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SOLUTIONS OF TRANSMISSION DYNAMICS MODEL FOR HBV USING HPM WITH RESIDUAL ERRORS

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Abstract. The transmission dynamics mathematical model of infectious disease is a significant disease controlling technique, which is being used on the occurrence of hepatitis B to value the varying immunization strategies. In this paper, we analyze the transmission dynamics models through mathematical model using Homotopy perturbation method (HPM) which defines how to control the impact of Hepatitis B virus (HBV). To get the solution for nonlinear ordinary differential equations, Homotopy Perturbation Method (HPM) has been used. We have discussed the numerical simulations up to six order approximation and residual error analysis with the help of Matlab software in this paper.

Keywords: hepatitis B virus; homotopy perturbation method; transmission dynamics; residual errors; vaccination; mathematical modeling.

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1. INTRODUCTION

Infection by Hepatitis B virus is a life threatening disease which may or may not show any symptoms to identify. According to the statistical survey, it has been reported that more than 350 million people globally carriers Hepatitis B and die 0.6 million per year [1]. For protecting and prohibiting from this disease, timing vaccination is the possible solution to control the disease [2, 3] which has been proved by various research works. Particularly, Mathematical models help to understand the dynamics of HBV [4]. Anderson et al. have studied the compartmental model for dynamics of sex transmission using a partial differential equation model. Its numerical solution gives the occurrence of infection [5]. The mathematical model has been developed by Williams et al. for assessing the effectiveness of immunization incorporating the age- dependent fertility and vertical (perinatal) transmission which provides the possibilities for mass vaccination especially for infected infants during the confinement [6]. Muhammad Altaf Khan et al. developed a model to understand the effectiveness of immigrants' host population [7]. Reza Akbari analyzed in his study that by giving vaccination and treatment the dynamics of hepatitis B virus is controlled [8].

The motivation of this paper is helping the scholars of Medical as well as Science finding a solution relating to HBV. In this paper, we have found the solution for HBV using HPM method. Mojtaba Hajipour., et al analyzed the non-linear problem [9, 10] and their works propose sixth-order approximation [11]. This paper proposes sixth order approximation and higher order for the same problem. The numerical simulations obtained up to sixth-order approximations and error analysis was done using Matlab. It will open for new futuristic avenues in nonlinear modeling in different angles for further research.

2. MATHEMATICAL MODELING OF HBV TRANSMISSION MODEL DYNAMICS

We consider the epidemiological compartmental model for hepatitis virus dynamics. Peifeng Liang et al's work is extended for this study [1] for a set of population who has been taken for this study on the basis of S-I-D-B-R-A epidemic model and divided them into six compartmental classes. They are Susceptible (S) denoted the population at precarious of infection with HBV; Latent (I) denoted the population infected but not yet infectious; Acute (D) denoted

the population at early high infectious stage of HBV; Carrier (B) denoted the population with continuing HBV infection; Recovery (R) denoted the recovered population for the lifetime; Vaccinated (A) denoted the immunity that monitors vaccination may disappear over time [12, 13]. Parameter values are presented in Table1. The horizontal transmission is fewer effective compared with the vertical transmission [14]. In this model, the vaccinated compartment is added because there is a possibility for a reduction of immunity after HBV recovery of population. As there is a chance for infection after the recovery of the individual in the life time, vaccination is important. So this model is feasible. Here, a represents age and t is time. λ represents the force of infection, v is the rate of successful vaccination, natural mortality rate is μ, age dependent mortality is μ_c and the recovery rate is δ are assumed as constant. The analytical solution is found for equation (2.1) using HPM method.

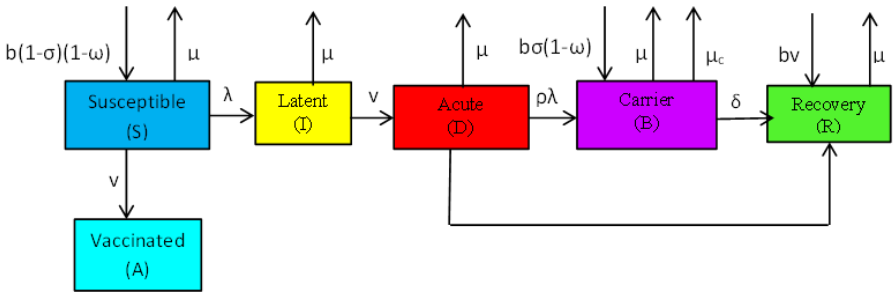

FIGURE 1. Compartmental diagram for HBV

The model becomes:

$$
\begin{aligned}
\frac{dS}{dt} &= b-\omega B-b\sigma+b\sigma\omega B+\phi A-(\mu+\lambda+v)S\\
\frac{dI}{dt} &= \lambda S-(\mu+v)I\\
\frac{dD}{dt} &= vI-(\mu+\gamma)D\\
\frac{dB}{dt} &= b\sigma(1-\omega)B+\rho\lambda D-(\mu+\mu_c+\delta)B\\
\frac{dR}{dt} &= (1-\rho)\gamma D+\delta B-\mu R\\
\frac{dA}{dt} &= b\omega+vS-(\phi+\mu)A
\end{aligned}
\tag{2.1}
$$

The initial and boundary conditions are:

$S(0)=1\times10^{-1}$;$I(0)=1\times10^{-2}$;$D(0)=1\times10^{-3}$;$B(0)=1\times10^{-4}$;$R(0)=1\times10^{-5}$;$A(0)=1\times10^{-6}$

TABLE 1

Parameter	Explanation	Range
μ	Natural mortality rate	0.055 – 0.016
λ	Force of HBV infection	0.013 – 0.159
v	Rate of individuals leave the latent class	6 – 8 per year
β	Coefficient of transmission	0.8 – 20.49
ω	The birth proportion with successful vaccination	0.055 – 0.095
σ	proportion of perinatal infection	0.7 – 0.9
δ	The recovery rate of carrier	0.0005 – 0.03
b	Birth rate	0.0121 – 0.05
ϕ	Rate of waning vaccine-induced immunity	0.001 – 0.039
γ	Rate of individuals leave the acute class	3 – 4 per year
ρ	Probability of one's suffering from acute HBV to chronic bearer.	0.05 – 0.09

3. HOMOTOPY PERTURBATION METHOD

To explain the primary concept of HPM, we take the non-linear functional equation as follows:

$$P(x) - g(a) = 0, \quad a \in \Omega \tag{3.1}$$

The boundary conditions are:

$$Q = \left(x, \frac{\partial x}{\partial n}\right) = 0, a \in \dot{\Gamma} \tag{3.2}$$

Here P is an arbitrary functional operator, Q is a boundary operator, g(a) is an analytic function $\dot{\Gamma}$ and is the boundary of the domain Ω. Generally, the operator P can be divided into two parts T_L and T_N, where T_L is a linear and T_N is a non-linear operator. Therefore Eq (3.1) can be rewritten as follows:

$$T_L(u) + T_N(u) - g(a) = 0 \tag{3.3}$$

We construct a homotopy $h(a,p) : \Omega \times [0,1] \to R$ which satisfies

$$M(h,p) = (1-p)\left[T_L(h) - T_L(x_0)\right] + p[P(h) - g(a)] = 0 \tag{3.4}$$

or

$$M(h,p) = T_L(h) - T_L(x_0) + pT_L(x_0) + p\left[T_N(h) - g(a)\right] = 0 \tag{3.5}$$

Where p $\in [0,1]$ is an embedding parameter and x_0 is an initial approximation for the solution of Eq. (3.1), which satisfies the boundary conditions. By using HPM, let us use p, and take the solution of Eq. (3.5) should be written as a power series in p:

$$h = h_0 + h_1 p + h_2 p^2 + \ldots = \sum_{i=0}^{\infty} h_i p^i \tag{3.6}$$

Let us consider p =1, the approximate solution of Eq. (3.2) should be established as follows:

$$x = \lim_{p \to 1} h = h_0 + h_1 + h_2 + \ldots \tag{3.7}$$

4. APPLICATIONS

The analytical solution of this model using the LHAM is

$$S(t) = 1 \times 10^{-1} e^{-(\mu+\lambda+v)t} + \left(1 \times 10^{-1} - \frac{A_2}{1-\beta} + \frac{A_3}{1-A_4}\right) e^{-t} + \frac{A_2 e^{-\beta t}}{1-\beta} - \frac{A_3 e^{-A_4 t}}{1-A_4}$$

Here $A_2 = 5.91 \times 10^{-6}, A_3 = 1 \times 10^{-9}, A4 = 6 \times 10^{-3}$

$$\therefore S(t) = 1 \times 10^{-1} e^{-6.018t} + 0.09997 e^{-t} + 2.955 \times 10^{-5} e^{-0.8t} - 1 \times 10^{-9} e^{-0.006t} \tag{4.1}$$

$I(t) = 1 \times 10^{-2} e^{-(\mu+r)t} + \left(1 \times 10^{-2} - \frac{\alpha_1}{1-\alpha_2}\right) e^{-(\mu+v)t} + \frac{\alpha_1 e^{-\alpha_2 t}}{1-\alpha_2}$

Here $\alpha_1 = 1.3 \times 10 - 3, \alpha_2 = 6.018$

$$\therefore I(t) = 2 \times 10^{-2} e^{-6.005t} + 1 \times 10^{-2} e^{-6.018t} \tag{4.2}$$

$$\begin{aligned} D(t) &= 2\left(1 \times 10^{-3}\right) e^{-(\mu+\gamma)t} + tv\left(1 \times 10^{-2}\right) e^{-(\mu+\gamma)t} \\ \therefore D(t) &= (0.002 + 0.06t) e^{-3005t} \end{aligned} \tag{4.3}$$

$$\begin{aligned} B(t) &= \left(1 \times 10^{-4}\right) e^{-\beta} + \left(\left(1 \times 10^{-4}\right) - \psi\right) e^{-\mu} + \psi e^{-(\mu+\gamma)t} \\ \psi &= 3.47 \times 10^{-7} \\ \therefore B(t) &= 1.99653 \times 10^{-4} e^{-0.8t} + 3.47 \times 10^{-7} e^{-3.005t} \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
&R(t) = 1\times10^{-5}e^{-\mu t} + \left(1\times10^{-5} + \xi - \kappa\right)e^{-\mu t} - \xi e^{-(\mu+\gamma)t} + \kappa e^{-\beta t} \\
&\xi = 9.5\times10^{-4}, \mathrm{K} = -6.29\times10^{-7} \\
&\therefore R(t) = 9.70629\times10^{-4}e^{-0.005t} - 9.5\times10^{-4}e^{-3.005t} - 6.29\times10^{-7}e^{-0.08t}
\end{aligned} \tag{4.5}
$$

$$
\begin{aligned}
&A(t) = 1\times10^{-6}e^{-(\phi+\mu)t} + \left(\left(1\times10^{-6}\right) - \frac{b\omega}{A_4} - \varphi\right)e^{-.4,t} + \frac{b\omega}{A_4} + \varphi e^{-\alpha_2 t} \\
&\phi = -35.29 \\
&\therefore A(t) = 0.45833 + 34.832e^{-0.006t} - 35.29e^{-6.018t}
\end{aligned} \tag{4.6}
$$

5. Numerical Results

Let us consider the values [17] for numerical results are,

$S_0 = 1\times10^{-1}; I_0 = 1\times10^{-2}; D_0 = 1\times10^{-3}; B_0 = 1\times10^{-4}; R_0 = 1\times10^{-5}; A_0 = 1\times10^{-6}$ $\mu = 0.005, \lambda = 0.013, v = 6, \beta = 0.8, \omega = 0.055, \sigma = 0.7, \quad \delta = 0.005, b = 0.05\ \phi = 0.001, \gamma = 3, \rho = 0.05$

Let us use MatLab software to obtain the sixth-order expansions for $S(t), I(t), D(t), B(t)\ R(t)$ and $A(t)$

$$
\begin{aligned}
&S(t) = 0.1 + 0.01595ht + 0.001798h^2t + 0.00278h^3t + 0.0001487h^4t + 0.0003585h^5t \\
&+0.005785h^6t + 0.0002486h^2t^2 + 0.0004324h^3t^2 + 0.006788h^4t^2 + 0.083721h^5t^2 + \\
&0.0094426h^6t^2 + 0.0008022h^3t^3 + 0.0000037878h^4t^3 + 0.076555656h^5t^3 + 0.0056488h^6t^3 \ldots
\end{aligned} \tag{5.1}
$$

$$
\begin{aligned}
&I(t) = 0.01 + 0.0014855ht + 0.0045568h^2t + 0.0005568h^3t + 0.000854846h^4t + 0.000045544532h^5t \\
&+0.0005496 + h^6t + 0.0004668h^2t^2 + 0.0000545155h^3t^2 + 0.00055669h^4t^2 + 0.004047956h^5t^2 + \\
&0.000054876599h^6t^2 + 0.000045956531h^3t^3 + 0.000098462h^4t^3 + \\
&0.00005857656h^5t^3 + 0.00054167989h^6t^3 \ldots
\end{aligned} \tag{5.2}
$$

$$
\begin{aligned}
&D(t) = 0.001 + 0.0006555ht + 0.0004862h^2t + 0.00015463h^3t + 0.00046387h^4t + 0.00065452h^5t \\
&+0.00005697h^6t + 0.000077956h^2t^2 + 0.0006713h^3t^2 + 0.000338425h^4t^2 + 0.0008952h^5t^2 \\
&+0.00075199h^6t^2 + 0.00097123h^3t^3 + 0.00003541h^4t^3 + 0.00087661h^5t^3 + 0.0008553h^6t^3 \ldots
\end{aligned} \tag{5.3}
$$

$$
\begin{aligned}
&B(t) = 0.0001 + 0.00005456ht + 0.00005469h^2t + 0.000056545h^3t + 0.00003221h^4t + 0.00001546h^5t \\
&+0.000067556h^6t + 0.00008165h^2t^2 + 0.00009482h^3t^2 + 0.00000697h^4t^2 + 0.000089945h^5t^2 \\
&+0.00009561h^6t^2 + 0.0000248968h^3t^3 + 0.00003786h^4t^3 + 0.00007665h^3t^3 + 0.000086551h^6t^3 \ldots
\end{aligned} \tag{5.4}
$$

(5.5)
$$R(t) = 0.00001 + 0.00000545ht + 0.000005853h^2t + 0.000007626h^3t + 0.000007543h^4t + 0.000007236h^5t$$
$$+0.000006512 + h^6t + 0.0000076565h^2t^2 + 0.000009225h^3t^2 + 0.0000006481h^4t^2 + 0.000007551h^5t^2$$
$$+0.000009125h^6t^2 + 0.000002756h^3t^3 + 0.000003912h^4t^3 + 0.00000794h^5t^3 + 0.000004665h^6t^3 \ldots$$

(5.6)
$$A(t) = 0.000001 + 0.0000003551ht + 0.000000466h^2t + 0.000000658h^3t + 0.0000008484h^4t +$$
$$0.0000008493h^5t + 0.0000007555 + h^6t + 0.0000007561h^2t^2 + 0.00000031124h^3t^2$$
$$+0.000000076551h^4t^2 + 0.00000007552h^5t^2 + 0.00000004565h^6t^2 + 0.0000000452h^3t^3$$
$$+0.0000000465h^4t^3 + 0.0000005545h^5t^3 + 0.00000078954h^6t^3 \ldots$$

6. Results and Discussion

In this article, an error was done and optimal values of h obtained [18]. For that, we substitute eqns (5.1) to (5.6) in (2.1).

(6.1)
$$ER_1(S,I,D,B,R,A;h_1) = \frac{d\phi_S(t;h_1)}{dt} - b + \omega_B(t;h_1) + b\sigma - b\sigma\omega_B(t;h_1) - \phi_A(t;h_1) + (\mu+\lambda+v)_S(t;h_1)$$

(6.2)
$$ER_2(S,I,D,B,R,A;h_2) = \frac{d\phi_I(t,h_2)}{dt} - \lambda_5(t,h_2) + (\mu+v)_I(t;h_2)$$

(6.3)
$$ER_3(S,I,D,B,R,A;h_3) = \frac{d\phi_D(t,h_3)}{dt} - v_I(t;h_3) + (\mu+\gamma)_D(t;h_3)$$

(6.4)
$$ER_4(S,I,D,B,R,A;h_4) = \frac{d\phi_B(t;h_4)}{dt} - b\sigma(1-\omega)_B(t,h_4) - \rho\lambda_D(t,h_4) + (\mu+\mu_c+\delta)_B(t;h_4)$$

(6.5)
$$ER_5(S,I,D,B,R,A;h_5) = \frac{d\phi_R(t,h_5)}{dt} - (1-\rho)\gamma_D(t;h_5) - \delta_B(t;h_5) + \mu_R(t;h_5)$$

(6.6)
$$ER_6(S,I,D,B,R,A;h_6) = \frac{d\phi_A(t;h_6)}{dt} - b\omega - v_S(t;h_6) + (\phi+\mu)_A(t;h_6)$$

Let us consider the square residual error for sixth order approximation:

(6.7)
$$RS(h_1) = \int_0^1 (ER_1(S,I,D,B,R,A;h_1))^2\,dt$$

TABLE 2. The h value is

S(t)	-1.1 $\leq$ h $\leq$ -0.2
I(t)	-1.2 $\leq$ h $\leq$ -0.4
D(t)	-1.3 $\leq$ h $\leq$ -0.5
B(t)	-1.4 $\leq$ h $\leq$ -0.7
R(t)	-1.5 $\leq$ h $\leq$ -0.8
A(t)	-1.6 $\leq$ h $\leq$ -0.9

TABLE 3. The minimum values of $\mathrm{RS}(\mathrm{h_1}^*), \mathrm{RI}(\mathrm{h_2}^*), \mathrm{RD}(\mathrm{h_3}^*), \mathrm{RB}(\mathrm{h_4}^*), \mathrm{RR}(\mathrm{h_5}^*), \mathrm{RA}(\mathrm{h_6}^*)$

	h*	Minimum value
$RS(h_1)$	-0.875466	6.36552510^{-6}
$RI(h_2)$	-0.756595	5.53215710^{-8}
$RD(h_3)$	-0.655558	4.23545910^{-12}
$RB(h_4)$	-0.535258	3.57878610^{-13}
$RR(h_5)$	-0.478622	2.98775210^{-15}
$RA(h_6)$	-0.345762	1.72636410^{-16}

$$RI(h_2) = \int_0^1 (ER_2(S,I,D,B,R,A;h_2))^2\, dt \tag{6.8}$$

$$RD(h_3) = \int_0^1 (ER_3(S,I,D,B,R,A;h_3))^2\, dt \tag{6.9}$$

$$RB(h_4) = \int_0^1 (ER_4(S,I,D,B,R,A;h_4))^2\, dt \tag{6.10}$$

$$RR(h_5) = \int_0^1 (ER_5(S,I,D,B,R,A;h_5))^2\, dt \tag{6.11}$$

$$RA(h_6) = \int_0^1 (ER_6(S,I,D,B,R,A;h_6))^2\, dt \tag{6.12}$$

The minimal values of $\mathrm{RS}(\mathrm{h_1}), \mathrm{RI}(\mathrm{h_2}), \mathrm{RD}(\mathrm{h_3}), \mathrm{RB}(\mathrm{h_4}), \mathrm{RR}(\mathrm{h_5}), \mathrm{RA}(\mathrm{h_6})$ are shown below.

$$\frac{dRS(h_1^*)}{dh_1} = 0, \frac{dRI(h_2^*)}{dh_2} = 0, \frac{dRD(h_3^*)}{dh_3} = 0, \quad \frac{dRB(h_4^*)}{dh_4} = 0, \frac{dRR(h_5^*)}{dh_5} = 0, \frac{dRA(h_6^*)}{dh_6} = 0.$$

TABLE 4. The residual errors for $ER_1, ER_2, ER_3, ER_4, ER_5$ and ER_6 for te (0,1)

t	ER1 (S,I,D,B,R,A;h_1*)	ER2 (S,I,D,B,R,A;h_2*)	ER3 (S,I,D,B,R,A;h_3*)	ER4 (S,I,D,B,R,A;h_4*)	ER5 (S,I,D,B,R,A;h_5*)	ER6 (S,I,D,B,R,A;h_6*)
0.0	8.4657610^{-6}	5.5363710^{-5}	3.6467710^{-8}	5.4547710^{-6}	2.4687710^{-5}	8.7478910^{-5}
0.1	3.5498210^{-3}	9.7452710^{-6}	8.7478910^{-9}	3.4655310^{-4}	1.4785410^{-6}	4.8888910^{-2}
0.2	5.4885310^{-2}	2.7478910^{-4}	4.8898910^{-6}	7.5646310^{-5}	5.4785410^{-7}	9.3748910^{-8}
0.3	1.4785710^{-5}	6.7389310^{-3}	9.3743910^{-7}	1.4756810^{-3}	4.4785510^{-9}	7.8343910^{-7}
0.4	9.6734810^{-9}	1.988710^{-4}	7.8383910^{-6}	2.5455610^{-9}	8.4785110^{-3}	2.4737810^{-4}
0.5	6.6746310^{-8}	7.8998710^{-6}	2.4774810^{-5}	9.5477310^{-2}	9.7754110^{-4}	1.8389310^{-6}
0.6	9.7823510^{-7}	3.7489810^{-5}	1.8389310^{-7}	10.577710^{-3}	70.445710^{-8}	5.7378810^{-7}
0.7	6.6463810^{-1}	8.7388310^{-4}	5.7378810^{-6}	8.7764710^{-4}	45.854810^{-2}	6.7378210^{-2}
0.8	2.6638910^{-6}	1.7898810^{-6}	6.7378210^{-4}	6.4477510^{-8}	51.478510^{-3}	2.3743710^{-3}
0.9	8.6536710^{-4}	7.8938910^{-7}	7.3743810^{-9}	11.477810^{-5}	81.447710^{-4}	3.7478910^{-7}
1	4.7789810^{-5}	4.7889710^{-6}	3.7478910^{-5}	2.7478910^{-6}	14.785710^{-5}	7.6474710^{-8}

We consider the optimal values of $h_1{}^*, h_2{}^*, h_3{}^*, h_4{}^*, h_5{}^*$ and $h_6{}^*$ for all of the cases are:

$h_1^* = -0.875466, h_2^* = -0.756595, h_3{}^* = -0.655558$

$h_4^* = -0.535258, h_5{}^* = -0.478622, h_6{}^* = -0.345762$

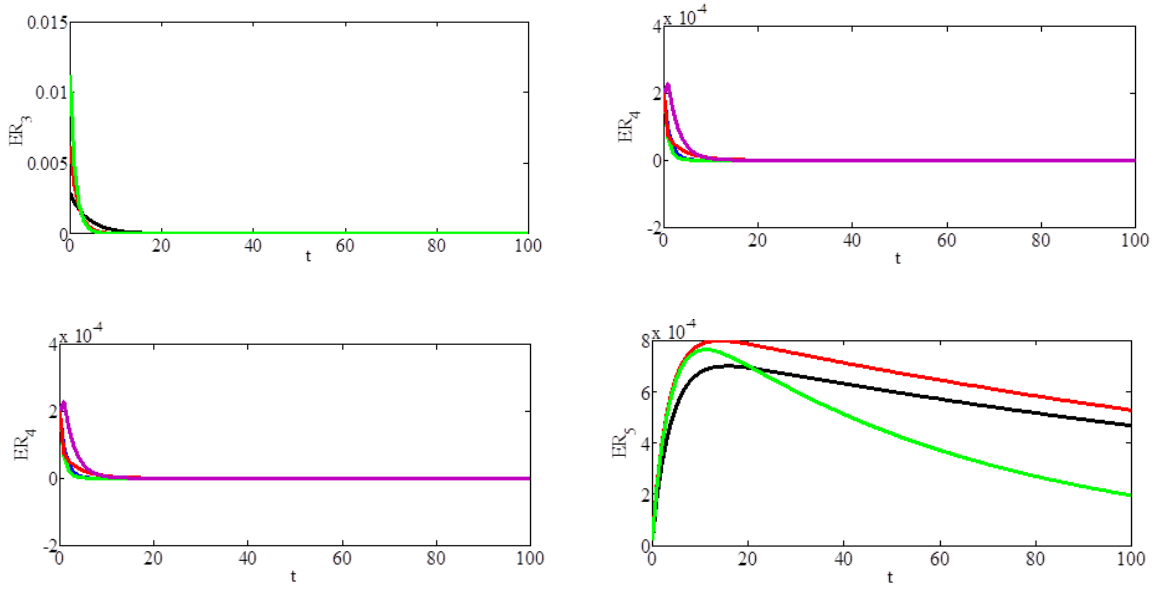

FIGURE 2. The residual errors for $ER_1, ER_2, ER_3, ER_4, ER_5$ and ER_6 for $t \in (0,1)$

The infectious condition of dangerous class will be decreased when the natural mortality and v values are increased. When the continuous treatment is given, the impact of HBV infection decreased compared with dangerous class. This process is continued till the patient will be recovered. After recovering, some patients are advised to take vaccine. Because of that protection

after HBV recovered keep up the lifetime of the population, though the protection of vaccination may vanish over time. In figure 2, shows the residual errors for $ER_1, ER_2, ER_3, ER_4, ER_5$ and ER_6 for $t \in (0,1)$.

7. Conclusion

The transmission dynamics mathematical model of HBV is individual from the macroscopic observation to put on the spread of hepatitis B in the population. Age-factor is the significant cause for the spread of HBV. The realistic simulations and the effect of vaccination strategies have been analyzed including the newborn. The result is that this compartmental model projected the vaccination criteria for controlling HBV transmission. The numerical simulation provides numerical understanding of the transmission of HBV which is obtained up to six order approximations and error analysis was done using Matlab.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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