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PERTURBED LEAST SQUARES TECHNIQUE FOR SOLVING VOLTERRA FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS BASED ON CONSTRUCTED ORTHOGONAL POLYNOMIALS

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Abstract: A numerical method is presented in this paper to solve fractional integro-differential equations in the sense of Caputo, the fractional derivative is considered. The proposed method is perturbed Least Squares Method (PLSM) with the aid of constructed orthogonal polynomials as basis functions. The suggested method reduces this type of problem to a solution of system of linear algebraic equations and then solved using Maple 18 software. Some numerical examples are provided to show the accuracy and applicability of the presented method, numerical results show that when applied to fractional integro-differential equations, the method is easy to implement and accurate.

Keywords: orthogonal polynomial; perturbed least squares; numerical method.

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1. INTRODUCTION

Fractional calculus is an area that encompasses the integration, derivatives and applications of arbitrary orders in science, engineering and other fields. The application of fractional differentiation in mathematical modelling of real-life problem has been significantly increased in recent years, such as earthquake modelling, decreased viral transmission, regulation of electrical socket memory behaviour, etc. There are many fascinating or exciting books about fractional calculus and fractional differential equations [1], [2]. Many fractional integro-differential equations (FIDEs) are often difficult to solve and hence may not have analytical or exact solutions in the interval of consideration, so approximate and numerical methods must be used. Several numerical methods to solve the FIDEs have been given, such as, Adomian Decomposition Method [3], Standard Least Squares Method [4-6], Homotopy Analysis Transform Method [7], Collocation Method [8], Fractional Order Model with Caputo–Fabrizio Operator [9], Homotopy Perturbation Method [10], Differential Transform Method [11], Parameter Expansion Method [12], Variational Iteration Method [13].

Rawashdeh [8] proposed a numerical solution of integro differential fractional equations using th e method of collocation in which polynomial spline functions was used to find the approximate solution. Momani and Qaralleh [14] suggested an efficient method for the solution of the systems of fractional integro-differential equations solution using Adomian Decomposition Method (ADM). Also, Mittal and Nigam [3], employed Adomian Decomposition Method for the solution of fractional Integro-differential equations. ADM requires the construction of Adomian polynomials which was reported demanding to construct. Mohammed [4], applied least squares method and shifted Chebyshev polynomial for the solution of fractional integro-differential equations. In the work, the author employed shifted Cheyshev polynomial of the first kind as basis function and the result was presented graphically. Taiwo and Fesojaye, [15] applied perturbation least-Squares Chebyshev method for solving fractional order integro-differential equations. In their work, an approximate solution taken together with the Least - Squares

Method is utilized to reduce the fractional integro-differential equations to system of algebraic equations, which are solved for the unknown constants associated with the approximate solution. [7] applied homotopy analysis transform method for the solving fractional integro-differential equation in the work, Laplace transforms was used to reduce a differential equation to an algebraic equation. [8] suggested a numerical method called Numerical studies for the resolution of fractional Integro-differential equations using the least square method and polynomials of Berntein. Also, [6] applied standard least squares method for solving fractional integrodifferential equations using constructed orthogonal as basis function The main objective of this work is to find the numerical solution of the volterra type fractional integro-differential equation using the standard least square method based on the orthogonals constructed as basis functions. The general form of the problem class considered in this work is as follow

(1) $D^{\alpha}u(x) = p(x)u(x) + f(x) + \int_0^x k(x,t)u(x)dt, \;\; o \leq x, t \leq 1,$

With the following supplementary conditions:

(2) $u^{(i)}(0) = \delta_{i,} i = 0,1,2, \dots, n-1,$ n -1 $< \alpha \leq n, \; n \in, N$

Where $D^{\alpha}u(x)$ indicates the $\propto th$ Caputo fractional derivative of $u(x)$; $p(x)$,$f(x)$, $K(x,t)$ are given smooth functions, δ_i are real constant, x and t are real variables varying [0, 1] and $u(x)$ is the unknown function to be determined.

2. PRELIMINARIES

2.1 Some relevant basic definitions.

Definition 1.

Fraction Calculus involves differentiation and integration of arbitrary order (all real numbers and complex values). Example $D^{\frac{1}{2}}, D^{\pi}, D^{2+i}$ etc.

Definition 2.

The Caputor Factional Derivative is defined as

(3) $$D^{\alpha} f(x) = \frac{1}{\Gamma(\mathrm{m}-\propto)} \int_0^x (x-s)^{m-\propto-1} f^m(s) ds$$

Where m is a positive integer with the property that $m-1 <\propto< m$

For example, if $0 <\propto< 1$ the caputo fractional derivative is

(4) $$D^{\alpha} f(x) = \frac{1}{\Gamma(1-\propto)} \int_0^x (x-s)^{-\propto} f^1(s) ds$$

Hence, we have the following properties:

(1) $J^{\alpha} J^{v} f = j^{\alpha+v} f, \alpha, v > 0, f \in C_{\mu}, \mu > 0$

(2) $J^{\alpha} x^{\gamma} = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma,}\ \alpha > 0, \gamma > -1, x > 0$

(3) $J^{\alpha} D^{\alpha} f(x) = f(x) - \sum_{k=0}^{m-1} f^k(0) \frac{x^k}{k!}, \qquad x > 0, m-1 < \alpha \leq m$

(4) $D^{\alpha} J^{\alpha} f(x) = f(x), \quad x > 0, m-1 < \alpha \leq m,$

(5) $D^{\alpha} C = 0, C$ is the constant,

(6) $$\begin{cases} 0, & \beta \in N_0, \beta < [\alpha], \\ D^{\alpha} x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha}, & \beta \in N_0, \beta \geq [\alpha], \end{cases}$$

Where $[\alpha]$ denoted the smallest integer greater than or equal to α and $N_0 = \{0, 1, 2, ...\}$

Definition 3**.**

Shifted Chebyshev polynomial of the first kind denoted by $T_n(x)$ is denoted by the following:

(5) $$T^*{}_n(x) = \cos\{n \cos^{-1}(2x-1)\}; \ \ n \geq 0$$

and the recurrence relation is given by

(6) $$T^*{}_{n+1}(x) = 2(2x-1)T^*{}_n(x) - T^*{}_{n-1}(x)\ ; n = 1,2$$

with the initial condition

(7) $$T^*{}_0(x) = 1, T^*{}_1(x) = 2x-1$$

Definition 4.

Orthogonality: Two functions say $u_p(x)$ and $u_q(x)$ defined on the interval $a \leq x \leq b$

are said to be orthogonal if

(8) $< u_p(x), u_q(x) = \int_a^b u_p(x)\, u_q(x)\, dx = 0$

If, on the other hand, a third function $w(x) > 0$ exists such that

(9) $< u_p(x), u_q(x) > = \int_a^b w(x) u_p(x)\, u_q(x)\, dx = 0$

Then, we say that $u_p(x)$ and $u_q(x)$ are mutually orthogonal with respect to the weight function $w(x)$.

Generally, we write:

(10) $\int_a^b w(x) u_p(x)\, u_q(x) = \begin{cases} 0 & p \neq q \\ \int_a^b w(x) u^2{}_p(x) dx & p = q\,, \end{cases}$

Definition 5.

We defined absolute error as:

(11) Absolute Error $= |Y(x) - y_n(x)|; \quad 0 \leq x \leq 1,$

where $Y(x)$ is the exact solution and $y_n(x)$ is the approximate solution.

3. Materials and Methods

In this section, we constructed our orthogonal polynomials using the general weight function of the form: $w(x) = (a + bx^i)^k$.

This corresponds to quartic functions for $a = 1, b = -1, k = 1$ and $i = 4$ respectively, satisfying the orthogonality conditions in the interval $[a, b]$ under consideration.

According to Gram-Schmidt orthogonalization process, the orthogonal polynomial $u_j(x)$ Valid in the interval [a,b] with the leading term x^j, is given as

(12) $u_j(x) = x^j - \sum_{i=0}^{j-1} a_{j,i} u_i(x) \quad i = 0,1,2 \ldots \ldots . j - 1$ and $j \geq 1$

Where $u_j(x)$ is an increasing polynomial of degree j and $u_i(x)$ are the Corresponding values of the approximating functions in x. Then, starting with $u_0(x) = 1$, we find that the linear polynomial $u_j(x)$ with leading term x, is written as

(13) $u_1(x) = x + a_{1,0} u_0(x)$

Where $a_{1,0}$ is a constant to be determined. Since $u_1(x)$ and $u_0(x)$ are orthogonal, we have

(14) $$\int_a^b w(x)u_1(x)u_0(x)\,dx = 0 = \int_a^b xw(x)u_0(x)\,dx + a_{1,0}\int_a^b w(x)u^2{}_0(x)\,dx$$

Using (10) and (14). From the above, we have

(15) $$a_{1,0} = \frac{\int_a^b w(x)xu_0(x)}{\int_a^b w(x)u^2{}_0(x)dx}$$

Hence, substituting (15) into (12) gives

(16) $$u_1(x) = x + \frac{\int_a^b w(x)xu_0(x)}{\int_a^b w(x)u^2{}_0(x)dx}$$

Proceeding in this way, the method is generalized and is written as

(17) $$u_j(x) = x^j + a_{j,0}u_0(x) + a_{j,1}u_1(x) + a_{j,2}u_2(x) + \cdots a_{j,j-1}u_{j-1}(x)$$

where the constants $a_{j,0}$ are so chosen such that $u_j(x)$ is orthogonal to $u_0(x), u_1(x), u_2(x) \ldots\ldots u_{j-1}(x)$. These conditions yield

(18) $$a_{j,i} = -\frac{\int_a^b x^j w(x)u_0(x)}{\int_a^b w(x)u^2{}_0(x)dx}$$

For $k = 1, a = 1, b = -1$ and $i = 4$ valid in $[0, 1]$

(19) $$w(x) = 1 - x^4$$

(20) $$u_0(x) = 1$$

We have $\mathrm{k} = 1, \mathrm{j} = 1$ and$u_0(x) = 1$, we write equation (12) as

(21) $$u_1(x) = x - a_{1,0}u_0(x)$$

Simplifying the above equation, we have

(22) $$u_1(x) = x, u_2(x) = x^2 - \frac{5}{21}$$

The shifted equivalent of the (22) that is valid in $[0, 1]$ are given as:

(23) $$u_0{}^*(x) = 1, u_1{}^*(x) = 2x - 1\,, u_2{}^*(x) = 4t^2 - 4x + \frac{16}{21}$$

In this work the method assumed an approximate solution with the orthogonal polynomial as basis function as

(24) $$u(x) \cong u_n(x) = \sum_{i=0}^{n} a_i u_i^*(x)$$

Where $u_i^*(x)$ denotes the orthogonal polynomial of degree N wherea_i, $\mathrm{i} = 0,1,2,\ldots$ are constants.

4. DEMONSTRATION OF THE PROPOSED METHOD

In this section, we demonstrate the proposed method mentioned above

4.1 Perturbed Least Squares Method (PLSM)

The perturbed least squares method is based on the constructed orthogonal polynomials as basis function and used to find the numerical solution of fractional integro-differential equation given in (1). The basis idea of the perturbed least squares method as conceived by [17] by substituting (24) into a slightly perturbed (1) to obtain

(25) $$D^{\alpha}\left[\sum_{i=0}^{n} a_i u_i^*(x)\right] = f(x) + \int_0^x k(x,t) \sum_{i=0}^{n} a_i u_i^*(t)\, dt + H_n(x)$$

Where,

(26) $$H_n(x) = \sum_{i=1}^{\lceil\alpha\rceil} \tau_i T_{n-i+1}^*(x) \qquad x \in [a,b]$$

And $\lceil\alpha\rceil$ is the smallest integer which is bigger than the real number, which is the order of the fractional integro-differential equation. N is the degree of the approximation, $\tau_i(i1(2)n)$ is free tau parameters to be determined, $T_{n-i+1}^*(x)$ are Chebyshev polynomials defined in (5). Operating $J^{\propto}$ on both sides of (25) as follows:

(27) $$\sum_{i=0}^{n} a_i u_i^*(x) = \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + J^{\propto} f(x) + J^{\propto}\left[\int_0^x k(x,t) \sum_{i=0}^{n} a_i u_i^*(t)\, dt\right] + J^{\propto} \sum_{i=1}^{\lceil\alpha\rceil} \tau_i T_{n-i+1}^*(x)$$

Hence, the residual equation is obtained as

(28) $$R\left(a_0, a_1, \ldots\ldots\ldots, a_n, \tau_1, \tau_2, \ldots \tau_\alpha\right) = \sum_{i=0}^{n} a_i u_i^*(x) - \left\{\sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + J^{\propto} f(x) + J^{\propto}\left[\int_0^x k(x,t) \sum_{i=0}^{n} a_i u_i^*(t)\, dt\right]\right\} + J^{\propto} \sum_{i=1}^{\lceil\alpha\rceil} \tau_i T_{n-i+1}^*(x)$$

Let

(29) $S\left(a_{0,}a_1, \dots \dots \dots, a_n\right) = \int_0^1 \left[R\left(a_{0,}a_1, \dots \dots \dots, a_n, \tau_1, \tau_2, \dots \tau_\alpha\right)\right]^2 w(x)dx$

Where $w(x)$ is the positive weight function defined in the interval, $[\mathrm{a}, \mathrm{b}]$. In this work, we take $w(x) = 1$ for simplicity. Thus,

(30) $S\left(a_{0,}a_1, \dots \dots \dots, a_n\right) = \int_0^1 \left\{\sum_{i=0}^{n} a_i u_i{}^*(x) - \{\sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + J^\propto f(x) + \right.$

$\left.[\int_0^x k(x,t)\sum_{i=0}^{n} a_i u_i{}^*(t)\, dt]\} + J^\propto \sum_{i=1}^{\lceil\alpha\rceil} \tau_i T_{n-i+1}^*(x)\right\}^2 dx$

In order to minimize equation (30), we obtained the values of a_i $(i \geq 0)$ by finding the minimum value of S as:

(31) $$\frac{\partial S}{\partial a_i} = 0, i = 0,1,2 \dots, n$$

(32) $$\frac{\partial S}{\partial \tau_i} = 0, i = 0,1,2 \dots, \lceil\alpha\rceil$$

Applying (31) and (32) on (30) to have

(33) $\int_0^1 \left\{\sum_{i=0}^{n} a_i u_i{}^*(x) - \{\sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!} + J^\propto f(x) + J^\propto[\int_0^x k(x,t)\sum_{i=0}^{n} a_i u_i{}^*(t)\, dt]\} + \right.$

$\left. J^\propto \sum_{i=1}^{\lceil\alpha\rceil} \tau_i T_{n-i+1}^*(x)\right\} dx \times \int_0^1 \left\{u_i{}^*(x) - J^\propto(\int_0^x k(x,t)u_i{}^*(t)dt) + T_{n-i+1}^*(x)\right\}dx$

Thus, (33) are then simplified for $i = 0,1, \dots n$ to obtain $(n + 1 + \lceil\alpha\rceil)$ algebraic system of equations in $(n + 1 + \lceil\alpha\rceil)$ unknown a'_i s which are put in matrix form as follow:

$$A = \begin{pmatrix} \int_0^1 R(x,a_0)h_0 dx & \int_0^1 R(x,a_1)h_0 dx \cdots & \int_0^1 R(x,a_n)h_0 dx & \int_0^1 R(x,\tau_1)h_0 dx \dots & \int_0^1 R\left(x,\tau_{\lceil\alpha\rceil}\right)h_0 dx \\ \int_0^1 R(x,a_0)h_1 dx & \int_0^1 R(x,a_1)h_1 dx \cdots & \int_0^1 R(x,a_n)h_1 dx & \int_0^1 R(x,\tau_1)h_1 dx \dots & \int_0^1 R\left(x,\tau_{\lceil\alpha\rceil}\right)h_1 dx \\ \vdots & \vdots & \ddots & \vdots & \\ \int_0^1 R(x,a_0)h_n dx & \int_0^1 R(x,a_1)h_n dx \dots & \int_0^1 R(x,a_n)h_n dx & \int_0^1 R(x,\tau_1)h_n dx \dots & \int_0^1 R\left(x,\tau_{\lceil\alpha\rceil}\right)h_n dx \end{pmatrix},$$

(34)

(35) $$B = \begin{pmatrix} \int_0^1 \left[J^\propto f(x) + \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}\right] h_0 dx \\ \int_0^1 \left[J^\propto f(x) + \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}\right] h_1 dx \\ \vdots \\ \int_0^1 \left[J^\propto f(x) + \sum_{k=0}^{m-1} u^k(0)\frac{x^k}{k!}\right] h_n dx \end{pmatrix}$$

where $h_i = u_i{}^*(x) - J^{\propto}[\int_0^x k(x,t)u_i{}^*(t)dt + T^*_{n-i+1}(x)], i = 0,1,\dots,n$

$$R(x,a_i) = \sum_{i=0}^{n} a_i u_i{}^*(x) - J^{\propto}[\int_0^x k(x,t)\sum_{i=0}^{n} a_i u_i{}^*(t)\,dt + \sum_{i=1}^{\lceil\alpha\rceil} \tau_i T^*_{n-i+1}(x)]\,, i = 0,1,\dots,n$$

The $(\mathrm{n}+1\lceil\alpha\rceil)$ linear equation are then solved using Gaussian elimination method or any suitable computer package like maple 18 to obtain the unknown constants $a_i(i = 0(1)(n + \lceil\alpha\rceil)$ and $\tau_{\lceil\alpha\rceil}$which are then substituted back into the assumed approximate solution to give the required approximation solution

4.2 Numerical Examples

In this section, we show some examples the method discussed above on the general integration - differential equations. The problems are solved using the constructed orthogonal polynomials as basic function. The examples are solved to illustrate the computational cost accuracy and efficiency of the proposed methods using Maple 18.

Example 1**:** Consider the following fractional Integro-differential [16]:

$$D^{\frac{3}{4}}u(x) = -\frac{x^2e^x}{5}u(x) + \frac{6x^{2.25}}{\Gamma(3.25)} + e^x\int_0^x tu(t)dt \tag{36}$$

Subject to $u(0) = 0$. The exact solution is

$$u(x) = x^3 \tag{37}$$

Applying the above method on (36) , we got the exact solution as:

$$u(x) = x^3 \tag{38}$$

Example 2: Consider the following fractional Integro-differential [7]:

$$D^{\frac{1}{2}}u(x) = u(x) + \frac{8x^{2.25}}{3\Gamma(0.5)} - x^2 - \frac{1}{2}x^3 + \int_0^x tu(t)dt \tag{39}$$

Subject to $u(0) = 0$. The exact solution is

$$U(x) = x^2 \tag{40}$$

Applying the above method on (39) to have the required approximate solution as:

$$u(x) = 2.651 \times 10^{-12} + x^2 \tag{41}$$

Example 3: Consider the following fractional Integro-differential [16]:

(42) $D^{\frac{1}{2}}u(x) = (\cos(x) - \sin(x))\,u(x) + f(x) + \int_0^x x\sin(t)\,u(t)dt$

(43) $f(x) = \frac{2x^{1.5}}{\Gamma(2.5)} + \frac{1}{\Gamma(1.5)}x^{0.5} + x(\cos(x) - x\sin(x) + x^2\cos(x))$

Subject to $u(0) = 0$. The exact solution is

(44) $U(x) = x^2 + x$

Applying the above method on (42) to have the required approximate solution as:

(45) $u(x) = -4 \times 10^{-10} + x + x^2$

5. RESULTS

5.1 Tables of Results

Table 1: Numerical results of Example 1.

x	Exact Solution	Approximate Solution	Absolute Error
0.0	0.000	0.00000000000000	0.00E+00
0.1	0.001	0.00100000000000	0.00E+00
0.2	0.008	0.00800000000000	0.00E+00
0.3	0.273	0.02700000000000	0.00E+00
0.4	0.064	0.06400000000000	0.00E+00
0.5	0.125	0.12500000000000	0.00E+00
0.6	0.216	0.21600000000000	0.00E+00
0.7	0.343	0.34300000000000	0.00E+00
0.8	0.512	0.51200000000000	0.00E+00
0.9	0.729	0.72900000000000	0.00E+00
1.0	1.000	1.00000000000000	0.00E+00

Table 2: Numerical results of Example 2.

x	Exact Solution	Approximate Solution	Absolute Error
0.0	0.00	0.00000000000265	2.651E-12
0.1	0.01	0.01000000000000	0.00E+00
0.2	0.04	0.04000000000000	0.00E+00
0.3	0.09	0.09000000000000	0.00E+00
0.4	0.16	0.16000000000000	0.00E+00
0.5	0.25	0.25000000000000	0.00E+00
0.6	0.36	0.36000000000000	0.00E+00
0.7	0.49	0.49000000000000	0.00E+00
0.8	0.64	0.64000000000000	0.00E+00
0.9	0.81	0.81000000000000	0.00E+00
1.0	1.00	1.00000000000000	0.00E+00

Table 3: Numerical results of Example 3.

x	Exact Solution	Approximate Solution	Absolute Error
0.0	0.000	0.00000000000040	4.000E-12
0.1	0.110	0.00000000000000	0.00E+00
0.2	0.240	0.00000000000000	0.00E+00
0.3	0.390-	0.00000000000000	0.00E+00
0.4	0.560	0.00000000000000	0.00E+00
0.5	0.750	0.00000000000000	0.00E+00
0.6	0.960	0.00000000000000	0.00E+00
0.7	1.190	0.00000000000000	0.00E+00
0.8	1.440	0.00000000000000	0.00E+00
0.9	1.710	0.00000000000000	0.00E+00
1.0	2.000	0.00000000000000	0.00E+00

5.2 Graphical Representation of the Method.

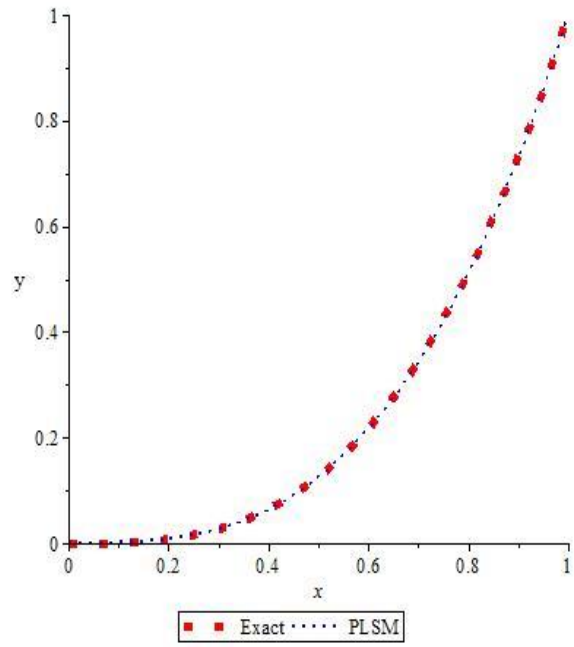

FIGURE 1: The graph of approximation solution and exact of example 1

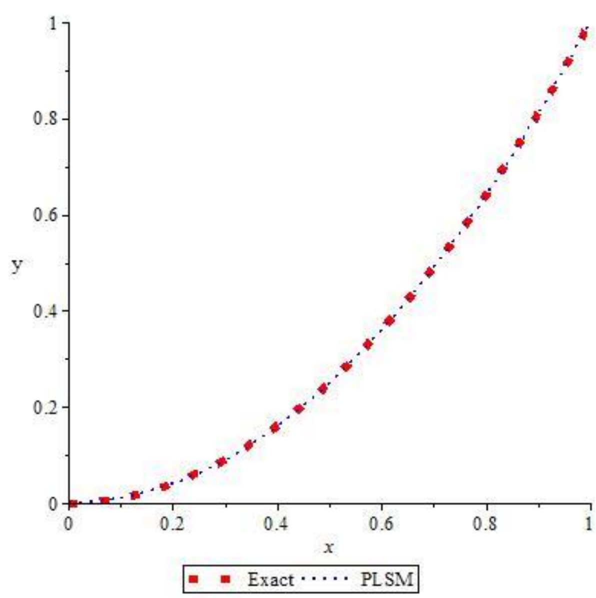

FIGURE 2: The graph of approximation solution and exact of example 2

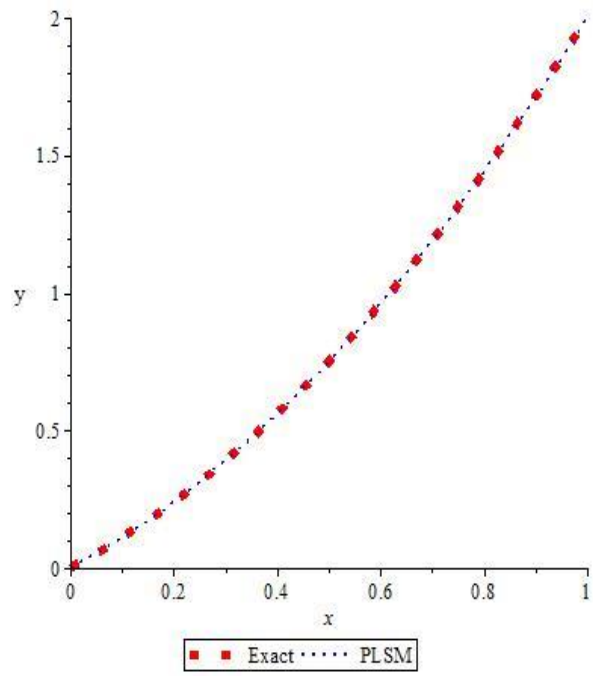

FIGURE 3: The graph of approximation solution and exact of example 3

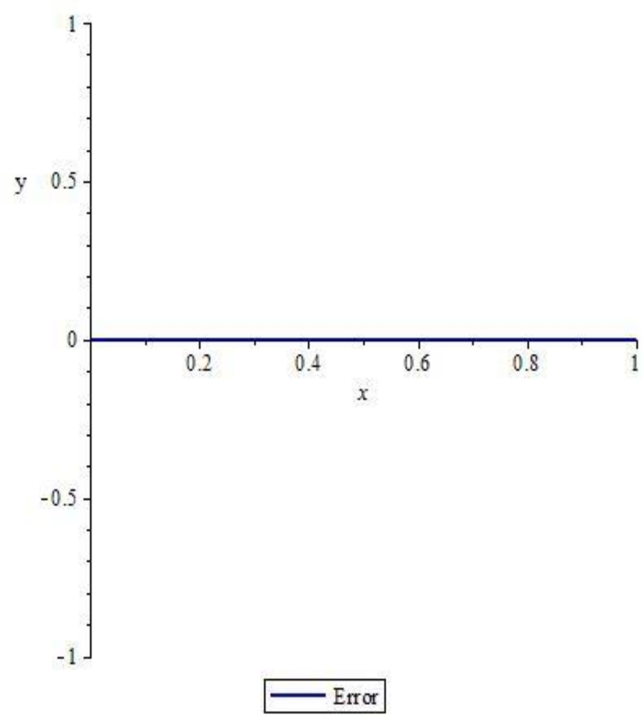

FIGURE 4: The error graph of example 1

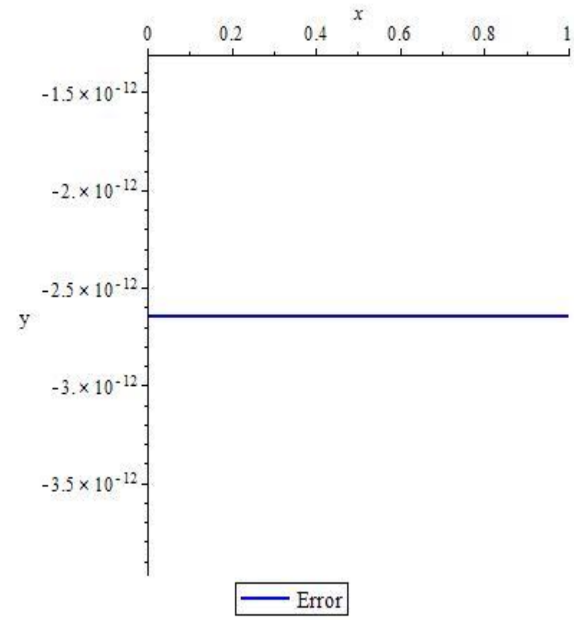

FIGURE 5: The error graph of example 2

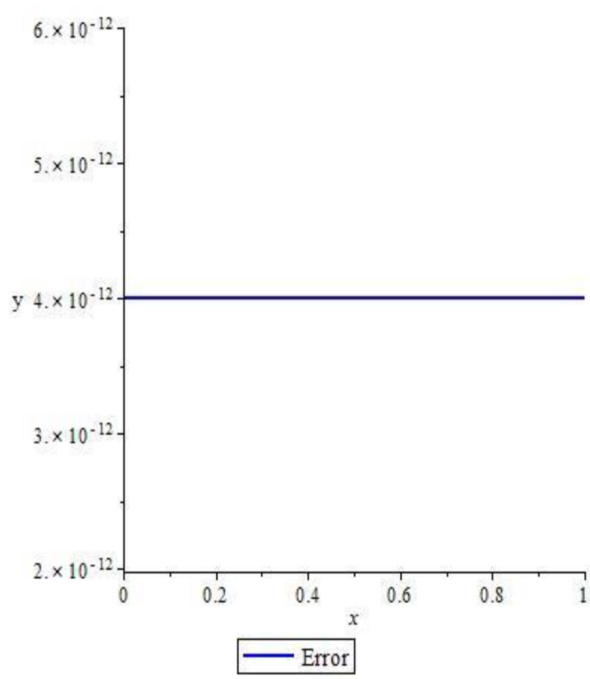

FIGURE 6: The error graph of example 3

6. DISCUSSION OF RESULTS

All the three numerical examples presented in this study were solved using maple 18 software. The Tables of error for the examples shows that the method with the constructed orthogonal polynomials is accurate and converges at the lower numbers of the approximate. Also, for the graphs of the three examples when compared the approximate solution with the exact equations, we have exact equation graphs.

7. CONCLUSION

The study showed that the method with the constructed orthogonal polynomials is successfully used for solving FIDEs in a wide range with three examples. The method gives more realistic series solutions that converge very rapidly in fractional equations. The results obtained showed that the method is powerful when compared with the exact solutions and also show shown that there is a similarity between the exact and the approximate solution. The results showed that PLSM is a powerful and efficient technique to find a very good solution for this type of equation as well as analytical solutions to numerous physical problems in science and engineering. Also, the results were presented in graphical forms to further demonstrate of the method.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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