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TOTAL DOMINATION COLORING OF GRAPHS

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Abstract. A total domination coloring of a graph G is a proper coloring of G in which open neighbourhood of each vertex contains at least one color class and each color class is dominated by at least one vertex. The minimum number of colors required for a total domination coloring of G is called the total domination chromatic number of G and is denoted by $\chi_{td}(G)$. In this paper, we study the total domination chromatic number of some graph classes. The bounds of total domination chromatic number with respect to the graph parameters such as the domination number, chromatic number, total dominator chromatic number and total domination number are also studied.

Keywords: domination coloring; dominator coloring; total dominator coloring; total domination coloring.

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1. INTRODUCTION

For the terminology and results of graph theory, we refer to [4], for more about domination in graphs refer to [5] and for the terminology of graph coloring, we refer [2]. All graphs mentioned in this article are simple, connected, finite and undirected.

Let $G = (V, E)$ be a graph on n vertices and m edges. The *degree* of a vertex v is denoted by $deg(v)$. The minimum degree and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. The *open neighbourhood* of a vertex v of G , denoted by $N(v)$, is the set of all

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adjacent vertices of v . Similarly, the *closed neighbourhood* of v , denoted by $N[v]$, is defined to be $N(v) \cup \{v\}$. Note that any subset of $N[v]$ is dominated by v .

We denote a path on n vertices by P_n , a cycle on n vertices by C_n and a complete graph on n vertices by K_n . A complete multi-partite graph with partitions containing r and s vertices is denoted by $K_{r,s}$ and a complete multi-partite graph with k partitions is represented by $K_{r_1,r_2,r_3,\dots,r_k}$, where the i -th partition contains r_i vertices.

A set $D \subset V(G)$ is said to be a *dominating set* of a graph G if every vertex in $V - D$ is adjacent to at least one vertex in D . A *minimal dominating set* of G is a dominating set of G such that no proper subset of it is a dominating set of G . The minimum cardinality of a minimal dominating set is called the *domination number* of G , denoted by $\gamma(G)$. A dominating set D of G is called a *total dominating set* of G if every element in $V(G)$ is adjacent to some vertex in D . The minimum cardinality of a minimal total dominating set is called the *total domination number* of G , $\gamma_t(G)$.

Like domination in graphs, graph coloring is also an interesting area of research in graph theory. In graph coloring, we assign colors to vertices, edges or both subject to some predefined conditions. In a proper vertex coloring c of a graph G , we color the vertices in such a way that the adjacent vertices receive different colors and the minimum number of colors required to color G is called the *chromatic number* of G and is denoted by $\chi(G)$. The set of all vertices of a graph G with same color is known as a *color class* of G . Throughout this paper, we denote color of a vertex by $c_i, i \leq n$ and color class by V_i which is the set consisting of all vertices with color c_i . If $v \in V_i$ and V_i contains only one vertex v , then v is called a *solitary vertex*.

Many types of graph colorings based on domination properties of graphs have been introduced and studied extensively in the literature. The notion of *dominator coloring* of graphs is the first of all such colorings, which is defined to be a proper vertex coloring in which each vertex dominates at least one color class and the minimum number of colors used in a dominator coloring of G is called the *dominator chromatic number* of G , denoted by $\chi_d(G)$, was introduced by Gera *et al.* [3]. If a vertex v dominates a color class, then we refer to that color

class as the *dom-color class* of v . A dom-color class is a subset of $N[v]$. Here, note that $\{v\}$ can possibly be the dom-color class of a vertex v .

If we impose one more condition to dominator coloring that a dom-color class of a vertex v does not contain the vertex v , then the new coloring is called a *total dominator coloring* of the graph G . In other words, total dominator coloring of a graph G is a dominator coloring such that a dom-color class of a vertex v is a subset of $N(v)$ with respect to a given domination coloring of G and the minimum cardinality of a minimal total dominating set is called the *total dominator chromatic number* of G and is denoted by χ_d^t and in this case a dom color class is known as a *proper dom-color class*. This coloring has been introduced and explored by Kazemi [6]. In the above mentioned colorings every vertex dominates some color class and it is not necessary that all the color classes are dominated by some vertex. Imposing this condition in dominator coloring, Zhuo and Zhao ([7]) introduced the concept of *domination coloring* as a proper coloring in which every vertex of G dominates some color class and every color class is dominated by some vertex of G . In this paper, we introduce another variation of domination coloring, called total domination coloring, obtained by imposing some new conditions to the coloring protocol.

2. TOTAL DOMINATION COLORING

The domination coloring of a given graph need not be unique. For example, FIGURE 1 illustrate domination coloring of P_4 in two different ways.

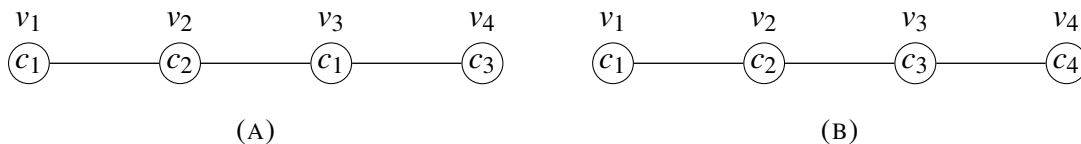


FIGURE 1. Domination coloring of P_4 .

In FIGURE 1(A) the dom-color class of v_4 is $\{v_4\}$ whereas in FIGURE 1(B) the dom-color class of v_4 is $\{v_3\}$. Note that in the second case, v_4 has a proper dom-color class. In this context, the type of domination coloring of a graph G with respect to which all the vertices of G have a proper dom-color classes would of interest for further study. Motivated by this fact, we introduce the notion of the total domination coloring of a graph as follows.

Definition 2.1. A *total domination coloring* of a graph G is a proper vertex coloring of G in which each vertex v of G dominates at least one color class (other than $\{v\}$) and each color class of G is dominated by at least one vertex of G . The minimum number of colors required for the total domination coloring of G is called the *total domination chromatic number* of G and is denoted by $\chi_{td}(G)$.

In other words, a total domination coloring of G is a total dominator coloring in which each color class is dominated by at least one vertex of G . Therefore, the graphs which admit total domination coloring should be graphs without isolated vertices. The following theorem characterizes graphs admitting total domination coloring. It follows from the fact that an n -coloring of a graph of order n with $\delta(G) \geq 1$, in which all vertices of G assume distinct colors, is a total domination coloring of G .

Theorem 2.2. Any graph G with $\delta(G) \geq 1$ admits a total domination coloring.

As consequence of the above theorem, we have the following result.

Corollary 2.3. For any graph of order n with $\delta(G) \geq 1$, $2 \leq \chi_{td}(G) \leq n$.

It can also be observed that the total domination chromatic number $\chi_{td}(G)$ of a graph G will be an upper bound for the chromatic number $\chi(G)$, the dominator chromatic number $\chi_d(G)$, the total dominator chromatic number $\chi'_d(G)$ and the domination chromatic number $\chi_{dd}(G)$ of the graph.

Our next result is a necessary and sufficient condition for a graph to have total domination chromatic number 2.

Theorem 2.4. For any graph G , $\chi_{td}(G) = 2$ if and only if G is a complete bipartite graph.

Proof. First assume that G is a complete bipartite graph. Then G is 2-colorable. It can be noted that every vertex in V_1 (with color c_1) dominates the color class V_2 (of the color c_2) and vice versa. Thus, we have $\chi_{td}(G) = 2$.

Conversely, assume that $\chi_{td}(G) = 2$. Therefore, G is 2-colorable and hence is bipartite. We now have to prove that G is complete bipartite. Assume the contrary. Let G be a graph with

$\chi_{td} = 2$, but not a complete bipartite graph. Consider the bipartition (V_1, V_2) of G . If a vertex $v \in V_1$ is not adjacent to some vertices in V_2 , then, v does not dominate a color class contradicting to the fact that the coloring concerned is a total domination coloring. Thus, every vertex in V_1 is adjacent to all vertices in V_2 . Using the same argument, we can show that every vertex in V_2 is adjacent to all vertices in V_1 as well. Therefore, G is complete bipartite. Hence the theorem. \square

The following theorem provides an expression for the total domination chromatic number of a disconnected graph in terms of the total domination chromatic number of its components.

Theorem 2.5. *Let G be a disconnected graph with $\delta(G) \geq 1$ and let $H_1, H_2, H_3, \dots, H_k$ be the components of G . Then, $\chi_{td}(G) = \sum_{i=1}^k \chi_{td}(H_i)$.*

Proof. Consider a total domination coloring c of G . Then vertices belonging to different components cannot have the same color with respect to c . Therefore, $\chi_{td}(G) = \sum_{i=1}^k \chi_{td}(H_i)$. \square

3. RELATION BETWEEN $\chi_{td}(G)$ AND OTHER GRAPH PARAMETERS

In this section, the relation between total domination chromatic number and certain other graph parameters such as domination number, total domination number and chromatic number are discussed.

Theorem 3.1. *Let G be a graph with $\delta(G) \geq 1$. Then $\max\{\chi(G), \gamma(G)\} \leq \chi_{td}(G) \leq \chi(G)\gamma(G)$.*

Proof. Since the total domination coloring is a proper coloring, we have, $\chi(G) \leq \chi_{td}(G)$. We know that in a total domination coloring all color classes are dominated by at least one vertex. Consider a χ_{td} -coloring of G . Let u_i be a vertex dominating the color class V_i . Let $D = \{u_1, u_2, u_3, \dots, u_{\chi_{td}}\}$, be a dominating set of G . Then $\gamma(G) \leq \chi_{td}(G)$. Therefore, $\max\{\chi(G), \gamma(G)\} \leq \chi_{td}(G)$.

To prove the remaining part, consider a γ -set D of G . Color each of the vertices in D by distinct colors. Then, in order to color the neighbourhood of one vertex in D , we require at most $\chi(G) - 1$ colors. Therefore, in order to color the neighbourhood of D , at most $\gamma(G)(\chi(G) - 1)$ colors are required. This coloring can easily be verified as a total domination coloring. Thus, any total domination coloring of G has at most $\gamma(G) + \gamma(G)(\chi(G) - 1) = \chi(G)\gamma(G)$ colors, as required. \square

The bounds of the above theorem are sharp. For example, consider the complete graph $K_n; n \geq 2$, for which $\chi(K_n) = \chi_{td}(K_n) = n$. Next we obtain bounds for the total domination chromatic number for triangle-free graphs in terms of their total domination number.

Theorem 3.2. *For any triangle-free graph G of order n , $\gamma_t(G) \leq \chi_{td}(G) \leq 2\gamma_t(G)$.*

Proof. Let G be a graph with total domination number $\gamma_t(G) = r$. Note that all total domination colorings of G are total dominator colorings and hence $\gamma_t(G) \leq \chi_{td}(G)$.

Consider the following coloring pattern of G . First color the r vertices in a minimum total dominating set, say D , with r distinct colors and then take any one vertex v in D and color all the vertices in the open neighbourhood $N(v)$ which are not yet colored with $(r+1)$ -th color (which is possible since G is triangle-free). Then, take the open neighbourhood of another vertex in D and color all the vertices in the open neighbourhood which are not already colored with the $(r+2)$ -th color. Proceed with this coloring procedure until all vertices of G are colored. Therefore, at most $2r$ colors are required to color G .

Now, it remains to prove that the above-mentioned coloring is a total domination coloring. It is obvious that each vertex of G dominates at least one vertex in the set D and since all vertices in D are solitary, all vertices of G will dominate at least one color class. Since each color class is a part of an open neighbourhood of some vertices in D , they will be dominated by some vertices of D . That is, the above defined coloring is a total domination coloring thus proving that $\gamma_t(G) \leq \chi_{td}(G) \leq 2\gamma_t(G)$. \square

The bounds of the above theorem are sharp. For example, consider the star graph G , having $\gamma_t(G) = \chi_{td}(G) = 2$ and the bi-star graph, for which $\gamma_t(G) = 2$ and $\chi_{td}(G) = 4$. If we consider a complete 4-partite graph K_{r_1, r_2, r_3, r_4} , we have $\gamma_t(G) = 2$ and $\chi_{td}(G) = 4$. Thus, for complete 4-partite graphs also, the above upper bound is sharp.

Proposition 3.3. *If G is a graph of order $n \geq 2$ and $\gamma(G) = 1$, then $\chi_{td}(G) = \chi(G)$.*

Proof. Since total domination coloring of a graph is a proper vertex coloring, we have $\chi_{td}(G) \geq \chi(G)$. Let G be a graph of order $n \geq 2$ and $\gamma(G) = 1$ and c be a χ -coloring of G . Since $\gamma(G)$ is 1, the graph G has a universal vertex, say v . Then, v is a solitary vertex that dominates all color

classes other than $\{v\}$ and is dominated by all vertices in $V(G)$ other than v . Furthermore, all color classes other than $\{v\}$ are dominated by v . Thus, c is a total domination coloring of G and hence $\chi_{td}(G) = \chi(G)$. \square

The next theorem gives the total domination chromatic number of the join of two graphs G_1 and G_2 in terms of their chromatic numbers.

Theorem 3.4. *Let G_1 and G_2 be any two graphs with chromatic numbers $\chi(G_1)$ and $\chi(G_2)$ respectively. Then $\chi_{td}(G_1 + G_2) = \chi(G_1) + \chi(G_2)$.*

Proof. Let G_1 and G_2 be any two graphs and also let $\chi(G_1) = k_1$ and $\chi(G_2) = k_2$. Consider the graph $G_1 + G_2$. If we color the vertices of G_1 in $G_1 + G_2$ with k_1 colors and the vertices of G_2 in $G_1 + G_2$ with k_2 colors, which are different from the already used k_1 colors, then we can prove that $\chi_{td}(G_1 + G_2) = k_1 + k_2$. In this coloring of $G_1 + G_2$, each vertex in G_1 dominates all color classes in G_2 and vice versa. Here all color classes are dominated by a vertex. That is all the requirements of total domination coloring are satisfied and hence $\chi_{td}(G_1 + G_2) = \chi(G_1) + \chi(G_2)$. \square

We can generalize the above theorem as follows.

Corollary 3.5. *Let $G_1, G_2, G_3, \dots, G_k$ be k graphs with chromatic numbers $\chi(G_1), \chi(G_2), \chi(G_3), \dots, \chi(G_k)$ respectively. Then $\chi_{td}(G_1 + G_2 + G_3 + \dots + G_k) = \chi(G_1) + \chi(G_2) + \chi(G_3) + \dots + \chi(G_k)$.*

The following theorem is a graph realisation problem.

Theorem 3.6. *For integers k and n with $2 \leq k \leq n$, there exists a connected graph G of order n such that $\chi_{td}(G) = k$.*

Proof. Consider a complete graph K_k with $k \geq 2$ vertices and construct a new graph G by adding $n - k$ pendant vertices at the k^{th} vertex of the complete graph. Now, we have to show that $\chi_{td}(G) = k$.

Since $\chi(K_k) = k$, at least k colors are required for χ_{td} -coloring of G . Let us now examine the following coloring pattern: Color all vertices of K_k with distinct colors and the remaining pendant vertices with the color of first vertex. Then, the above defined coloring is a total dominating coloring of G . Therefore, we have $\chi_{td}(G) = k$. \square

4. TOTAL DOMINATION CHROMATIC NUMBER OF SOME GRAPH CLASSES

The total domination chromatic number of some fundamental graphs and graph classes are discussed in this section. We begin by examining the total domination chromatic number of paths .

Theorem 4.1. For $n \geq 3$, $\chi_{td}(P_n) = 2\lceil \frac{n}{3} \rceil$

Proof. Consider a path P_n with $n \geq 3$ and vertex set $V(P_n) = \{v_1, v_2, \dots, v_n\}$ labeled consecutively from left to right. Since each color class is dominated by at least one vertex with respect to (any) total domination coloring c , note that any two vertices in the same color class will be at a distance 2 and hence every color class can have at most two vertices of P_n . Now, the following cases are to be considered.

Case-1: $n \equiv 0 \pmod{3}$.

Here, the vertices v_1 and v_3 can have the same color, say c_1 and the vertex v_2 can have the color c_2 . Note that no other vertex in P_n can take the color c_2 because otherwise, the vertex v_1 will not dominate any color class in P_n . In a similar manner, both v_4 and v_6 (if exist) can assume the same but new color, say c_3 , and the vertex v_5 can assume the color c_4 . This coloring pattern can be followed until all vertices are colored as per requirement. It is to be noted that this coloring partitions the vertex set $V(P_n)$ in to $\frac{n}{3}$ subsets of consecutive vertices, where each of these partitions requires two new colors. Hence, $\chi_{td}(P_n) = \frac{2n}{3}$.

Case-2: $n \equiv 1 \pmod{3}$.

Here, we partition the vertex set $V(P_n)$ into $\lfloor \frac{n}{3} \rfloor - 1$ partitions of three consecutive vertices and one partition with four vertices. Each partition of cardinality 3 can be colored as explained in *Case-1* which yield $2(\lfloor \frac{n}{3} \rfloor - 1)$ distinct colors. Now it remains to color the four vertices $v_{n-3}, v_{n-2}, v_{n-1}$ and v_n which belong to the last partition. If v_{n-3} and v_{n-1} have the same color, then the vertex v_n will not have a dominating color class, as it is adjacent only to v_{n-1} . If

v_{n-2} and v_n have the same color, then the vertex v_{n-3} will not have a dominating color class. Therefore, all these four vertices should assume four distinct colors. Thus, the total number of colors required is $2(\lfloor \frac{n}{3} \rfloor - 1) + 4 = 2(\lfloor \frac{n}{3} \rfloor + 1) = 2\lceil \frac{n}{3} \rceil$.

Case-3: $n \equiv 2 \pmod{3}$.

In this case, we partition the vertex set $V(P_n)$ into $\lfloor \frac{n}{3} \rfloor$ partitions of three consecutive vertices and one partition with 2 vertices. Each partition of cardinality 3 can be colored as explained in Case-1 which yield $2\lfloor \frac{n}{3} \rfloor$ distinct colors. The two vertices in the last partition should assume two distinct colors. Thus, the total number of colors required is $2\lfloor \frac{n}{3} \rfloor + 2 = 2(\lfloor \frac{n}{3} \rfloor + 1) = 2\lceil \frac{n}{3} \rceil$, completing the proof. \square

For the cycles C_3 and C_4 , we can observe that any minimal proper coloring will be a total domination coloring as well. For $n = 7$, we can easily verify that the coloring v_i with c_i for $i = 1$ to 4; v_5, v_6 and v_7 with colors c_3, c_4 and c_5 respectively is a minimum total domination coloring. That is $\chi_{td}(C_7) = 5$. Therefore, the following theorem describes the total domination chromatic number of cycles C_n ; $n \geq 5, n \neq 7$.

Theorem 4.2. For $n \geq 5, n \neq 7$, $\chi_{td}(C_n) = 2\lceil \frac{n}{3} \rceil$.

Proof. The proof of the theorem is similar to that of Theorem 4.1. \square

The total domination chromatic number of the complements of paths is determined in the following theorem. When $n \leq 3$, the complement of the path P_n will have isolated vertices, and hence we consider complements of paths with more than 3 vertices.

Theorem 4.3. For $n \geq 4$, $\chi_{td}(\overline{P}_n) = \begin{cases} 4, & \text{if } n = 4 \\ \lceil \frac{n}{2} \rceil, & \text{otherwise.} \end{cases}$

Proof. Since P_4 is self-complementary and $\chi_{td}(P_4) = 4$, the first part of the result follows immediately. Now, consider a path of order n with $n \geq 5$ with vertex set $V(P_n) = \{v_1, v_2, \dots, v_n\}$ labelled consecutively from left to right and let c be a total domination coloring of G . Consider \overline{P}_n and let c be a proper coloring of \overline{P}_n . Since any independent set of \overline{P}_n will have at most two elements, at most two vertices in \overline{P}_n can have the same color with respect to c . Then, color the vertices of \overline{P}_n in such a manner that the vertices v_{2i-1} and v_{2i} have the same color, say c_i , where

$1 \leq i \leq \lfloor \frac{n}{2} \rfloor$. This coloring c satisfies all the conditions of total domination coloring. Thus, if n is even, we have $\frac{n}{2}$ colors in the color set and if n is odd, we have $\lfloor \frac{n}{2} \rfloor + 1 = \lceil \frac{n}{2} \rceil$ colors, completing the proof. \square

The result for total domination chromatic number of the complements of cycles is presented in the next theorem.

Theorem 4.4. For $n \geq 4$, $\chi_{td}(\overline{C_n}) = \begin{cases} 4, & n = 4, 5 \\ \lceil \frac{n}{2} \rceil, & \text{otherwise.} \end{cases}$

Proof. Consider $\overline{C_4}$ and if we color all the 4 vertices in it with 4 different colors we can see that $\chi_{td}(\overline{C_4}) = 4$. We know that $\overline{C_5}$ is C_5 and we have $\chi_{td}(\overline{C_5}) = 4$. For $n \geq 6$, the proof is similar to that of $\overline{P_n}$. Hence the result. \square

In view of Proposition 3.3, it is clear that the total domination chromatic numbers of the graph classes such as complete graphs, wheel graphs, and star graphs are the same as their chromatic numbers.

Let $K_{r_1, r_2, r_3, \dots, r_k}$ represent a complete k -partite graph. The following theorem determines total domination chromatic number of a k -partite graph.

Theorem 4.5. For $k \geq 2$, $\chi_{td}(K_{r_1, r_2, r_3, \dots, r_k}) = k$.

Proof. Note that every partition in $K_{r_1, r_2, r_3, \dots, r_k}$ is a color class under any minimal proper coloring (k -coloring) of it and every vertex in one partition dominates all vertices in all other partitions (color classes). That is, every k -coloring of $K_{r_1, r_2, r_3, \dots, r_k}$ is also a total domination coloring of it. Thus, $\chi_{td}(K_{r_1, r_2, r_3, \dots, r_k}) = k$. \square

From the above theorem we find a counter example for the converse of Proposition 3.3. Consider a complete multi-partite graph G with none of its partitions a singleton set. In this case, $\chi_{td}(G) = \chi(G)$, while $\gamma(G) \neq 1$.

A *wheel graph* is a graph obtained by joining all vertices of a cycle to an external vertex. Then, this external vertex becomes the central vertex of the wheel graph with degree $n - 1$. Since the central vertex of wheel graph $W_{1, n}$ is a universal vertex, we have $\chi_{td}(W_{1, n}) = \chi(W_{1, n})$.

A *helm graph*, denoted by $H_{1,n,n}$, is a graph obtained by attaching one pendant vertex to every rim vertex of a wheel graph $W_{1,n}$. The next theorem discusses the total domination chromatic number of a helm graph.

Theorem 4.6. For $n \geq 3$, $\chi_{td}(H_{1,n,n}) = 2n$.

Proof. Note that every vertex of $H_{1,n,n}$ is a support vertex of distinct pendant vertices. Therefore, all rim vertices should have distinct colors. Otherwise, the pendant vertices will not dominate any color class. Also, no two pendant vertices should have the same color, as otherwise, there will not be a vertex in the graph which dominates the color class of these pendant vertices. The central vertex of the graph can assume any one of the colors of the pendant vertices. Thus, the coloring mentioned above will satisfy all requirements of a total domination coloring. Therefore, $\chi_{td}(H_{1,n,n}) = 2n$. \square

A *bi-star* G is a graph obtained by adding to each vertex of K_2 at least one pendant vertex. The following theorem determines χ_{td} of bi-stars.

Theorem 4.7. For any bi-star G , $\chi_{td}(G) = 4$.

Proof. Let u and v be the two support vertices of the bi-star graph G . As the color of a support vertex with respect to a total domination coloring is distinct, two different colors are required to color the two support vertices of G and these colors cannot be repeated anywhere with respect to the coloring of G . As the distance between pendant vertices, which are adjacent to distinct support vertices is three, these vertices cannot be in the same color class. Therefore, at least two colors are required to color pendant vertices of G . Thus, at least four colors are required for a total domination coloring of G . Now, we can observe the coloring mentioned above satisfies all the conditions of total domination coloring of G . Therefore, the total domination chromatic number of a bi-star is 4. \square

A *multistar graph* is a graph formed by joining at least one pendant vertex to each vertex of a complete graph K_l , $l \geq 2$ and is denoted by $K_l(a_1, a_2, a_3, \dots, a_l)$, where a_i denote the number of pendant vertices at the vertex v_i . In the following theorem, we determine the total domination chromatic number of a multistar graph.

Theorem 4.8. *If $G = K_l(a_1, a_2, a_3, \dots, a_l), a_i \geq 1$, then $\chi_{td}(K_l(a_1, a_2, a_3, \dots, a_l)) = 2l$.*

Proof. Let $G = K_l(a_1, a_2, a_3, \dots, a_l), a_i \geq 1$ be a multistar graph. We need l colors to color the clique in G and these colors cannot be repeated as each of these vertices being support vertices. Next, we have to color the pendant vertices. It is to be noted that color of pendant vertices at two different support vertices cannot be same as the distance between them is not 2. Therefore, at least l more colors are required to color pendant vertices. If we color all pendant vertices such that same color for all pendant vertices which are adjacent. Then, the above defined coloring is a total domination coloring. Hence the theorem. □

Proposition 4.9. *Total domination chromatic number of a Petersen graph is 6.*

Proof. Let P denote the Petersen graph. From FIGURE 2 we can easily verify that the coloring given in the figure is a total domination coloring. Therefore, $\chi_{td}(P) \leq 6$. We have for any graph G , $\chi_{td}(G) \geq \chi_d^t(G)$, as all total domination coloring is also a total dominator coloring. It has been proved in [1] that $\chi_d^t(P) = 6$. Therefore, we have $\chi_{td}(P) \geq 6$. Hence the result. □

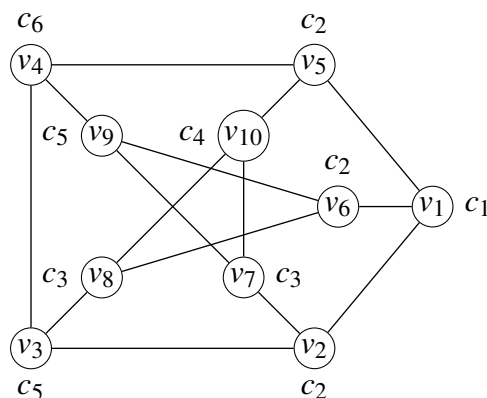


FIGURE 2. Total domination coloring of the Petersen graph

Observation 4.10. *If G is an $(n - 2)$ -regular graph of order n , then $\chi_{td}(G) = \frac{n}{2}$.*

Theorem 4.11. *Let K_n be a complete graph and let graph G be obtained by adding pendant vertices at vertices of K_n . Then, $\chi_{td}(G) = \chi(K_n) = n$ if and only if the number of support vertices in G is less than or equal to $\lfloor \frac{n}{2} \rfloor$.*

Proof. We have, $\chi_{td}(K_n) = n$. Let G be a graph obtained by adding pendant vertices at some vertices of K_n . Let s be the number of support vertices of G . Now we have to find the maximum value of s such that $\chi_{td}(G) = n$. We know that the color of each support vertices are distinct and cannot assign to any other vertex and color of pendant vertices at any two distinct support vertices are different. We can color the pendant vertices at a particular support vertex with same color. Therefore, $2s$ different colors are required to color the support vertices and pendant vertices. Therefore to have n as the total domination G , $2s \geq n$. That is, $s \leq \lfloor \frac{n}{2} \rfloor$. \square

Next we present a characterization theorem for unicyclic graphs whose total domination number is the same as that of its chromatic number.

Theorem 4.12. *Let G be a unicyclic graph. $\chi_{td}(G) = \chi(G)$ if and only if G is C_3 or C_4 or graph obtained by adding pendant vertices at one of the vertices of C_3 .*

Proof. We know that if G is a unicyclic graph, then $\chi(G)$ is either 2 or 3. We know that the cycles having total domination chromatic number 2 or 3 are C_4 and C_3 . If we add pendant vertices at any one of its vertices of C_4 , then a third color is required to color that support vertex as the color of support vertices are distinct and cannot assign to any other vertex with respect to a total domination coloring of a graph. \square

5. TREES

In this section, we discuss the total domination coloring and the corresponding total domination chromatic number of certain types of trees.

Theorem 5.1. *Let T be any tree with s support vertices. Then, $\chi_{td}(T) \geq s + 1$. Moreover, the bound is sharp if and only if T is a star.*

Proof. Let T be a tree with s support vertices. We know that with respect to total domination coloring of graphs, all support vertices should necessarily be solitary vertices and hence s colors are required to color the support vertices of T . Therefore, at least one more color is required to color the pendant vertices of T . That is, $\chi_{td}(T) \geq s + 1$.

Next, we have to prove that $\chi_{td}(T) = s + 1$ if and only if T is a star. Assume that T is a star graph. Then, $s = 1$. We know that $\chi_{td}(T) = 2 = s + 1$. To prove the sufficient condition let us

assume that T is not a star, now it remains to prove that $\chi_{td}(T) \neq s + 1$. In this case, T will have at least two support vertices and hence two distinct colors are required to color these support vertices. The distances between the pendant vertices adjacent to these support vertices are at least 3 and hence all the pendant vertices at these vertices will not belong to the same color class. That is, $\chi_{td}(T) \geq s + 2$. Hence the result. \square

The following theorem determines a lower bound for the total domination chromatic number of a tree in terms of number of support vertices of the tree.

Theorem 5.2. *Let T be a tree with s support vertices. $\chi_{td}(T) \geq 2s$.*

Proof. We know that color of support vertices are distinct and cannot be repeated with respect to a total domination coloring. Consider a support vertex and its adjacent pendant vertices. Therefore, at least two colors are required to color these vertices. There are s set of such vertices and hence $\chi_{td}(T) \geq 2s$. \square

The complete caterpillar graph is a tree obtained by attaching at least one pendant vertex to each vertex of a path. The following observation shows that the bound of the above theorem is sharp.

Observation 5.3. *If T is a complete caterpillar graph, then it can be seen that $\chi_{td}(T) = 2s$.*

The following theorem characterises all trees which satisfies $\chi_{td}(T) = 2s$.

Theorem 5.4. *For a tree T with s support vertices, $\chi_{td}(T) = 2s$ if and only if every non-support vertex of T is adjacent to at least one support vertex of T .*

Proof. Let T be a tree with s support vertices. By Theorem 5.2, we have $\chi_{td}(T) \geq 2s$. Assume that every non-support vertex of T is adjacent to at least one support vertex of T . Color all the s support vertices with s different colors and corresponding pendant vertices with another s colors. Remaining vertices can be colored using the color of one of its adjacent support vertex. This coloring is a total domination coloring of T . Therefore, $\chi_{td}(T) = 2s$.

If possible, assume that there exist a vertex v in $V(T)$, that is not adjacent to any of the support vertices of T . As the graph has s support vertices, we know that s colors are required to color the

support vertices and another s distinct colors to color the pendant vertices and no two pendant vertices adjacent to different support vertices can have the same color. Let u be an arbitrary support vertex in $V(T)$. Since the distance between v and u is more than 2, we cannot color the vertex v with the color of s . That is, $\chi_{td}(T) = 2s + 1$, which is a contradiction. Therefore, any non-support vertex of T is adjacent to a support vertex. \square

Observation 5.5. *Let T be a tree with n vertices. Then, $\chi_{td}(T) = n$ if and only if the number of support vertices is equal to the number of leafs equal to $\frac{n}{2}$.*

Now we proceed to determine the exact value of the total domination chromatic number of trees of diameter 4. Note that trees of diameter 2 and 3 are stars and bistars for which we the value of χ_{td} are 2 and 4 respectively.

The trees with diameter 4 can be classified into two types (see FIGURE 3).

Type 1: The centers of two or more stars with at least two vertices are joined to a new vertex.

Type 2: The centers of two or more stars with at least two vertices are joined to the center of another star with at least two vertices.



FIGURE 3. Two types of trees with diameter 4.

These structural properties of trees with diameter 4, are used to determine the total domination chromatic number of such trees, as discussed in the following theorem.

Theorem 5.6. *If T is a tree with diameter 4 and s support vertices, then $\chi_{td}(T) = 2s$.*

Proof. Let T be a tree with diameter 4 and s support vertices. We know that the color of each support vertex is distinct and thus, s colors are required to color the support vertices of T . We also have, pendant vertices at different support vertices should be of different colors. Two pendant vertices will have same color if they are adjacent to the same support vertex. Since the

number of support vertices is s , we have to use another s colors to color the pendant vertices. Therefore, we have to use at least $2s$ colors in the χ_{td} -coloring of T . Next, we have to prove that there exist a χ_{td} -coloring of T with $2s$ colors.

Case 1: Let T be a tree of Type 1 mentioned above (see FIGURE 3(A)). Color the s support vertices with s different colors and color the pendant vertices with another s colors in such a way that pendant vertices at a support vertex receive same color and pendant vertices attached to different support vertices receive different colors. Color the common vertex used to join the center of the stars with color of any one of the pendant vertices. Then we can see that the above defined coloring is a total domination coloring. Hence, $\chi_{td}(T) = 2s$.

Case 2: Let T be a tree of Type 2 (see FIGURE 3(B)). Then, color the s support vertices with s different colors and color the pendant vertices with another s colors in such a way that pendant vertices at a support vertex receive same color and pendant vertices attached to different support vertices receive different colors. Then, we can see that the above defined coloring is a total domination coloring. Hence $\chi_{td}(T) = 2s$. \square

An upper bound for the total domination chromatic number of a tree in terms of the domination number is determined in the following result.

Theorem 5.7. *For any tree T , $\chi_{td}(T) \leq 2\gamma(T)$.*

Proof. We know that for a tree T , $\chi(T) = 2$ and therefore, by Proposition 3.1, we have $\chi_{td}(T) \leq 2\gamma(T)$. \square

In the above theorem, we can verify the equality by considering the trees such as stars, bi-stars, trees with diameter 4 or complete caterpillar graphs.

6. CONCLUSION

In this paper, we have initiated a study on total domination coloring of graphs. There is a large scope for further studies on this new coloring. Some of the areas that demand further investigation are given below.

- (i) Find the graphs with $\chi(G) = \chi_d^t(G)$.
- (ii) Characterize graphs having $\chi_{td}(G) = \chi_d^t(G)$.

- (iii) Characterize graphs having $\chi_{td}(G) = \gamma(G)$.
- (iv) Find graphs for which $\chi_{td}(G) = \chi(G) \cdot \gamma(G)$.
- (v) Study the complexity of total domination coloring of graphs.
- (vi) Study the criticality concepts in total domination coloring of graphs.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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