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ON COMMON α-FIXED POINT THEOREMS

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Abstract: In this research article, we give a new concept of common α-fixed point, α-compatible mappings, weakly α-compatible mappings, α-commuting mappings, weakly α-commuting mappings, and α-continuous mappings and then prove some common α-fixed point theorems for these mappings under new contractive conditions. Further, we generalize the results of Singh and Chouhan [17] for common α-fixed points and give some results for common α-fixed points under this newly introduced concept along with α- contractive conditions. Many examples have also been given and proved in support of our concept and results.

Keywords: common fixed point; common α-fixed point; weakly α-compatible maps; α-continuous mappings.

2010 AMS Subject Classification: 47H10.

1. INTRODUCTION

The fixed point theory is an important area in the fast-growing fields of non-linear analysis and non-linear operators. Using fixed point techniques, it is possible to analyze several concrete problems from science and engineering, where one is concerned with a system of

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differential/integral/functional equations. Fixed point theorems are the most important tools for proving the existence and the uniqueness of the solutions to various mathematical models (differential, integral and partial differential equations and variational inequalities, etc.), representing phenomena arising in a different field such as steady-state temperature distribution, chemical reactions, Neutron transport theories, economic theories, epidemic and flow of fluids. In 1886, the notion of fixed point theory was first introduced by Poincare[12]. He established the first result on a fixed point on using a continuous function.

Some interesting results on fixed point theorems in metric spaces are given by Edelstein[4], Assad and Kirk[1], Sehgal[13], Iseki et al.[5], Kubiak [11] and Sehgal and Bharucha-Reid[14]. Browder [3] proved the following useful theorem in1912. Browder's fixed point theorems are fundamental theorems in the area of fixed point theory and its applications. The Browder's fixed point theorem states that "If C is a unit ball in E^n (Euclidean n-dimensional space) and $T: C \to C$ a continuous function. Then T has a fixed point in C or $Tx = x$ has a solution."

In 1922 the concept of contraction type mappings in metric space was investigated by Banach [2]. He proved an interesting result in metric space, using the condition of contraction mapping. This result is known as the Banach contraction principle which states that "Every contraction mapping of a complete metric space into itself has a unique fixed point".

Banach contraction principle has many fruitful applications but it has one serious drawback that it requires the continuity of the function throughout the space. Avoiding this drawback Kannan [10] proved the modified result in fixed point theory. He proved that "let T be a self-mapping of complete metric space X satisfying the following inequality $d(Tx, Ty) \leq k\, [d\,(x, Tx) + d\,(y, Ty)]$ for all $x, y \in X, 0 < k < 1/2$, then T has a unique fixed point".

Jungck [6] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem. Banach fixed point theorem has many applications but suffers from one drawback, the definition requires continuity of the function. There then follows a flood of papers involving contractive definitions that do not require the continuity of the function. This result was further generalized and extended in various ways by many authors.

In 1982, Sessa [15] generalized the result of Jungck [6] and introduced the concept of weakly commuting mappings in metric space and proved that the two commuting mappings also commute weakly, but two weakly commuting mappings are not necessarily commuting.

In 1986, Jungck [7], introduced the concept of compatible mappings which is the generalization of commuting mappings. After this result, Jungck et al. [8], generalized his own result and proved some common fixed point theorem under the condition of compatible mappings of type (A). Jungck and Rhoades [9] introduced the concept of weakly compatible mappings and proved fixed point theorems for set-valued mappings without continuity.

Many mathematicians have studied in great detail fixed point theorems for compatible mappings, compatible mappings of type(A), weak compatible mappings, and weak compatible mappings of type (A).

The purpose and objective of this paper is to give a new concept of common α-fixed point, α-compatible mappings, weakly α-compatible mappings, α-commuting mappings, weakly α-commuting mappings, and α-continuous mappings and then proving some common α-fixed point theorems for these mappings under new contractive conditions and generalizing results of Singh and Chouhan [17] for the common α-fixed point under these concepts and α-contractive conditions.

2. PRELIMINARIES

Definition 2.1 ([4]): Let $T: X \to X$ is a function on a set X. A point $x \in X$ is called a fixed point of T if $T(x) = x$, i.e. a point, which remains invariant under the transformation T, is called a fixed point and theorems concerning with the properties and existence of fixed points are known as fixed point theorems.

Definition 2.2 ([6]): Two self maps f and g of a metric space X are said to be commuting if $fgx = gfx$ for all $x \in X$.

Definition 2.3 ([15]): A pair of self maps f and g of a metric space (X, d) is called weakly commuting maps if $d(fgx, gfx) \le d(fx, gx)$ for all $x \in X$.

Definition 2.4 ([7]): If (X,d) is a metric space,two self maps f and g of X is called compatible maps if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Definition 2.5 ([16]): Two self maps f and g are called weakly compatible or coincidently **commuting** if f and g commute at coincidence points.

Definition 2.6 ([8]): Two self maps S and T of a metric space (X,d) are called compatible of type (A) if $\lim_{n\to\infty} d(TSx_n, SSx_n) = 0$ and $\lim_{n\to\infty} d(STx_n, TTx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Definition 2.7 ([2]): Let X be a non-empty set. A function $d: X \times X \to R$ is said to be metric on X if it satisfies the followings:

(i) $d(x,y) \geq 0$, for all $x,y \in X$

(ii) $d(x,y) = 0$ iff $x = y$ for all $x,y \in X$

(iii) $d(x,y) = d(y,x)$, for all $x,y \in X$

(iv) $d(x,y) \leq d(x,z) + d(y,z)$ for all $x,y,z \in X$

The ordered pair (X,d) is called a metric space.

A map $d: R^n \times R^n \to R$ defined by $d(x,y) = |x_i - y_i|$ where $x = \{x_i\}_{i=1}^{\infty}$ and $y = \{y_i\}_{i=1}^{\infty}$ is a metric on R^n,(R^n,d) is called Euclidean metric space.

Definition 2.8 ([2]): A sequence $\{x_n\}$ in a metric space (X,d) is said to converge to a point, if for $\varepsilon > 0$ there exists $n_0 \in N$ such that $d(x_n, x) < \varepsilon$ for all $n \geq n_0$.

(or) symbolically we write $\lim_{n\to\infty} x_n = x$.

Definition 2.9 ([2]): Let $\{x_n\}$ be a sequence in a metric space (X,d). Then $\{x_n\}$ is called a Cauchy sequence if for given any $\varepsilon > 0$ there exists $n_0 \in N$ such that $d(x_m, x_n) \leq \varepsilon$, $m,n \geq n_0$.

Definition 2.10 ([2]): A metric space (X,d) is said to be complete if every Cauchy sequence in (X,d) is convergent in (X,d).

For all $x,y \in R$, let $d(x,y) = |x - y|$.Then the metric space (R,d) is complete.

Definition 2.11 ([2]): A mapping T from a metric space (X,d) into itself is called contraction if $d(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$ and $0 \leq k < 1$.

Now, we give the notion of common α-fixed point, α-compatible mappings, weakly α-compatible mappings, α-commuting mappings, weakly α-commuting mappings and α-continuous mappings under the followings.

Definition 2.12: A point $z \in X$ is said to be an α-fixed point of a map $T: X \to X$ if $(\alpha oT)z = z$.

Remark 2.13: A fixed point is not necessarily α-fixed point and the α-fixed point is not necessarily a fixed point If $\alpha = 1$, the identity map, the notions coincide.

Example 2.14: Let $T, \alpha: R \to R$ be defined by $T(x) = x + 1$ and $\alpha(x) = x^2 - 1$.

Two points 0 and -1 are α-fixed points but not fixed points.

Example 2.15: Let $X = [0,1]$.Define map $T, \alpha: X \to X$ be defined by $T(x) = \frac{x}{2}$ and $\alpha(x) = x^2$

$(\alpha oT)(x) = \alpha[T(x)] = \alpha\left(\frac{x}{2}\right) = \frac{x^2}{4}$.

0 is the α-fixed point of T which is a fixed point of T.

Example 2.16: Let $T, \alpha: R \to R$ be defined by $T(x) = \sqrt{x}$ and $\alpha(x) = \frac{x^2}{2}$. 0 is the α-fixed point of T which is a fixed point of T.

Definition 2.17: A pair of self maps f and g of a metric space X is said to be α-commuting if (αof) and (αog) are commuting.

i.e. $\big((\alpha of)o(\alpha og)\big)x = \big((\alpha og)o(\alpha of)\big)x$ for all $x \in X$.

The following examples show the relation between commuting and α-commuting maps.

Example 2.18: Let $f, g, \alpha: R \to R$ be defined by $f(x) = x^2$, $g(x) = \sqrt{x}$, and $\alpha(x) = 2x$ for all $x \in R$.

$$fg(x) = f\big(\sqrt{x}\big) = x \ , \ gf(x) = g(x^2) = x \text{ .Therefore } fg(x) = gf(x).$$

Hence, f and g are commuting maps.

$$\text{Also for } x \in R, \quad (\alpha of)(x) = \alpha(x^2) = 2x^2 \ , \ (\alpha og)(x) = \alpha\big(\sqrt{x}\big) = 2\sqrt{x}$$

$$\big((\alpha of)o(\alpha og)\big)(x) = (\alpha of)\big(2\sqrt{x}\big) = (\alpha of)\big(2\sqrt{x}\big) = 2(2\sqrt{x})^2 = 8x$$

$$\big((\alpha og)o(\alpha of)\big)(x) = (\alpha og)(2x^2) = 2\sqrt{2x}$$

Therefore, $\big((\alpha of)o(\alpha og)\big)(x) \neq \big((\alpha og)o(\alpha of)\big)(x)$.

Hence, f and g are α-commuting maps.

Example 2.19: Let $f, g, \alpha: (R - \{0\}) \to (R - \{0\})$ be defined by $f(x) = x^2$, $g(x) = x^3$ and $\alpha(x) = 1/x$ for all $x \in R$.

$fg(x) = f(x^3) = x^6$, $gf(x) = g(x^2) = x^6$. Therefore, $fg(x) = gf(x)$.

Hence, f and g are commuting maps.

Also for $x \in R - \{0\}$, $(\alpha of)(x) = \alpha(x^2) = \frac{1}{x^2}$, $(\alpha og)(x) = \alpha(x^3) = \frac{1}{x^3}$.

$$\big((\alpha of)o(\alpha og)\big)(x) = (\alpha of)\left(\frac{1}{x^3}\right) = \frac{1}{(\frac{1}{x^3})^2} = x^6$$

$$\big((\alpha og)o(\alpha of)\big)(x) = (\alpha og)\left(\frac{1}{x^2}\right) = \frac{1}{(\frac{1}{x^2})^3} = x^6.$$

Therefore, $\big((\alpha of)o(\alpha og)\big)(x) = \big((\alpha og)o(\alpha of)\big)(x)$.

Hence, f and g are α-commuting maps.

Example 2.20: Let $f, g, \alpha: R \to R$ be defined by $f(x) = x^2$, $g(x) = 2x$ and $\alpha(x) = x/2$ for all $x \in R$.

$fg(x) = f(2x) = 4x^2$, $gf(x) = g(x^2) = 2x^2$. Therefore $fg(x) \neq gf(x)$.

Hence, f and g are not commuting maps.

Also for $x \in R$, $(\alpha of)(x) = \alpha(x^2) = x^2/2$, $(\alpha og)(x) = \alpha(2x) = \frac{2x}{2} = x$

$$\big((\alpha of)o(\alpha og)\big)(x) = (\alpha of)(x) = x^2/2$$

$$\big((\alpha og)o(\alpha of)\big)(x) = (\alpha og)(x) = x^2/2.$$

Therefore, $\big((\alpha of)o(\alpha og)\big)(x) = \big((\alpha og)o(\alpha of)\big)(x)$.

Hence, f and g are α-commuting maps.

Example 2.21: Let $f, g, \alpha: (R - \{0\}) \to (R - \{0\})$ be defined by $f(x) = x^2$, $g(x) = 2x$ and $\alpha(x) = 1/x$ for all $x \in R$.

$fg(x) = f(2x) = 4x^2$, $gf(x) = g(x^2) = 2x^2$. Therefore, $fg(x) \neq gf(x)$.

Hence, f and g are not commuting maps.

Also for $x \in R - \{0\}$, $(\alpha o f)(x) = \alpha(x^2) = \frac{1}{x^2}$, $(\alpha o g)(x) = \alpha(2x) = \frac{1}{2x}$.

$$\big((\alpha o f)o(\alpha o g)\big)(x) = (\alpha o f)\left(\frac{1}{2x}\right) = \frac{1}{(\frac{1}{2x})^2} = 4x^2$$

$$\big((\alpha o g)o(\alpha o f)\big)(x) = (\alpha o g)\left(\frac{1}{x^2}\right) = \frac{1}{2(\frac{1}{x^2})} = \frac{x^2}{2}.$$

Therefore, $\big((\alpha o f)o(\alpha o g)\big)(x) \neq \big((\alpha o g)o(\alpha o f)\big)(x)$.

Hence, f and g are not α-commuting maps also.

Definition 2.22: A pair of self maps f and g of a metric space (X, d) is called weakly α-commuting maps if $(\alpha o f)$ and $(\alpha o g)$ are weakly commuting maps.

i.e. $d\big(((\alpha o f)o(\alpha o g))(x), ((\alpha o g)o(\alpha o f))(x)\ \big) \leq d\big((\alpha o f)(x), (\alpha o g)(x)\big)$ for all $x \in X$.

Definition 2.23: The self maps f, g : $(X, d) \rightarrow (X, d)$ are called α-compatible maps if $(\alpha o f)$ and $(\alpha o g)$ are compatible if whenever $\{x_n\}$ is a sequence in X such that $(\alpha o f)(x_n), (\alpha o g)(x_n) \rightarrow t \in X$, then $d\big(((\alpha o f)o(\alpha o g))(x_n), ((\alpha o g)o(\alpha o f))(x_n)\ \big) \rightarrow 0$.

Definition 2.24: Two self maps f and g are called weakly α-compatible if $(\alpha o f)$ and $(\alpha o g)$ are weakly compatible, i.e $(\alpha o f)$ and $(\alpha o g)$ commute at coincidence points.

Remark 2.25: It is clear that $\boldsymbol{\alpha}$-commuting maps are weakly $\boldsymbol{\alpha}$-commuting maps, weakly $\boldsymbol{\alpha}$-commuting maps are α-compatible maps and α-compatible maps are weakly α-compatible maps. But the converse is not true in any case. These facts are elaborated in the following example:

Example 2.26: Let $f, g, \alpha: (R - \{0\}) \rightarrow (R - \{0\})$ be defined by $f(x) = x^3$, $g(x) = x^2$, and $\alpha(x) = 1/x$ for all $x \in R$.

Also for $x \in R - \{0\}$, $(\alpha o f)(x) = \alpha(x^3) = \frac{1}{x^3}$, $(\alpha o g)(x) = \alpha(x^2) = \frac{1}{x^2}$.

$$\big((\alpha o f)o(\alpha o g)\big)(x) = (\alpha o f)\left(\frac{1}{x^2}\right) = \frac{1}{(\frac{1}{x^2})^3} = x^6$$

$$\big((\alpha o g)o(\alpha o f)\big)(x) = (\alpha o g)\left(\frac{1}{x^3}\right) = \frac{1}{\left(\frac{1}{x^3}\right)^2} = x^6.$$

Therefore, $\big((\alpha of)o(\alpha og)\big)(x) = \big((\alpha og)o(\alpha of)\big)(x)$.

Hence, f and g are α-commuting maps. Also for $x \in R - \{0\}$

$$d\big(((\alpha of)o(\alpha og)\big)(x), \big((\alpha og)o(\alpha of)\big)(x)\ ) = |x^6 - x^6| = 0$$

$$d\big((\alpha of)(x), (\alpha og)(x)\big) = \left|\frac{1}{x^3} - \frac{1}{x^2}\right| = \left|\frac{1-x}{x^3}\right|$$

$$d\big(((\alpha of)o(\alpha og)\big)(x), \big((\alpha og)o(\alpha of)\big)(x)\ ) = 0$$

$$\leq \left|\frac{1-x}{x^3}\right|$$

$$= d\big((\alpha of)(x), (\alpha og)(x)\big)$$

Hence, f and g are weakly α-commuting maps.

$$(\alpha of)(x_n) = \frac{1}{{x_n}^3}\ ,\ (\alpha og)(x_n) = \frac{1}{{x_n}^2}.$$

$$d\big((\alpha of)(x_n), (\alpha og)(x_n)\big) = \left|\frac{1}{{x_n}^3} - \frac{1}{{x_n}^2}\right| \to 0 \text{ as } x_n \to 1.$$

Hence, f and g are α-compatible maps.

Definition 2.27: The mapping $T:(X,d) \to (X,d)$ is said to be α-continuous if (αoT) is continuous. In other words for every $\varepsilon \geq 0, \forall \delta > 0$ such that

$$d(x,y) \leq \delta \Rightarrow d((\alpha oT)x, (\alpha oT)y) \leq \varepsilon.$$

3. MAIN RESULTS

In the present section, we prove four common α-fixed point theorems. Out of which the first two common α-fixed point theorems (3.1 and 3.2) have been proved by taking a new contraction inequality and the rest two common α-fixed point theorems generalized the theorem of Singh and Chouhan [17]. An example has also been given in support of our theorem.

Theorem 3.1. Let α, S and T be self-mappings of a complete metric space (X,d) satisfying:

(I) $\quad d\big((\alpha oS)x, (\alpha oT)y\big) \leq a\frac{[d(x,(\alpha oS)x]^2 + [d(x,(\alpha oT)y]^2}{d(x,(\alpha oS)x) + d(x,(\alpha oT)y)} + bd(x,y)$, where $d(x,(\alpha oS)x) + d(x,(\alpha oT)y) \neq 0$ for all x, y in X and $0 \leq 3a + b, 0 \leq a, b$. Then S and T have a unique common α-fixed point.

Proof. Let x_0 be any arbitrary point in X. We define a sequence $\{x_n\}$ in X such that

$x_n = (\alpha oS)x_{n-1}$ and $x_{n+1} = (\alpha oT)x_n$ for $n = 1,2,3, \ldots$

Then $d(x_n, x_{n+1}) = d((\alpha oS)x_{n-1}, (\alpha oT)x_n)$

$$\leq a\frac{[d(x_{n-1},(\alpha oS)x_{n-1}]^2 + [d(x_{n-1},(\alpha oT)x_n]^2}{d(x_{n-1},(\alpha oS)x_{n-1}) + d(x_{n-1},(\alpha oT)x_n)} + bd(x_{n-1},x_n)$$
$$= a\frac{[d(x_{n-1},x_n)]^2 + [d(x_{n-1},x_{n+1})]^2}{d(x_{n-1},x_n) + d(x_{n-1},x_{n+1})} + bd(x_{n-1},x_n)$$
$$\leq a\frac{[d(x_{n-1},x_n) + d(x_{n-1},x_{n+1})]^2}{d(x_{n-1},x_n) + d(x_{n-1},x_{n+1})} + bd(x_{n-1},x_n)$$
$$\leq a[d(x_{n-1},x_n) + d(x_{n-1},x_{n+1})] + bd(x_{n-1},x_n)$$
$$\leq ad(x_{n-1},x_n) + ad(x_{n-1},x_{n+1}) + bd(x_{n-1},x_n)$$
$$\leq (a+b)d(x_{n-1},x_n) + a[d(x_{n-1},x_n) + d(x_n,x_{n+1})]$$
$$d(x_n,x_{n+1}) \leq \frac{2a+b}{1-a}d(x_{n-1},x_n)$$
$$d(x_n,x_{n+1}) \leq \beta d(x_{n-1},x_n) \text{ , where } \beta = \frac{2a+b}{1-a} < 1.$$

Similarly, $d(x_{n-1}, x_n) \leq \beta d(x_{n-2}, x_{n-1})$ and so on.

Hence, $d(x_n, x_{n+1}) \leq \beta^n d(x_0, x_1) \to 0$ as $n \to \infty$, since $0 < \beta < 1$.

This proves that $\{x_n\}$ is a Cauchy sequence in X which is complete so it converges to a point z in X.

Now, $d(z, (\alpha oT)z) \leq d(z, x_n) + d(x_n, (\alpha oT)z)$

$$\leq d(z,x_n) + d((\alpha oS)x_{n-1}, (\alpha oT)z)$$
$$\leq d(z,x_n) + a\frac{[d(x_{n-1},(\alpha oS)x_{n-1})]^2 + [d(x_{n-1},(\alpha oT)z)]^2}{d(x_{n-1},(\alpha oS)x_{n-1}) + d(x_{n-1},(\alpha oT)z)} + bd(x_{n-1},z)$$
$$\leq d(z,x_n) + a\frac{[d(x_{n-1},(\alpha oS)x_{n-1}) + d(x_{n-1},(\alpha oT)z)]^2}{d(x_{n-1},(\alpha oS)x_{n-1}) + d(x_{n-1},(\alpha oT)z)} + bd(x_{n-1},z)$$
$$= d(z,x_n) + a[d(x_{n-1},(\alpha oS)x_{n-1}) + d(x_{n-1},(\alpha oT)z)] + bd(x_{n-1},z)$$
$$= d(z,x_n) + ad(x_{n-1},x_n) + ad(x_{n-1},(\alpha oT)z)] + bd(x_{n-1},z)$$

Taking limit as $n \to \infty$,

$$d(z,(\alpha oT)z) \leq ad(z,(\alpha oT)z)$$
$$(1-a)d(z,(\alpha oT)z) \leq 0 \text{ as } n \to \infty$$

Which is a contradiction.Hence, $d(z, (\alpha oT)z) = 0$.

This implies that $(\alpha oT)z = z$ i.e., z is the α-fixed point of T.

Now, $d(z,(\alpha oS)z) \leq d(z,x_n) + d(x_n,(\alpha oS)z)$

$$\leq d(z,x_n) + d((\alpha oS)z,(\alpha oT)x_{n-1})$$

$$\leq d(z,x_n) + a\frac{[d(z,(\alpha oS)z)]^2 + [d(z,(\alpha oT)x_{n-1})]^2}{d(z,(\alpha oS)z) + d(z,(\alpha oT)x_{n-1})} + bd(z,x_{n-1})$$

$$\leq d(z,x_n) + a\frac{[d(z,(\alpha oS)z) + d(z,(\alpha oT)x_{n-1})]^2}{d(z,(\alpha oS)z) + d(z,(\alpha oT)x_{n-1})} + bd(z,x_{n-1})$$

$$= d(z,x_n) + a[d(z,(\alpha oS)z) + d(z,(\alpha oT)x_{n-1})] + bd(z,x_{n-1})$$

$$= d(z,x_n) + ad(z,x_n) + ad(z,(\alpha oS)z)] + bd(z,x_{n-1})$$

Taking limit as $n \to \infty$,

$$d(z,(\alpha oS)z) \leq ad(z,(\alpha oS)z)$$

Which is a contradiction. Hence, $d(z,(\alpha oS)z) = 0$.

This implies that $(\alpha oS)z = z$ i.e., z is the α-fixed point of S.

Consequently, z is the common α-fixed point of S and T.

Now for the uniqueness of z, let z_1 be another common α-fixed point of S and T.

Then $d(z,z_1) = d((\alpha oS)z,(\alpha oT)z_1)$

$$\leq a\frac{[d(z,(\alpha oS)z)]^2 + [d(z,(\alpha oT)z_1)]^2}{d(z,(\alpha oS)z) + d(z,(\alpha oT)z_1)} + bd(z,z_1)$$

$$\leq a\frac{[d(z,(\alpha oS)z) + d(z,(\alpha oT)z_1)]^2}{d(z,(\alpha oS)z) + d(z,(\alpha oT)z_1)} + bd(z,z_1)$$

$$= a[d(z,(\alpha oS)z) + d(z,(\alpha oT)z_1)] + bd(z,z_1)$$

$$= ad(z,z) + ad(z,z_1)] + bd(z,z_1)$$

$$(1-a-b)d(z,z_1) \leq 0$$

This implies that $d(z,z_1) \leq 0$

Which is a contradiction, so $d(z,z_1) = 0 \Leftrightarrow z = z_1$.

This completes the proof.

Theorem 3.2: Let α, S and T be self-mappings of a complete metric space (X,d) satisfying:

(I) $d\big((\alpha oS)x,(\alpha oT)y\big) \leq a\frac{[d(x,(\alpha oS)x]^2+[d(y,(\alpha oT)y]^2}{d(x,(\alpha oS)x)+d(x,(\alpha oT)y)+bd(x,y)}$, where $d(x,(\alpha oS)x) + d(x,(\alpha oT)y) \neq 0$ for all x,y in X and $2a < b, 0 \leq a,b < 1$. Then S and T have a unique common α-fixed point.

Proof. Let x_0 be any arbitrary point in X. We define a sequence $\{x_n\}$ in X such that

$x_n = (\alpha oS)x_{n-1}$ and $x_{n+1} = (\alpha oT)x_n$ for $n = 1,2,3,...$

Then $d(x_n, x_{n+1}) = d((\alpha oS)x_{n-1}, (\alpha oT)x_n)$

$$\leq a\frac{[d(x_{n-1},(\alpha oS)x_{n-1}]^2 + [d(x_n,(\alpha oT)x_n]^2}{d(x_{n-1},(\alpha oS)x_{n-1}) + d(x_{n-1},(\alpha oT)x_n + bd(x_{n-1},x_n)}$$

$$= a\frac{[d(x_{n-1},x_n)]^2 + [d(x_n,x_{n+1})]^2}{d(x_{n-1},x_n) + d(x_{n-1},x_{n+1}) + bd(x_{n-1},x_n)}$$

$$\leq a\frac{[d(x_{n-1},x_n) + d(x_n,x_{n+1})]^2}{d(x_{n-1},x_n) + d(x_{n-1},x_{n+1}) + bd(x_{n-1},x_n)}$$

$$\leq a\frac{[d(x_{n-1},x_n) + d(x_n,x_{n+1})]^2}{d(x_n,x_{n+1}) + bd(x_{n-1},x_n)}$$

$$\leq (\frac{a}{b})\frac{[d(x_{n-1},x_n) + d(x_n,x_{n+1})]^2}{d(x_{n-1},x_{n+1}) + \frac{1}{b}\ d(x_n,x_{n+1})}$$

$$\leq (\frac{a}{b})\frac{[d(x_{n-1},x_n) + d(x_n,x_{n+1})]^2}{d(x_{n-1},x_n) +\ d(x_n,x_{n+1})}$$

$$\leq (\frac{a}{b})[d(x_{n-1},x_n) + d(x_n,x_{n+1})]$$

$$(1-\frac{a}{b})d(x_n,x_{n+1}) \leq (\frac{a}{b})d(x_{n-1},x_n)$$

$$d(x_n,x_{n+1}) \leq (\frac{a}{b-a})d(x_{n-1},x_n)$$

$$d(x_n,x_{n+1}) \leq \beta d(x_{n-1},x_n) \quad \text{where } \beta = (\tfrac{a}{b-a}) < 1.$$

Similarly $d(x_{n-1},x_n) \leq \beta d(x_{n-2},x_{n-1})$ and so on.

Hence, $d(x_n,x_{n+1}) \leq \beta^n d(x_0,x_1) \to 0$ as $n \to \infty$, since $0 < \beta < 1$.

This proves that $\{x_n\}$ is a Cauchy sequence in X which is complete so it converges to a point z in X.

Now, $d(z,(\alpha oT)z) \leq d(z,x_n) + d(x_n,(\alpha oT)z)$

$$\leq d(z,x_n) + d((\alpha oS)x_{n-1},(\alpha oT)z)$$

$$\leq d(z,x_n) + a\frac{[d(x_{n-1},(\alpha oS)x_{n-1})]^2 + [d(z,(\alpha oT)z)]^2}{d(x_{n-1},(\alpha oS)x_{n-1}) + d(x_{n-1},(\alpha oT)z) + bd(x_{n-1},z)}$$

$$\leq d(z, x_n) + a\frac{[d(x_{n-1}, x_n)]^2 + [d(z, (\alpha oT)z)]^2}{d(x_{n-1}, x_n) + d(x_{n-1}, (\alpha oT)z) + bd(x_{n-1}, z)}$$

$$\to a\frac{[d(z,(\alpha oT)z)]^2}{d(z,(\alpha oT)z)} \text{ as } n \to \infty$$

$$\text{i.e. } d(z, (\alpha oT)z) \leq ad(z, (\alpha oT)z)$$

$$(1 - a)d(z, (\alpha oT)z) \leq 0$$

$$d(z, (\alpha oT)z) \leq 0$$

Which is a contradiction so $d(z, (\alpha oT)z) = 0$.

This implies that $(\alpha oT)z = z$ i.e., z is the α-fixed point of T.

Now, $d(z, (\alpha oS)z) \leq d(z, x_{n+1}) + d(x_{n+1}, (\alpha oS)z)$

$$\leq d(z, x_{n+1}) + d((\alpha oS)z, (\alpha oT)x_n)$$

$$\leq d(z, x_{n+1}) + a\frac{[d(z, (\alpha oS)z)]^2 + [d(x_n, (\alpha oT)x_n)]^2}{d(z, (\alpha oS)z) + d(z, (\alpha oT)x_n) + bd(z, x_n)}$$

$$\leq d(z, x_{n+1}) + a\frac{[d(x_n, x_{n+1})]^2 + [d(z, (\alpha oS)z)]^2}{d(z, x_{n+1}) + d(z, (\alpha oS)z) + bd(z, x_n)}$$

$$\to ad(z, (\alpha oS)z) \text{ as } n \to \infty$$

$$(1 - a)d(z, (\alpha oS)z) \leq 0$$

Which is a contradiction so $d(z, (\alpha oS)z) = 0$.

This implies that $(\alpha oS)z = z$ i.e., z is the α-fixed point of S.

Consequently, z is the common α-fixed point of S and T.

Now for the uniqueness of z, let z_1 be another common α-fixed point of S and T.

Then $d(z, z_1) = d((\alpha oS)z, (\alpha oT)z_1)$

$$\leq a\frac{[d(z, (\alpha oS)z)]^2 + [d(z, (\alpha oT)z_1)]^2}{d(z, (\alpha oS)z) + d(z, (\alpha oT)z_1) + bd(z, z_1)}$$

$$\leq a\frac{[d(z, z)]^2 + d(z_1, z_1)]^2}{d(z, z) + d(z, z_1) + bd(z, z_1)}$$

$$d(z, z_1) \leq 0$$

Which is a contradiction, so $d(z, z_1) = 0$ .This implies that $z = z_1$.

This completes the proof.

Lemma l_1([17]): Let A and B be compatible maps from a metric space (X,d) into itself such that $lim_{n\to\infty}(A)x_n = lim_{n\to\infty}(B)x_n = t$, for some $t \in X$.

Then $lim_{n\to\infty}(BA)x_n = (A)t = t$, if A is continuous.

Lemma l_2 ([17]): Let A, B, S, and T be a mapping from a metric space (X,d) into itself satisfying the following conditions:

(1) $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$

(2) $[d(Ax,By)]^2 \leq k_1[d(Ax,Sx)d(By,Ty) + d(By,Sx)d(Ax,Ty)] + k_2[d(Ax,Sx)d(Ax,Ty) + d(By,Ty)d(By,Sx)]$

Where $0 \leq k_1 + 2k_2 < 1; k_1, k_2 \geq 0$.

(3) Let $x_0 \in X$ then by (1) there exists $x_1 \in X$ such that $Tx_1 = Ax_0$ and for x_1 there exists $x_2 \in X$ such that $Sx_2 = Bx_1$ and so on .Continuing this process we can define a sequence $\{y_n\}$ in X such that $y_{2n+1} = Tx_{2n+1} = Ax_{2n}$ and $y_{2n} = Sx_{2n} = Bx_{2n-1}$, then the sequence $\{y_n\}$ is Cauchy sequence in X.

By applying the concept of lemmas l_1 and l_2 Singh and Chouhan [17] proved two theorems (A and B) under as follows:

Theorem –A ([17]): Let A, B, S, and T be self maps of a complete metric space (X,d) satisfying conditions (1),(2) and (3) of lemma (l_2) and

(4) One of A, B, S, or T is continuous.

(5) (A,S) and (B,T) are compatible on X.

Then A, B, S, and T have a unique common fixed point.

Theorem –B ([17]): Let A , B, S, and T be self -maps of a complete metric space (X,d) with: $AS = SA$; $TB = BT$, satisfying the condition (4) of theorem (A) and there exists positive integers a, b, s and t such that

(6) $A^a(X) \subseteq T^t(X)$ and $B^b(X) \subseteq S^s(X)$

(7) $[d(A^a x, B^b y)]^2 \leq k_1[d(A^a x, S^s x)d(B^b y, T^t y) + d(B^b y, S^s x)d(A^a x, T^t y)] +$

$$k_2[d(A^a x, S^s x)d(A^a x, T^t y) + d(B^b y, T^t y)d(B^b y, S^s x)]$$

For all x, y in X; Where $0 \leq k_1, k_2 < 1$.

Then A, B, S, and T have a unique common fixed point in X.

Now under this section, we generalize above lemmas (l_1 & l_2) and theorems (A & B).

We generalize lemma l_1 under as follows:

Lemma 3.3: Let A and B be α-compatible maps from a metric space (X, d) into itself such that $lim_{n\to\infty}(\alpha oA)x_n = lim_{n\to\infty}(\alpha oB)x_n = t$, for some $t \in X$.

Then $lim_{n\to\infty}((\alpha oB)o(\alpha oA))x_n = (\alpha oA)t = t$, if A is α-continuous.

Proof. $$d\big(((\alpha oB)o(\alpha oA))x_n, (\alpha oA)t\big) \leq d\big(((\alpha oB)o(\alpha oA))x_n, ((\alpha oA)o(\alpha oB))x_n\big) + d\big(((\alpha oA)o(\alpha oB))x_n, (\alpha oA)t\big)$$

Letting $n \to \infty$, we have $lim_{n\to\infty}((\alpha oB)o(\alpha oA))x_n = (\alpha oA)t$.

Here, we generalize lemma l_2 under as follows:

Lemma 3.4: Let α, A, B, S, and T be a mapping from a metric space (X, d) into itself satisfying the following conditions:

(1) $(\alpha oA)(X) \subseteq (\alpha oT)(X)$ and $(\alpha oB)(X) \subseteq (\alpha oS)(X)$

(2) $$\big[d\big((\alpha oA)x, (\alpha oB)y\big)\big]^2 \leq k_1\big[d\big((\alpha oA)x, (\alpha oS)x\big)d\big((\alpha oB)y, (\alpha oT)y\big) + d\big((\alpha oB)y, (\alpha oS)x\big)d\big((\alpha oA)x, (\alpha oT)y\big)\big] + k_2\big[d\big((\alpha oA)x, (\alpha oS)x\big)d\big((\alpha oA)x, (\alpha oT)y\big) + d\big((\alpha oB)y, (\alpha oT)y\big)d\big((\alpha oB)y, (\alpha oS)x\big)\big]$$

where $0 \leq k_1 + 2k_2 < 1; k_1, k_2 \geq 0$.

(3) Let $x_0 \in X$ then by (1) there exists $x_1 \in X$ such that $(\alpha oT)x_1 = (\alpha oA)x_0$ and for x_1 there exists $x_2 \in X$ such that $(\alpha oS)x_2 = (\alpha oB)x_1$ and so on .Continuing this process we can define a sequence $\{y_n\}$ in X such that $y_{2n+1} = (\alpha oT)x_{2n+1} = (\alpha oA)x_{2n}$ and $y_{2n} = (\alpha oS)x_{2n} = (\alpha oB)x_{2n-1}$, then the sequence $\{y_n\}$ is Cauchy sequence in X.

Proof. By condition (2) and (3), we have

$$[d(y_{2n+1},y_{2n})]^2=\left[d\big((\alpha oA)x_{2n},(\alpha oB)x_{2n-1}\big)\right]^2$$
$$\leq k_1\left[d\big((\alpha oA)x_{2n},(\alpha oS)x_{2n}\big)d\big((\alpha oB)x_{2n-1},(\alpha oT)x_{2n-1}\big)\right.$$
$$\left.+\,d\big((\alpha oB)x_{2n-1},(\alpha oS)x_{2n}\big)d\big((\alpha oA)x_{2n},(\alpha oT)x_{2n-1}\big)\right]$$
$$+\,k_2\left[d\big((\alpha oA)x_{2n},(\alpha oS)x_{2n}\big)d\big((\alpha oA)x_{2n},(\alpha oT)x_{2n-1}\big)\right.$$
$$\left.+\,d\big((\alpha oB)x_{2n-1},(\alpha oT)x_{2n-1}\big)d\big((\alpha oB)x_{2n-1},(\alpha oS)x_{2n}\big)\right]$$
$$\leq k_1[d(y_{2n+1},y_{2n})d(y_{2n},y_{2n-1})]+k_2[d(y_{2n+1},y_{2n})d(y_{2n+1},y_{2n-1})]$$
$$[d(y_{2n+1},y_{2n})]^2\leq[k_1d(y_{2n},y_{2n-1})+k_2d(y_{2n+1},y_{2n-1})]d(y_{2n+1},y_{2n})]$$
$$\text{i.e., } d(y_{2n+1},y_{2n})\leq k_1d(y_{2n},y_{2n-1})+k_2d(y_{2n+1},y_{2n-1})$$
$$\leq k_1d(y_{2n},y_{2n-1})+k_2[d(y_{2n+1},y_{2n})+d(y_{2n},y_{2n-1})]$$
$$d(y_{2n+1},y_{2n})-k_2d(y_{2n+1},y_{2n})\leq(k_1+k_2)d(y_{2n},y_{2n-1})$$
$$d(y_{2n+1},y_{2n})\leq\left(\frac{k_1+k_2}{1-k_2}\right)d(y_{2n},y_{2n-1})=p\,d(y_{2n},y_{2n-1})$$
$$d(y_{2n+1},y_{2n})\leq p\,d(y_{2n},y_{2n-1})\quad\text{where } p=\left(\frac{k_1+k_2}{1-k_2}\right)<1.$$

Similarly, $d(y_{2n},y_{2n-1})\leq p\,d(y_{2n-1},y_{2n-2})$ and so on.

Hence, $d(y_{2n+1},y_{2n})\leq p^n\,d(y_1,y_0)\to 0$ as $n\to\infty$ since $0<p<1$.

Hence $\{y_n\}$ is a Cauchy sequence.

Now by applying the concept of lemmas 3.3 and 3.4 we give our main results as theorems 3.5 and 3.7.

We generalize Theorem-A under as follows:

Theorem 3.5: Let α, A, B, S, and T be self maps of a complete metric space (X,d) satisfying conditions (1),(2) and (3) of lemma (3.4) and

(4) One of A,B,S, or T is α-continuous.

(5) (A,S) and (B,T) are α-compatible on X.

Then A, B, S, and T have a unique common α-fixed point.

Proof. By lemma 3.4, $\{y_n\}$ is Cauchy sequence, and since X is complete so there exists a point $z\in X$ such that $\lim y_n=z$ as $n\to\infty$. Consequently, the subsequences

$(\alpha oA)x_{2n}, (\alpha oS)x_{2n}, (\alpha oB)x_{2n-1}$ and $(\alpha oT)x_{2n+1}$ also converge to z.

Suppose S is α-continuous then by α-compatibility of (A, S) and by Lemma 3.3, we have $((\alpha oS)o(\alpha oS))x_{2n} \to (\alpha oS)z$ and $((\alpha oA)o(\alpha oS))x_{2n} \to (\alpha oS)z$ as $n \to \infty$.

Now by condition (2) of lemma 3.4, we have

$$[d(((\alpha oA)o(\alpha oS))x_{2n}, (\alpha oB)x_{2n-1})]^2$$

$$\leq k_1\big[d(((\alpha oA)o(\alpha oS))x_{2n}), \big((\alpha oS)o(\alpha oS))x_{2n}\big)d\big((\alpha oB)x_{2n-1}, (\alpha oT)x_{2n-1}\big)$$

$$+ d\big((\alpha oB)x_{2n-1}, ((\alpha oS)o(\alpha oS))x_{2n}\big)d(((\alpha oA)o(\alpha oS))x_{2n}, (\alpha oT))x_{2n-1})\big]$$

$$+ k_2\left[d\big(((\alpha oA)o(\alpha oS)\big)x_{2n}, ((\alpha oS)o(\alpha oS))x_{2n})d\big(((\alpha oA)o(\alpha oS)\big)x_{2n}, (\alpha oT))x_{2n-1}\right.$$

$$\left.+ d\big((\alpha oB)x_{2n-1}, (\alpha oT)x_{2n-1}\big)d\left((\alpha oB)x_{2n-1}, \big((\alpha oS)\big)o(\alpha oS)x_{2n}\right)\right].$$

Letting $n \to \infty$, we have

$$[d((\alpha oS)z, z)]^2$$

$$\leq k_1\big[d((\alpha oS))z, (\alpha oS)z)d(z, z) + d(z, (\alpha oS)z)d\big((\alpha oS)z, z\big)\big]$$

$$+ k_2\big[d\big((\alpha oS)z, (\alpha oS)z\big)d((\alpha oS)z, z) + d(z, z)d(z, (\alpha oS)z)\big]$$

$$\text{i.e., } [d((\alpha oS)z, z)]^2 \leq k_1\big[d\big((\alpha oS)z, z\big)\big]^2$$

Which is a contradiction. Hence, $d\big((\alpha oS)z, z\big) = 0$. So z is α-fixed point of S.

Now, $$\big[d\big((\alpha oA), (\alpha oB)x_{2n-1}\big)\big]^2 \leq k_1\big[d\big((\alpha oA)z, (\alpha oS)z\big)d\big((\alpha oB)x_{2n-1}, (\alpha oT)x_{2n-1}\big) +$$

$$d\big((\alpha oB)x_{2n-1}, (\alpha oS)z\big)d\big((\alpha oA)z, (\alpha oT)x_{2n-1}\big)\big] +$$

$$k_2\big[d\big((\alpha oA)z, (\alpha oS)z\big)d\big((\alpha oA)z, (\alpha oT)x_{2n-1}\big) +$$

$$d\big((\alpha oB)x_{2n-1}, (\alpha oT)x_{2n-1}\big)d\big((\alpha oB)x_{2n-1}, (\alpha oS)z\big)\big]$$

Letting $n \to \infty$, we have

$$[d((\alpha oA)z, z)]^2 \leq k_2\, d\big((\alpha oA)z, z\big)d\big((\alpha oA)z, z\big)$$

$$[d((\alpha oA)z, z)]^2 \leq k_1\big[d\big((\alpha oA)z, z\big)\big]^2.$$

Hence $(\alpha oA)z = z$.

Now since $(\alpha oA)z = z$, by condition (1) $z \in (\alpha oT)(X)$ so there exists a point $u \in X$ such that $z = (\alpha oA)z = (\alpha oT)u$.

Moreover by condition (2) we obtain

$$[d(z,(\alpha oB)u)]^2 = [d((\alpha oA)z,(\alpha oB)u)]^2$$
$$\leq k_1[d((\alpha oA))z,(\alpha oS)z)d((\alpha oB)u,(\alpha oT)u)$$
$$+ d((\alpha oB)u,(\alpha oS)z)d((\alpha oA)z,(\alpha oT)u)]$$
$$+ k_2[d((\alpha oA)z,(\alpha oS)z)d((\alpha oA)z,(\alpha oT)u)$$
$$+ d((\alpha oB)u,(\alpha oT)u)d((\alpha oB)u,(\alpha oS)z)]$$

$$\text{i.e. } [d(z,(\alpha oB)u)]^2 \leq k_2\left[d\big((\alpha oB)u,z\big)\right]^2.$$

Hence $(\alpha oB)u = z$ i.e., $z = (\alpha oT)u = (\alpha oB)u$.

By condition (5), we have

$$d((\alpha oT)o(\alpha oB))u,((\alpha oB)o(\alpha oT))u) = 0.$$

Hence $d\big((\alpha oT)z,(\alpha oB)z\big) = 0$ yields $(\alpha oT)z = (\alpha oB)z$.

Now using (2), we have

$$[d(z,(\alpha oT)z)]^2 = \left[d\big((\alpha oA)\big)z,\big((\alpha oB)z\big)\right]^2$$
$$\leq k_1\left[d((\alpha oA))z,(\alpha oS)z)d\big((\alpha oB)z,(\alpha oT)z\big) + d((\alpha oB)z,(\alpha oS)z)d((\alpha oA)z,(\alpha oT)z)\right]$$
$$+ k_2[d((\alpha oA)z,(\alpha oS)z)d((\alpha oA)z,(\alpha oT)z)$$
$$+ d((\alpha oB)z,(\alpha oT)z)d((\alpha oB)z,(\alpha oS)z)]$$
$$\text{i.e. } [d(z,(\alpha oT)z)]^2 \leq k_1[d(z,(\alpha oT)z)]^2.$$

Which is a contradiction. Hence $z = (\alpha oT)z$ yields $z = (\alpha oB)z = (\alpha oT)z$.

Therefore z is a common α-fixed point of A, B, S, and T.

Similarly, we can prove this when any one of A, B, or T is α-continuous.

Finally in order to prove the uniqueness of z, suppose ω be another common α-fixed point of A, B, S and T. Then we have

$$[d(z,\omega)]^2 = [d((\alpha oA)z,(\alpha oB)\omega)]^2$$
$$\leq k_1[d((\alpha oA))z,(\alpha oS)z)d((\alpha oB)\omega,(\alpha oT)\omega)$$
$$+ d((\alpha oB)\omega,(\alpha oS)z)d((\alpha oA)z,(\alpha oT)\omega)]$$
$$+ k_2\big[d((\alpha oA)z,(\alpha oS)z)d((\alpha oA)z,(\alpha oT)\omega)$$
$$+ d((\alpha oB)\omega,(\alpha oT)\omega)d\big((\alpha oB)\omega,(\alpha oS)z\big)\big]$$
$$\leq k_1[d(z,z)d(\omega,\omega)+d(\omega,z)d(z,\omega)]+k_2[d(z,z)d(z,\omega)+d(\omega,\omega)d(\omega,z)]$$

i.e., $[d(z,\omega)]^2 \leq k_1[d(z,\omega)]^2$. Hence $d(z,\omega)=0 \Leftrightarrow z=\omega$

This completes the proof.

Example 3.6: Let $X=[0,1]$, $d(x,y)=|x-y|$.Define maps $\alpha,A,B,S,T\colon X\to X$ such that $\alpha(x)=x^2, A(x)=0, T(x)=\sqrt{\frac{x}{2}}, B(x)=\sqrt{\frac{x}{8}}, S(x)=\sqrt{x}$,

$$(\alpha oA)(x)=0, (\alpha oT)(x)=\frac{x}{2}, (\alpha oB)(x)=\frac{x}{8}, (\alpha oS)(x)=x.$$

Here, $(\alpha oA)(X)\subseteq(\alpha oT)(X), (\alpha oB)(X)\subseteq(\alpha oS)(X)$.

$$(\alpha oA)o(\alpha oS)(x)=(\alpha oA)(x)=0 \text{ and } (\alpha oS)o(\alpha oA)(x)=0$$
$$d\big((\alpha oA)x_n,(\alpha oS)x_n\big)=d(0,x_n)=|0-x_n|\to 0 \text{ as } x_n\to 0$$
$$d\Big(\big((\alpha oA)o(\alpha oS)\big)x_n,((\alpha oS)o(\alpha oA)x_n\Big)=0 \text{ as } x_n\to 0.$$

Hence, A,S are α-compatible maps. Similarly B,T are α-compatible maps.

$$d\big((\alpha oB)x_n,(\alpha oT)x_n\big)=|x_n/8-x_n/2|=|-6x_n/16|\to 0 \text{ as } x_n\to 0$$
$$\big((\alpha oB)o(\alpha oT)\big)x_n=\frac{(\alpha oB)x_n}{2}=\frac{\frac{x_n}{2}}{8}=\frac{x_n}{16}$$
$$\big((\alpha oT)o(\alpha oB)\big)x_n=\frac{(\alpha oT)x_n}{8}=\frac{x_n}{16}$$
$$d\Big(\big((\alpha oB)o(\alpha oT)\big)x_n,((\alpha oT)o(\alpha oB)x_n\Big)\to 0 \text{ as } x_n\to 0$$
$$\big[d\big((\alpha oA)x,(\alpha oB)y\big)\big]^2=\left|0-\frac{y}{8}\right|^2=\left|\frac{y^2}{64}\right|=\left|\frac{y^2}{32}+\frac{xy}{94}\right| \;\forall x,y\in X$$
$$=\frac{1}{4}\big[d((\alpha oA))x,(\alpha oS)x)d\big((\alpha oB)y,(\alpha oT)y\big)+d\big((\alpha oB)y,(\alpha oS)x\big)d\big((\alpha oA)x,(\alpha oT)y\big)\big]+$$

$\frac{1}{3}\left[d((\alpha oA)x,(\alpha oS)x)d((\alpha oA)x,(\alpha oT)y)+d((\alpha oB)y,(\alpha oT)y)d\big((\alpha oB)y,(\alpha oS)y\big)\right]$.

Hence all the assumptions of Theorem 3.5 are satisfied with $k_1=\frac{1}{4}, k_2=\frac{1}{3}$, and 0 is a unique α-fixed point.

Here, we generalize Theorem-B under as follows:

Theorem 3.7: Let α, A , B, S, and T be self -maps of a complete metric space (X,d) with:

$(\alpha oA)o(\alpha oS)=(\alpha oS)o(\alpha oA)$; $(\alpha oT)o(\alpha oB)=(\alpha oB)o(\alpha oT)$, satisfying the condition (4) of theorem 3.5 and there exists positive integers a,b,s and t such that

(6) $(\alpha oA^a)(X)\subseteq(\alpha oT^t)(X)$ and $(\alpha oB^b)(X)\subseteq(\alpha oS^s)(X)$

(7) $$\left[d\big((\alpha oA^a)x,(\alpha oB^b)y\big)\right]^2$$

$$\le k_1\left[d\big((\alpha oA^a)x,(\alpha oS^s)x\big)d\big((\alpha oB^b)y,(\alpha oT^t)y\big)\right.$$

$$\left.+\,d\big((\alpha oB^b)y,(\alpha oS^s)x\big)d\big((\alpha oA^a)x,(\alpha oT^t)y\big)\right]$$

$$+\,k_2\left[d\big((\alpha oA^a)x,(\alpha oS^s)x\big)d\big((\alpha oA^a)x,(\alpha oT^t)y\big)\right.$$

$$\left.+\,d\big((\alpha oB^b)y,(\alpha oT^t)y\big)d\big((\alpha oB^b)y,(\alpha oS^s)x\big)\right]$$

for all x,y in X; where $0\le k_1,k_2<1$.

Then A,B, S, and T have a unique common α-fixed point in X.

Proof. Since A and B are α-commute with S and T and so A^a and B^b also α-commute with S^s and T^t respectively. Thus by Theorem 3.5, there exists z in X such that, $z=(\alpha oA^a)z=(\alpha oB^b)z=(\alpha oS^s)z=(\alpha oT^t)z$.

From this we obtain

$$(\alpha oA)z=(\alpha oA)o(\alpha oA^a)z=(\alpha oA^a)o(\alpha oA)z$$

$$\text{and }(\alpha oA)z=(\alpha oA)o(\alpha oS^s)z=(\alpha oS^s)o(\alpha oA)z.$$

Therefore $(\alpha oA)z$ is a common α-fixed point of A^a and S^s.

Similarly, we can show that $(\alpha oB)z$ is a common α-fixed point of B^b and T^t.

Now putting $x=(\alpha oA)z$ and $y=(\alpha oB)z$ in (7) we have,

$$[d((\alpha oA)z,(\alpha oB)z)]^2 = \left[d\left((\alpha oA^a)x,(\alpha oB^b)x\right)\right]^2$$

$$\leq k_1\left[d((\alpha oA^a)x,(\alpha oS^s)x)d\left((\alpha oB^b)y,(\alpha oT^t)y\right)\right.$$

$$\left.+d\left((\alpha oB^b)y,(\alpha oS^s)x\right)d((\alpha oA^a)x,(\alpha oT^t)y)\right]$$

$$+k_2\left[d((\alpha oA^a)x,(\alpha oS^s)x)d((\alpha oA^a)x,(\alpha oT^t)y)\right.$$

$$\left.+d\left((\alpha oB^b)y,(\alpha oT^t)y\right)d\left((\alpha oB^b)y,(\alpha oS^s)x\right)\right]$$

$$= k_1[d((\alpha oA)z,(\alpha oA)z)d((\alpha oB)z,(\alpha oB)z) + d((\alpha oB)z,(\alpha oA)z)d((\alpha oA)z,(\alpha oB)z)]$$

$$+k_2[d((\alpha oA)z,(\alpha oA)z)d((\alpha oA)z,(\alpha oB)z)$$

$$+d((\alpha oB)z,(\alpha oB)z)d((\alpha oB)z,(\alpha oA)z)]$$

$$= k_1[d((\alpha oA)z,(\alpha oB)z)]^2$$

i.e.,$[d((\alpha oA)z,(\alpha oB)z)]^2 \leq k_1[d((\alpha oA)z,(\alpha oB)z)]^2$

Which is a contradiction. Hence $d((\alpha oA)z,(\alpha oB)z) = 0$. So that z is common α-fixed point of A^a, B^b, S^s and T^t.

Further, we obtain $(\alpha oS)z = (\alpha oS)o(\alpha oA^a)z = (\alpha oA^a)o(\alpha oS)z$

and $(\alpha oS)z = (\alpha oS)o(\alpha oS^s)z = (\alpha oS^s)o(\alpha oS)z$.

Therefore $(\alpha oS)z$ is a common α-fixed point of A^a and S^s. Also, $(\alpha oT)z$ is a common α-fixed point of B^b and T^t.

Now putting $x = (\alpha oS)z$ and $y = (\alpha oT)z$ in (7) we have,

$$[d((\alpha oS)z,(\alpha oT)z)]^2 = \left[d\left((\alpha oA^a)x,(\alpha oB^b)x\right)\right]^2$$

$$\leq k_1\left[d((\alpha oA^a)x,(\alpha oS^s)x)d\left((\alpha oB^b)y,(\alpha oT^t)y\right)\right.$$

$$\left.+d\left((\alpha oB^b)y,(\alpha oS^s)x\right)d((\alpha oA^a)x,(\alpha oT^t)y)\right]$$

$$+k_2\left[d((\alpha oA^a)x,(\alpha oS^s)x)d((\alpha oA^a)x,(\alpha oT^t)y)\right.$$

$$\left.+d\left((\alpha oB^b)y,(\alpha oT^t)y\right)d\left((\alpha oB^b)y,(\alpha oS^s)x\right)\right]$$

$$= k_1\left[d\big((\alpha oS)z,(\alpha oS)z\big)d\big((\alpha oT)z,(\alpha oT)z\big)+d\big((\alpha oT)z,(\alpha oS)z\big)d\big((\alpha oS)z,(\alpha oT)z\big)\right]$$

$$+ k_2\left[d\big((\alpha oS)z,(\alpha oS)z\big)d\big((\alpha oS)z,(\alpha oT)z\big)\right.$$

$$\left.+ d\big((\alpha oT)z,(\alpha oT)z\big)d\big((\alpha oT)z,(\alpha oS)z\big)\right]$$

$$\text{i.e.,}\left[d\big((\alpha oS)z,(\alpha oT)z\big)\right]^2 \le k_1\left[d\big((\alpha oS)z,(\alpha oT)z\big)\right]^2$$

Which is a contradiction. Hence $(\alpha oS)z = (\alpha oT)z$.

Therefore $(\alpha oS)z = (\alpha oT)z$ is a common α- the fixed point of A^a, B^b, S^s, and T^t. But we have seen that A^a, B^b, S^s, and T^t have a unique common α- fixed point z.

Hence $z = (\alpha oA)z = (\alpha oB)z = (\alpha oS)z = (\alpha oT)z$.

This completes the proof.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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