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NEW LAPLACE VARIATIONAL ITERATIVE METHOD FOR SOLVING TWO-DIMENSIONAL TELEGRAPH EQUATIONS

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Abstract: In this paper, we present a new semi analytical technique based on Laplace transform and modified variational iterative method for solving two- dimensional telegraph equation arising in many applications of sciences and engineering. Some numerical examples have been presented to illustrate the accuracy of the proposed technique.

Keywords: Laplace transform; modified variational iterative method; 2D telegraph equation; numerical examples.

2010 AMS Subject Classification: 44A10, 35E15, 47J30.

1. INTRODUCTION

There are many initial and boundary value problems which involve partial differential equations. Only a few of these equations can be solved by analytical methods. In most cases, we depend on the numerical solution of such partial differential equations such as finite difference methods, finite element methods, wavelet methods and decomposition methods. There are many semi analytical or analytical methods which play a significant role in computational mathematics such as Homotopy perturbation method (HPM), Adomian decomposition method (ADM), Variational

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iteration methods (VIM) and Homotopy analysis methods (HAM). Two- dimensional partial differential equations have many applications in sciences and engineering. In mathematics, Laplace transform based numerical methods plays a major role in the field of computational and applied mathematics. To make it iterative, variational methods have attracts the attention of various researchers and scientists. Combination of these two methods gives numerical results with significant convergence. Variational iteration method has been developed for solving delay differential equations in [1]. Variational iteration technique for nonlinear equations has been presented in [2]. Laplace-variational iteration method has been used for solving the homogeneous Smoluchowski coagulation equation in [3]. New modified variational iteration transform method has presented for solving eighth order boundary value problems in [4]. In [5], reconstruction of variational iteration algorithms with the aid of the Laplace transform is presented. Numerical solutions of time-fractional diffusion equation in porous medium have presented in [6], by using variational iteration method. Some limitations of the variation iteration method and how Laplace transform method overcome these, have been discussed in [7]. A semi- analytical method is presented for solving a family of Kuramoto- Shivshinsky equations in [8]. Laplace variational method is presented for modified fractional derivatives with non-singular kernel in [9]. New Laplace variational iteration method has been developed for solving nonlinear partial differential equations in [10]. Efficient numerical technique based on wavelets has been developed for solving three-dimensional partial differential equations in [11].

Consider the second-order linear two-dimensional hyperbolic telegraph equation in the region $\{(x,y): a \leq x \leq b,\ a \leq y \leq b\}$,

$$\frac{\partial^2 u}{\partial t^2}(x,y,t) + 2\rho \frac{\partial u}{\partial t}(x,y,t) + \sigma . u(x,y,t) = \xi \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + q(x,y,t), \quad 0 \leq t \leq T$$

with initial conditions

$$u(x,y,0) = f(x,y), \qquad \frac{\partial u}{\partial t}(x,y,0) = g(x,y)$$

where $\rho,\ \sigma,\ \xi$ are constants and $f,\ g$ are known functions.

The hyperbolic partial differential equations play an important role in formulating fundamental equations arising in atomic physics. These equations are very useful to understanding the various phenomena in many branches of applied sciences such as in engineering, industry, aerospace as well as in chemistry and biology too. There are many numerical techniques have been developed for solving two- dimensional telegraph equations and play a significant role to understand the behavior of the solutions of telegraph equations. In the literature, we found that many attempts have been taken for solving one-and two-dimensional telegraph equations. Many numerical techniques have been developed to find the numerical solutions of telegraph equation. In [12], Taylor matrix method has been developed for solving two- dimensional linear hyperbolic equation. Dual reciprocity boundary integral method has been developed for the solution of second-order one dimensional hyperbolic telegraph equations in [13]. In [14], unconditionally stable finite difference scheme has been presented to find the solution of one- dimensional linear hyperbolic equations. Haar wavelet method has been presented for solving two- dimensional telegraph equation in [15]. Interpolating scaling functions have been used to find the solution of telegraph equations in [16]. Cubic B-spline collocation method has presented for solving one-dimensional hyperbolic telegraph equations in [17]. Modified B-spline differential quadrature method has developed for solving two- dimensional telegraph equations in [18]. Chebyshev tau method is used for solving telegraph equation in [19].

2. LAPLACE TRANSFORM:

Suppose $y(t)$ be a function of t defined for all positive values of t. Then the Laplace transforms of $y(t)$, represented as $L\{y(t)\}$ and is defined as:

$$L\{y(t)\} = \int_0^\infty e^{-st} y(t) dt = \bar{y}(s),$$

provided the integral exists and $'s'$ is a parameter which may be a real or complex number. Therefore

$$L\{y(t)\} = \bar{y}(s),$$

that is

$$y(t) = L^{-1}\{\bar{y}(s)\}.$$

The term $L^{-1}\{\bar{y}(s)\}$, is called the inverse Laplace transform of $\bar{y}(s)$.

3. LINEARITY PROPERTY OF LAPLACE TRANSFORMS:

Laplace transform has linearity property. Let $x(t),\ y(t),\ z(t)$ be three functions of t defined for all positive values of t. Then

$$L\{c.x(t) + d.y(t) + e.z(t)\} = c.L\{x(t)\} + d.L\{y(t)\} + e.L\{z(t)\}$$

where c, d and e are arbitrary constants.

4. LAPLACE TRANSFORMS FOR DIFFERENTIATION:

Laplace transform is very useful to find the nth order differentiation of a function. Suppose that $x(t)$ be a function of t defined for all positive values of t. Then, the Laplace transform of n^{th} derivative of function $x(t)$ is

$$L\left[\frac{d^n(x(t))}{dt^n}\right] = s^n\bar{x}(s) - s^{n-1}x(0) - s^{n-2}x'(0) - s^{n-3}x''^{(0)} - \cdots - sx^{(n-2)}(0) - x^{(n-1)}(0)$$

where $\bar{x}(s) = L\{x(t)\}$.

5. LINEARITY PROPERTY OF INVERSE LAPLACE TRANSFORMS:

Linearity property is also applicable in case of inverse Laplace transform. Let $x(t),\ y(t),\ z(t)$ be any three functions of t defined for all positive values of t. Let $\bar{x}(s),\ \bar{y}(s)$ and $\bar{z}(s)$ be the corresponding functions of s such that $\bar{x}(s) = L\{x(t)\}$, $\bar{y}(s) = L\{y(t)\}$ and $\bar{z}(s) = L\{z(t)\}$. Then

$$L^{-1}\{c.\bar{x}(s) + d.\bar{y}(s) + e.\bar{z}(s)\} = c.L^{-1}\{\bar{x}(s)\} + d.L\{\bar{y}(s)\} + e.L\{\bar{z}(s)\}$$
$$= c.x(t) + d.y(t) + e.z(t)$$

where c, d and e are arbitrary constants.

6. VARIATIONAL ITERATIVE METHOD (VIM)

Variational iterative method is widely useful to evaluate the exact or approximate solutions of one-dimensional problems arising in many applications of sciences and engineering. The variational iteration method has fast rate of convergence and gives the solution in a rapidly infinite convergent series.

The nonlinear terms can be handled with the help of variational iteration method. Consider the differential equations,

$$\boldsymbol{p}u(x,y,t)+\boldsymbol{q}u(x,y,t)=\boldsymbol{r}(x,y,t) \tag{2}$$

with the initial conditions

$$u(x,y,0)=\boldsymbol{h}(x,y) \tag{3}$$

where $\boldsymbol{p}$ is a linear operator of the first order, $\boldsymbol{q}$ is nonlinear operator and $\boldsymbol{r}$ is a nonhomogeneous term. From variational iteration method, construct a correction functional as:

$$u_{m+1}=u_m+\int_0^t \lambda[\boldsymbol{p}u_m(x,y,s)+\boldsymbol{q}\tilde{u}_m(x,y,s)-\boldsymbol{r}(x,y,s)]ds \tag{4}$$

where λ is a known as Lagrange's multiplier and $\boldsymbol{m}$ denotes the m^{th} approximations, $\tilde{u}_m$ is restricted function, i.e. $\delta\tilde{u}_m=0$. The successive approximation u_{m+1} of the solution u will be obtained by using λ and u_0. The solution is

$$u=\lim_{m\to\infty} u_m$$

7. New Laplace Variational Iterative Method for Solving Two-Dimensional Telegraph Equations

In this section, combination of Laplace transform and modified variational iteration method is present to solve three- dimensional Schrodinger equations arising in quantum physics and physical chemistry. Approximate solution of this equation has been obtained in terms of convergent series with very easily computable components.

Assume that $\boldsymbol{p}$ is an operator of the first order $\frac{\partial}{\partial t}$. Equation (2) becomes

$$\frac{\partial}{\partial t}u(x,y,t)+\boldsymbol{q}u(x,y,t)=\boldsymbol{r}(x,y,t) \tag{5}$$

Taking Laplace transform on both sides of (5), we obtain

$$L\left\{\frac{\partial}{\partial t}u(x,y,t)\right\}+L\{\boldsymbol{q}u(x,y,t)\}=L\{\boldsymbol{r}(x,y,t)\} \tag{6}$$

$$sL\{u(x,y,t)\} - \boldsymbol{h}(x,y) = L\{\boldsymbol{r}(x,y,t)\} - L\{\boldsymbol{q}u(x,y,t)\} \tag{7}$$

Applying inverse Laplace transform on both sides of (7), we obtain

$$u(x,y,t) = \boldsymbol{R}(x,y,t) - L^{-1}\left[\frac{1}{s}L\{\boldsymbol{q}u(x,y,t)\}\right] \tag{8}$$

where $\boldsymbol{R}$ is the term arising from source term and given initial condition. From the correctional functional of the variational iteration method

$$u_{m+1}(x,y,t) = \boldsymbol{R}(x,y,t) - L^{-1}\left[\frac{1}{s}L\{\boldsymbol{q}u_m(x,y,t)\}\right] \tag{9}$$

Equation (9) represents the new modified correction functional of Laplace transform of variational iteration method, the solution is given by

$$u(x,y,t) = \lim_{m\to\infty} u_m(x,y,t)$$

8. NUMERICAL EXPERIMENTS

In this section, some numerical examples have been presented to illustrate the accuracy and efficiency of the proposed numerical technique.

Example 1: Consider the following two- dimensional telegraph equation

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} + 2\frac{\partial u(x,y,t)}{\partial t} + u(x,y,t) = \frac{1}{2}\nabla^2 u(x,y,t) \tag{10}$$

with initial conditions

$$u(x,y,0) = \text{sinhx sinhy} \quad \text{and} \quad u_t(x,y,0) = -2\ \text{sinhx sinhy}$$

Applying Laplace transform on both sides of (10), we obtain

$$L\left\{\frac{\partial^2 u(x,y,t)}{\partial t^2} + 2\frac{\partial u(x,y,t)}{\partial t} + u(x,y,t)\right\} = \frac{1}{2}L\{\nabla^2 u(x,y,t)\} \tag{11}$$

This implies

$$s^2L\{u(x,y,t)\} - su(x,y,0) - u_t(x,y,0) + 2sL\{u(x,y,t)\} - 2u(x,y,0) + L\{u(x,y,t)\}$$
$$= \frac{1}{2}L\{\nabla^2 u(x,y,t)\}$$

$$s^2L\{u(x,y,t)\} - s(\text{sinhx sinhy}) + 2\ \text{sinhx sinhy} + 2sL\{u(x,y,t)\} - 2\text{sinhx sinhy}$$
$$+ L\{u(x,y,t)\} = \frac{1}{2}L\{\nabla^2 u(x,y,t)\}$$

Applying initial conditions, we obtain

$$(s+1)^2 L\{u(x,y,t)\} = s(\text{sinhx sinhy}) + \frac{1}{2}L\{\nabla^2 u(x,y,t)\}$$

Divide by $(s+1)^2$, we obtain

$$L\{u(x,y,t)\} = \frac{s(\text{sinhx sinhy})}{(s+1)^2} + \frac{1}{2}\frac{1}{(s+1)^2}L\{\nabla^2 u(x,y,t)\} \tag{12}$$

Applying inverse Laplace transform on both sides of (12), we obtain

$$u = (1-t)\,e^{-t}(sinhx\ sinhy) + \frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}L\{\nabla^2 u(x,y,t)\}\right] \tag{13}$$

Using iteration method, from (13), we obtain

$$u_{m+1} = (1-t)\,e^{-t}(\text{sinhx sinhy}) + \frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}L\{\nabla^2 u_m\}\right] \tag{14}$$

From (14), we obtain

$$u_0 = \text{sinhx sinhy}$$

$$u_1 = \text{sinhx sinhy}\,(1 - 2t\,e^{-t})$$

$$u_2 = \text{sinhx sinhy}\left(1 - 2t\,e^{-t} - \frac{2t^3e^{-t}}{3!} - \frac{2t^5e^{-t}}{5!}\right)$$

$$u_3 = \text{sinhx sinhy}\left(1 - 2t\,e^{-t} - \frac{2t^3e^{-t}}{3!} - \frac{2t^5e^{-t}}{5!} - \frac{2t^7e^{-t}}{7!} - \frac{2t^9e^{-t}}{9!}\right)$$

and so on. The solution is

$$u = \lim_{n\to\infty} u_m$$

After simplification, we obtain

$$u = \text{sinhx sinhy}\left(1 - 2t\,e^{-t} - \frac{2t^3e^{-t}}{3!} - \frac{2t^5e^{-t}}{5!} - \frac{2t^7e^{-t}}{7!} - \frac{2t^9e^{-t}}{9!}\ldots\ldots\ldots\ldots\right)$$

$$u = \text{sinhx sinhy}\left(1 - e^{-t}\left(2t + \frac{2t^3}{3!} + \frac{2t^5}{5!} + \frac{2t^7}{7!} + \frac{2t^9}{9!}\ldots\ldots\ldots\ldots\right)\right)$$

$$u = \text{sinhx sinhy}\left(1 - e^{-t}(e^t - e^{-t})\right)$$

$$u = \text{sinhx sinhy}\,(1 - 1 + e^{-2t})$$

$$u = \text{sinhx sinhy}\,(e^{-2t}) \tag{15}$$

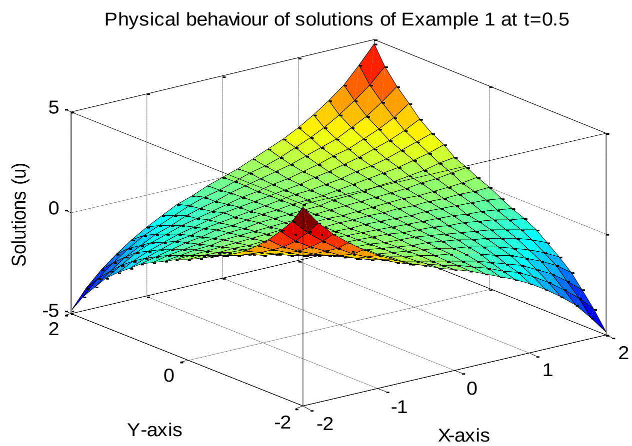

Figure 1: Description of solutions of Example 1 for $= 0.5$.

Figure 1 shows the physical behaviour of solutions of Example 1 at t=0.5.

Example 2: Consider the two- dimensional telegraph equation

$$\frac{\partial^2 u(x,y,t)}{\partial t^2} + 2\frac{\partial u(x,y,t)}{\partial t} + u(x,y,t) = \frac{1}{2}\nabla^2 u(x,y,t) \tag{16}$$

with initial conditions

$$u(x,y,0) = e^{x+y} \quad \text{and} \quad u_t(x,y,0) = -2\,e^{x+y}$$

Applying Laplace transform on both sides of (16), we obtain

$$L\left\{\frac{\partial^2 u(x,y,t)}{\partial t^2} + 2\frac{\partial u(x,y,t)}{\partial t} + u(x,y,t)\right\} = \frac{1}{2}\,L\{\nabla^2 u(x,y,t)\}$$

This implies

$$s^2 L\{u(x,y,t)\} - su(x,y,0) - u_t(x,y,0) + 2sL\{u(x,y,t)\} - 2u(x,y,0) + L\{u(x,y,t)\}$$
$$= \frac{1}{2}L\{\nabla^2 u(x,y,t)\}$$
$$s^2 L\{u(x,y,t)\} - s(e^{x+y}) + 2\,e^{x+y} + 2sL\{u(x,y,t)\} - 2\,e^{x+y} + L\{u(x,y,t)\}$$
$$= \frac{1}{2}L\{\nabla^2 u(x,y,t)\}$$

Applying initial conditions, we obtain

$$(s+1)^2 L\{u(x,y,t)\} = s(e^{x+y}) + \frac{1}{2}L\{\nabla^2 u(x,y,t)\}$$

Divide by $(s+1)^2$, we obtain

$$L\{u(x,y,t)\} = \frac{s(e^{x+y})}{(s+1)^2} + \frac{1}{2}\frac{1}{(s+1)^2}L\{\nabla^2 u(x,y,t)\} \tag{17}$$

Applying inverse Laplace transform on both sides of (17), we obtain

$$u = (1-t)\,e^{-t}(e^{x+y}) + \frac{1}{2}L^{-1}\left[\frac{1}{(s+1)^2}L\{\nabla^2 u(x,y,t)\}\right] \tag{18}$$

Using iteration method, from (18), we obtain

$$u_{m+1} = (1-t)\,e^{x+y-t} + \tfrac{1}{2}L^{-1}\left[\tfrac{1}{(s+1)^2}L\{\nabla^2 u_m\}\right] \tag{19}$$

From (19), we obtain

$$u_0 = e^{x+y}$$

$$u_1 = e^{x+y}(1 - 2t\,e^{-t})$$

$$u_2 = e^{x+y}\left(1 - 2t\,e^{-t} - \frac{2t^3e^{-t}}{3!} - \frac{2t^5e^{-t}}{5!}\right)$$

$$u_3 = e^{x+y}\left(1 - 2t\,e^{-t} - \frac{2t^3e^{-t}}{3!} - \frac{2t^5e^{-t}}{5!} - \frac{2t^7e^{-t}}{7!} - \frac{2t^9e^{-t}}{9!}\right)$$

and so on. The solution is

$$u = \lim_{n\to\infty} u_m$$

After simplification, we obtain

$$u = e^{x+y}\left(1 - 2t\,e^{-t} - \frac{2t^3e^{-t}}{3!} - \frac{2t^5e^{-t}}{5!} - \frac{2t^7e^{-t}}{7!} - \frac{2t^9e^{-t}}{9!}\dots\dots\dots\dots\right)$$

$$u = e^{x+y}\left(1 - e^{-t}\left(2t + \frac{2t^3}{3!} + \frac{2t^5}{5!} + \frac{2t^7}{7!} + \frac{2t^9}{9!}\dots\dots\dots\dots\right)\right)$$

$$u = e^{x+y}\left(1 - e^{-t}(e^t - e^{-t})\right)$$

$$u = e^{x+y}\,(1 - 1 + e^{-2t})$$

$$u = e^{x+y}\,(e^{-2t})$$

$$u = e^{x+y-2t}$$

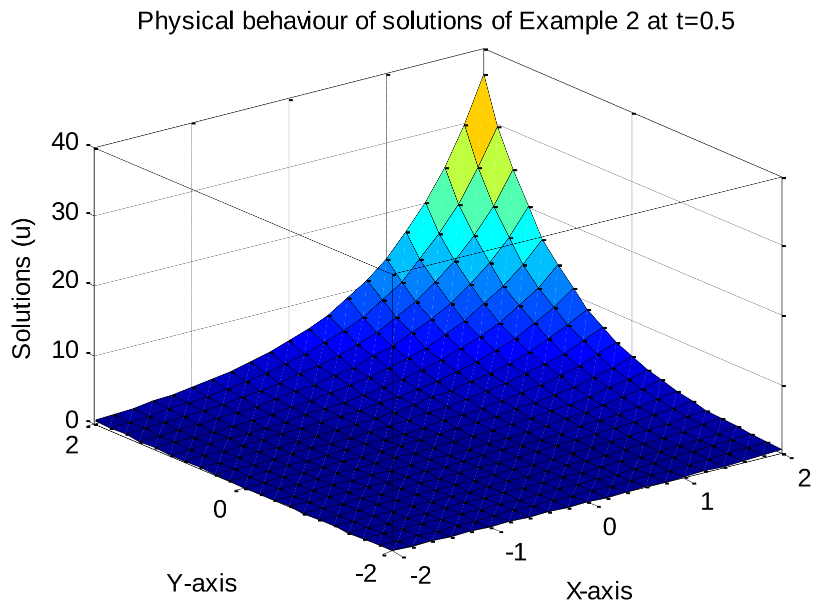

Figure 2: Description of solutions of Example 2 for $t = 0.2$

Figure 2 shows the physical behaviour of solutions of Example 2 at t=0.2.

CONCLUSION

From the above performed experiments, it is concluded that numerical technique based on the combination of two well-known classical methods such as Laplace transform method and variational iteration method, is a powerful semi analytical technique for solving two- dimensional telegraph equations. For future scope, this technique will be used for solving three-dimensional telegraph equations.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests

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