aut$G$, constitutes a group called the automorphism group of G.We call the isomorphism classes of graphs unlabeled graphs. If G is a graph with n vertices, $L(G)$ is the number of graphs isomorphic to G with vertex set $[n]$. It is well-known that

$$L(G) = \frac{l(G)!}{\mid aut(G) \mid}. \tag{1}$$

We use the notation $\sum\limits_{i=1}^{n} G_i = G_1 + G_2 + ... + G_n$ to mean the disjoint union of a set of graphs $\{G_i\}_{i=1,\ldots,n}$.

Definition 1. *The Cartesian product of graphs G_1 and G_2, denoted $G_1 \times G_2$, as defined by Sabidussi* [39] *under the name weak Cartesian product, is the graph whose vertex set is $V(G_1 \times G_2) = V(G_1) \times V(G_2) = \{(u, v) : u \in V(G_1), v \in V(G_2)\}$,in which (u, v) is adjacent to (w, z) if either $u = w$ and $\{v, z\} \in E(G_2)$ or $v = z$ and $\{u, w\} \in E(G_1)$.*

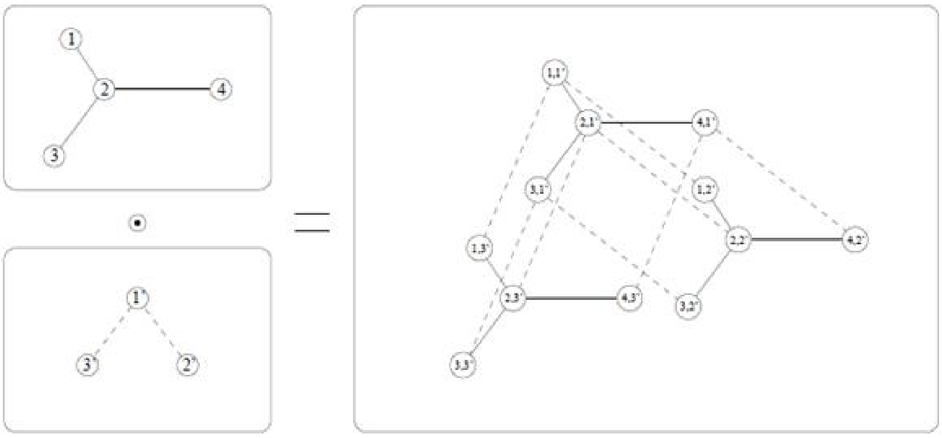

FIGURE 1. The Cartesian product of a graph with vertex set $\{1,2,3,4\}$ and a graph with vertex set $\{1',2',3'\}$ is a graph with vertex set $\{(i,j)\}$, where $i \in [4]$ and $j \in [3]'$.

Proposition 1. *Let G_1, G_2 and G_3 be graphs. Then (commutativity) $G_1 \times G_2 \cong G_2 \times G_1$; (associativity) $(G_1 \times G_2) \times G_3 \cong G_1 \times (G_2 \times G_3)$.*

These properties allow us to talk about the Cartesian product of a set of graphs $\{G_i\}_{i\in I}$, denoted $\times_{i\in I} G_i$. We denote by G^m the Cartesian product of n copies of G.

Definition 2. *A graph G is prime with respect to Cartesian product if G is a connected graph with more than one vertex such that $G \cong H_1 \times H_2$ implies that either H_1 or H_2 is a singleton vertex.*

Two graphs G and H are called relatively prime with respect to Cartesian product if and only if $G \cong G_1 \times J$ and $H \cong H_1 \times J$ imply that J is a singleton vertex.

If G is a connected graph, then G can be decomposed into prime factors, that is, there is a set $\{P_i\}_{i\in I}$ of prime graphs such that $G \cong \times_{i\in I} P_i$. In Figure, a connected graph G with 24 vertices is decomposed into the product of prime graphs P_1, P_2, P_3 with 3,2 and 4 vertices, respectively.

3.Sabidussi's Results

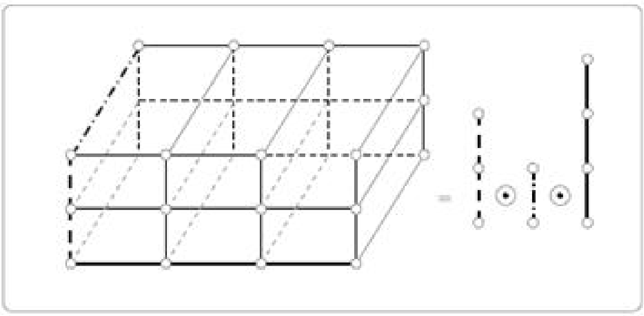

FIGURE 2. The decomposition of a connected graph into prime graphs.

Theorem 1 (Sabidussi[39]). *For any non-trivial connected graph G, the factorization of G into the Cartesian product of prime powers is unique up to isomorphism.*

The automorphism groups of the Cartesian product of a set of graphs was studied by Sabidussi [39] and Palmer [37].

Theorem 2 (Sabidussi). *Let $\{G_i\}_{i=1,\ldots,n}$ be a set of graphs. Then the automorphism group of the Cartesian product of $\{G_i\}_{i=1,\ldots,n}$ is isomorphic to the automorphism group of the sum of $\{G_i\}_{i=1,\ldots,n}$. That is,*

$$aut(\times_{i=1}^{n} G_i) \cong aut(\sum_{i=1}^{n} G_i).$$

Theorem 3 (Sabidussi). *Let $G_1, G_2, \ldots G_n$ be connected graphs which are pairwise relatively prime with respect to Cartesian multiplication. Then the automorphism group of the Cartesian product of $\{G_i\}_{i=1,\ldots,n}$ is isomorphic to the product of each of the automorphisms groups of $\{G_i\}_{i=1,\ldots,n}$. That is,*

$$aut(\times_{i=1}^{n} G_i) \cong \prod_{i=1}^{n} aut(G_i).$$

We introduce in the following an important theorem by Sabidussi about the automorphism group of a connected graph using its prime factorization.

Theorem 4 (Sabidussi). *Let G be connected graph with prime factorization*

$$G \cong P_1^{s_1} \times P_2^{s_2} \times \ldots \times P_k^{s_k},$$

where for $r = 1, 2, ..., k$, all P_r are distinct prime graphs, and all s_r are positive integers. Then we have

$$aut(G) \cong \prod_{r=1}^{k} aut(P_r^{s_r}) \cong \prod_{r=1}^{k} aut(P_r)^{\varrho_{s_r}}. \tag{2}$$

In other words, the automorphism group of G is generated by the automorphism groups of the factors and the transpositions of isomorphic factors.

4.Unlabeled Prime Graphs

In this section all graphs considered are unlabeled and connected.

Definition 3. *The (formal) Dirichlet series of a sequence $\{a_n\}_{n=1,2,...,\infty}$ is defined to be $\sum_{n=1}^{\infty} a_n/n^s$.*

The multiplication of Dirichlet series is given by

$$\sum_{n\geq 1} \frac{a_n}{n^s} . \sum_{m\geq 1} \frac{b_n}{n^s} = \sum_{n\geq 1} (\sum_{k|n} a_k b_{n/k}) \frac{1}{n^s}. \tag{3}$$

Definition 4. *A monoid is a semigroup with a unit. A free commutative monoid is a commutative monoid M with a set of primes $P \subseteq M$ such that each element $m \in M$ can be uniquely decomposed into a product of elements in P up to rearrangement.*

Let M be a free commutative monoid. We get a monoid algebra CM, in which the elements are all formal sums $\sum_{m\in M} c_m m$, where $c_m \in C$, with addition and multiplication defined naturally.

For each $m \in M$, we associate a length $l(m)$ that is compatible with the multiplication in M. That is, for any $m_1, m_2 \in M$, we have $l(m_1)l(m_2) = l(m_1 m_2)$.

It is well-known that the monoid algebra yields the following identity:

Proposition 2. *Let M be a free commutative monoid with prime set P. The following identity holds in the monoid algebra CM :*

$$\sum_{m\in M} m = \prod_{p\in P} \frac{1}{1-p}. \tag{4}$$

Furthermore, we can define a homomorphism from M to the ring of Dirichlet series under which each $m \in M$ is sent to $1/l(m)^s$, where l is a length function of M. Therefore,

$$\sum_{m\in M} \frac{1}{l(m)^s} = \prod_{p\in P} \frac{1}{1-l(p)^{-s}}. \tag{5}$$

Example 1. *Let N denote the set of all natural numbers, and let P denote the set of all prime numbers. Then N is a free commutative monoid with prime set P, and the length function is given by $l(n) = n$ for all $n \in N$. As an application of Proposition 2, we have the following well-known identity for expressing the zeta function:*

$$\zeta(s) = \sum_{n\in N} \frac{1}{n^s} = \prod_{p\in P} \frac{1}{1-p^{-s}}. \tag{6}$$

Recall that C is the set of unlabeled connected graphs under the operation of Cartesian product. The unique factorization theorem of Sabidussi gives C the structure of a commutative free monoid with a set of primes $\mathbb{P}$, where $\mathbb{P}$ is the set of unlabeled prime graphs. This is saying that every element of C has a unique factorization of the form $b_1^{e_1} b_2^{e_2} ... b_k^{e_k}$, where the b_i are distinct primes in $\mathbb{P}$. Let $l(G)$, the number of vertices in G, be a length function for C. We have the following proposition.

Lemma 1. *Let $\mathbb{C}$ and $\mathbb{P}$ be the set of unlabeled connected graphs and the set of unlabeled prime graphs, respectively. We have*

$$\sum_{G\in\mathbb{C}} \frac{1}{l(G)^s} = \prod_{P\in\mathbb{P}} \frac{1}{1-l(P)^{-s}}. \tag{7}$$

The enumeration of prime graphs was studied by Raphael Bellee [6]. *We use Dirichlet series to count unlabeled connected prime graphs.*

Theorem 5. *Let $\widetilde{c}_n$ be the number of unlabeled connected graphs on n vertices, and let b_m be the number of nlabeled prime graphs on m vertices. Then we have*

$$\sum_{n\geq 1} \frac{\widetilde{c}_n}{n^s} = \prod_{m\geq 2} \frac{1}{(1-m^{-s})^{b_m}}. \tag{8}$$

Furthermore, if we define numbers d_n for positive integers n by

$$\sum_{n\geq 1} \frac{d_n}{n^s} = \log \sum_{n\geq 1} \frac{\widetilde{c}_n}{n^s}, \tag{9}$$

then

$$d_n = \sum_{m^l=n} \frac{b_m}{l} \tag{10}$$

where the sum is over all pairs (m, l) of positive integers with $m^l = n$.

Remark 1. *A quick observation from Equation (10) is that $b_n = d_n$ whenever n is not of the form r^k for some $k > 1$.*

In what follows, we introduce an interesting recursive formula for computing d_n. To start with, we differentiate both sides of Equation (9) with respect to s and simplify. We get

$$\sum_{n\geq 2} \log n \frac{\widetilde{c}_n}{n^s} = (\sum_{n\geq 1} \frac{\widetilde{c}_n}{n^s})(\sum_{n\geq 2} \log n \frac{d_n}{n^s}),$$

which gives

$$\widetilde{c}_n \log n = \sum_{m^l=n} c_m d_l \log l. \tag{11}$$

Since c_1 is the number of connected graphs on 1 vertex, $c_1 = 1$. It follows from Equation (11) easily that $d_p = c_p$ when p is a prime number. Therefore, if p is a prime number, $b_p = d_p = c_p$. This fact can be seen directly, since a connected graph with a prime number of vertices is a prime graph.

Raphael Bellee used Equation (11) to find formulae for d_n where n is a product of two different primes or a product of three different primes:

If $n = pq$ where $p \neq q$,

$$d_n = \widetilde{c}_n - c_p c_q; \tag{12}$$

If $n = pqr$ where p, q and r are distinct primes,

$$d_n = \widetilde{c}_n + 2c_p c_q c_r - c_p c_{qr} - c_q c_{pr} - c_r c_{pq}. \tag{13}$$

In fact, Equations (12) and (13) are special cases of the following proposition.

Proposition 3. *Let $d_n, \widetilde{c}_n$ be defined as above. Then we have*

$$d_n = \widetilde{c}_n - \frac{1}{2}\sum_{n_1 n_2 = n} c_{n_1} c_{n_2} + \frac{1}{3}\sum_{n_1 n_2 n_3 = n} c_{n_1} c_{n_2} c_{n_3} - ...$$

Proof. We can use the identity

$$\log(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + ...$$

to compute from Equation (9) that

$$\begin{aligned} \sum_{n\geq 1} \frac{d_n}{n^s}) &= \log(1 + \sum_{n\geq 2} \frac{\widetilde{c}_n}{n^s}) \\ &= \sum_{n\geq 2} \frac{\widetilde{c}_n}{n^s} - \frac{1}{2}(\sum_{n\geq 2} \frac{\widetilde{c}_n}{n^s})^2 + \frac{1}{3}(\sum_{n\geq 2} \frac{\widetilde{c}_n}{n^s})^3 - .... \end{aligned}$$

Equating coefficients of n^{-s} on both sides, we get the desired result. □

Proof of Theorem 5. We start with

$$\sum_m \frac{1}{l(m)^s} = \prod_p \frac{1}{1 - l(p)^{-s}},$$

where the left-hand side is summed over all connected graphs, and the right-hand side is summed over all prime graphs. Regrouping the summands on the left-hand side with respect to the number of vertices in m, we get the left-hand side of Equation (8). Regrouping the factors on the right-hand side with respect to the number of vertices in p, we get the right-hand side of Equation (8).

Taking the logarithm of both sides of Equation (8), we get

$$\begin{aligned}\log \sum_{n\geq 1} \frac{\widetilde{c}_n}{n^s} &= \log \prod_{m\geq 2} \frac{1}{(1-m^{-s})^{b_m}} = \sum_{m\geq 2} b_m \log \frac{1}{(1-m^{-s})} \\ &= \sum_{m\geq 2} (b_m \sum_{l\geq 1} \frac{m^{-sl}}{l}) = \sum_{m\geq 2, l\geq 1} \frac{b_m}{lm^{st}},\end{aligned}$$

and Equation (10) follows immediately.

Next we will compute the numbers b_n in terms of the numbers d_n using the following lemma.

Lemma 2. *Let $\{D_i\}_{i=1,2,\ldots}$ and $\{J_i\}_{i=1,2,\ldots}$ be sequences of numbers satisfying*

$$D_k = \sum_{l|k} \frac{J_{k/l}}{l}, \tag{14}$$

and let μ be the Mobius function. Then we have

$$J_k = \frac{1}{k} \sum_{l|k} \mu(\frac{k}{l}) l D_l.$$

Proof. Multiplying by k on both sides of Equation (14), we get

$$kD_k = \sum_{l|k} \frac{k}{l} J_{k/l} = \sum_{l|k} l J_l.$$

Applying the Mobius inversion formula, we get

$$kJ_k = \sum_{l|k} \mu(\frac{k}{l}) l D_l.$$

Therefore,

$$J_k = \frac{1}{k} \sum_{l|k} \mu(\frac{k}{l}) l D_l.$$

□

Given any natural number n, let e be the largest number such that $n = r^e$ for some r. Note that r is not a power of a smaller integer. We let $D_k = d_{r^k}, J_k = b_{r^k}$. It follows that Equation (10) is equivalent to Equation (14).

Theorem 6. *For any natural number n, let e, r be as described in above. Then we have*

$$b_n = \frac{1}{e} \sum_{l|e} \mu(\frac{e}{l}) l d_{r^e}.$$

Proof. The result follows straightforwardly from Lemma 1.2. □

Table 1 gives the numbers of unlabeled prime graphs b_n compared with the numbers of unlabeled connected graphs $\widetilde{c}_n$ on no more than 12 vertices.

Table 1. Values of $\widetilde{c}_n$ and b_n, for $n \leq 12$.

n	$\widetilde{c}_n$	b_n
1	1	0
2	1	1
3	2	2
4	6	5
5	21	21
6	112	110
7	853	853
8	11117	11111
9	261080	261077
10	11716571	11716550
11	1006700565	1006700565
12	164059830476	164059830354

5.Applications

In practical applications, we observe perturbed product structures, so-called approximate graph products, since structures derived from real-life data are notoriously incomplete and/or plagued by measurement errors. As a consequence, the structures need to be analyzed in a way that is robust against inaccuracies, noise, and perturbations in the data.

The problem of computing approximate graph products was posed several years ago in a theoretical biology context [44]. The authors provided a concept concerning the topological theory of the relationships between genotypes and phenotypes. In this framework a so-called "character" (trait or Merkmal) is identified with a factor of a generalized topological space that describes the variational properties of a phenotype. The notion of a character can be understood as a property of an organism that can vary independently of other traits from generation to generation. Characters thus are not necessarily the same as observable properties such as arms, legs, fingers, a spinal chord, etc, although such observables of course often are instantiations of characters. The important biological distinction is whether such measurable attributes (or combinations thereof) form a "coordinate axis" along which the character states (e.g. the lengths of arms or fingers) can vary independently of other traits, or whether the underlying genetics dictates dependencies among the observables [33]. This question can be represented as a graph problem in the following way:

Consider a set X of "phenotypes", that is, representations of distinct organisms, each of which is characterized by a list of properties such as body shape, eye color, presence or absence of certain bones, etc. If one knows about the phylogenetic relationships between the members of X, we can estimate which combinations of properties are interconvertible over short evolutionary time-scales. This evolutionary "accessibility relation" introduces a graph-structure on X [8, 15, 16, 40]. In particular, a phenotype space inherits its structure from an underlying sequence space.

Sequence spaces are Hamming graphs, that is, Cartesian products of complete graphs, see [9, 10]. The structure of localized subsets turns out to be of particular interest. Gavrilets [17], Grner [19], and Reidys [38], for example, describe subgraphs in sequence spaces that correspond to the subset of viable genomes or to those sequences that give rise to the same phenotype. The structure of these subgraphs is intimately related to the dynamics of evolutionary processes [25, 42]. However, since characters are only meaningfully defined on subsets of phenotypes it is necessary to use a local definition [44]:A character corresponds to a factor in a factorizable induced subgraph with non-empty interior (where x is an interior vertex of $H \times G$ if x and all its neighbors within G are in H).

Other applications of graph products can be found in rather different areas as computer graphics and theoretical computer science. In [1, 2], the authors provide a framework, called TopoLayout, to draw undirected graphs based on the topological features they contain. Topological features are detected recursively, and their subgraphs are collapsed into single nodes, forming a graph hierarchy. The final layout is drawn using an appropriate algorithm for each topological feature [1]. Graph products have a well understood structure, that can be drawn in an effective way. Hence, for an extension of this framework in particular approximate graph products are of interest. Reasons and motivations to study graph products or graphs that have a product-like structure can be found in many other areas, e.g. for the formation of finite element models or construction of localized self-equilibrating systems in computational engineering [28, 30]. Other motivations can be found in discrete mathematics. A natural question is what can be said about a graph invariant of an (approximate) product if one knows the corresponding invariants of the factors. There are many contributions, treating this problem, e.g. [4, 7, 20, 21, 24, 34]. In all applications of practical interest, the graphs in question have to be either obtained from computer simulations (e.g. within the RNA secondary structure model as in [8, 15, 16]) or they need to be estimated from measured data. In both cases, they are known only approximately. In order to deal with such inaccuracies, a mathematical framework is needed that allows us to deal with graphs that are only approximate products. Given a graph G that has a product-like structure, the task is to find a graph H that is a nontrivial product and a good approximation of G, in the sense that H can be reached from G by a small number of additions or deletions of edges and vertices. In fact, a very small perturbation, such as the deletion or insertion of a single edge, can destroy the product structure completely, modifying a product graph to a prime graph [13, 48].In [47] Yegnanarayanan one of the two authors of this paper has obtained a number of results concerning applications of Analytical Number theory to Graph Theory via Product Graphs.

Before we conclude we would like to point out another vibrant area called Graph theory zetas. The graph theory zetas first appeared in work of Ihara on p-adic groups in the 1960s. Soon the connection with graphs was found and many papers appeared. The main authors in the 1980s and 90s were Sunada, Hashimoto [22], and Bass [5]. Other references are Venkov and Nikitin [43] and Northshield's paper in the volume of Hejhal et al [23] See [41, 35] for more history and references. The main properties of the Riemann zeta function have graph theory analogs, at least for regular graphs.

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