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MODIFIED SECOND ORDER SLOPE ROTATABLE DESIGNS USING SUPPLEMENTARY DIFFERENCE SETS

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Abstract: In this paper, following the methods of constructions of Mutiso et al. [7-8], Chiranjeevi and Victorbabu [2-3], a new method of construction of modified second order slope rotatable designs using supplementary difference sets is suggested. Some illustrative examples are also presented.

Keywords: response surface designs, second order slope rotatable designs, modified second order slope rotatable designs, supplementary difference sets.

2010 AMS Subject Classification: 62K05.

1. INTRODUCTION

The concept of rotatability, which is very important in response surface designs, was proposed by Box and Hunter [1]. Das and Narasimham [4] developed second and third order rotatable designs and constructed rotatable designs using balanced incomplete block designs (BIBD). Seberry [9] studied some remarks on supplementary difference sets (SDS) and applications of SDS. Koukouvinos et al. [15] suggested a general construction method for five level second order

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rotatable designs (SORD) using SDS. Hader and Park [5] introduced slope rotatability for second order response surface designs and constructed slope rotatable central composite designs. Victorbabu and Narasimham [18] suggested conditions for slope rotatability in any general second order response surface designs and constructed second order slope rotatable designs (SOSRD) using BIBD. Victorbabu and Narasimham [19] constructed three level SOSRD using BIBD. Victorbabu and Narasimham [20] studied SOSRD using pairwise balanced designs. Victorbabu [10] constructed SOSRD using symmetrical unequal block arrangements (SUBA) with two unequal block sizes. Victorbabu [11-12] suggested a new restriction $\sum x_{iu}^4 = N \sum x_{iu}^2 x_{ju}^2$ to get modified slope rotatability for second order response surface designs. Further, they have constructed modified SOSRD using central composite designs and BIBD. Victorbabu [13-14] suggested reviews on second order rotatable and sloped rotatable designs. Victorbabu and Surekha [21] constructed a new method of three level SOSRD using BIBD. Victorbabu [15] suggested a bibliography on slope rotatable designs. Victorbabu [16] suggested a review on SOSRD over axial directions. Victorbabu [17] suggested a note on SOSRD using a pair of partially balanced incomplete block designs. Specially, Mustio et al. [7-8] suggested five level second order rotatable and modified second order rotatable designs using SDS. Chiranjeevi and Victorbabu [2] studied measure of slope rotatability for second order response surface designs and constructed measure of SOSRD using SDS. Chiranjeevi and Victorbabu [3] suggested a method of construction of SOSRD using SDS.

In this paper, following the methods constructions of Mutiso et al. [7-8], Chiranjeevi and Victorbabu [2-3], a new method of construction of modified second order slope rotatable designs using supplementary difference sets is suggested. Some illustrative examples are also presented.

2. PRELIMINARIES

Conditions for second order slope rotatable designs

Suppose we want to use the second order response surface designs D= ((x_{iu})) to fit the surface,

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i<j}\sum b_{ij} x_{iu} x_{ju} + e_u \tag{2.1}$$

where x_{iu} denotes the level of the i^{th} factor (i=1,2,…,v) in the u^{th} run (u=1,2,…,N) of the experiment and the e_u's are uncorrelated random errors with mean zero and variance σ^2.

A second order response surface design D is said to be SOSRD if the design points satisfy the following conditions (cf. Hader and Park [5], Victorbabu and Narasimham [18]).

$$\sum x_{iu} = 0, \sum x_{iu} x_{ju} = 0, \sum x_{iu} x_{ju}^2 = 0, \sum x_{iu} x_{ju} x_{ku} = 0,$$
$$\sum x_{iu}^3 = 0, \sum x_{iu} x_{ju}^3 = 0, \sum x_{iu} x_{ju} x_{ku}^2 = 0, \sum x_{iu} x_{ju} x_{ku} x_{lu} = 0.$$

for $i \neq j \neq k \neq l$; (2.2)

(i) $\sum x_{iu}^2 = \text{constant} = N\lambda_2$; (ii) $\sum x_{iu}^4 = \text{constant} = cN\lambda_4$; for all i (2.3)

$$\sum x_{iu}^2 x_{ju}^2 = \text{constant} = N\lambda_4; \text{for all } i \neq j \tag{2.4}$$

$$\frac{\lambda_4}{\lambda_2^2} > \frac{v}{(c+v-1)} \tag{2.5}$$

$$\lambda_4\left[v(5-c)-(c-3)^2\right] + \lambda_2^2\left[v(c-5)+4\right] = 0 \tag{2.6}$$

Where c, λ_2 and λ_4 are constants and the summation is over the design points.

The variances and co-variances of the estimated parameters are,

$$V(\hat{b}_0) = \frac{\lambda_4(c+v-1)\sigma^2}{N[\lambda_4(c+v-1)-v\lambda_2^2]},$$

$$V(\hat{b}_i) = \frac{\sigma^2}{N\lambda_2},$$

$$V(\hat{b}_{ij}) = \frac{\sigma^2}{N\lambda_4},$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2}{(c-1)N\lambda_4}\left[\frac{\lambda_4(c+v-2)-(v-1)\lambda_2^2}{\lambda_4(c+v-1)-v\lambda_2^2}\right],$$

$$Cov(\hat{b}_0, \hat{b}_{ii}) = \frac{-\lambda_2\sigma^2}{N[\lambda_4(c+v-1)-v\lambda_2^2]},$$

$$\mathrm{Cov}(\hat{b}_{ii},\hat{b}_{jj})=\frac{(\lambda_2^2-\lambda_4)\sigma^2}{(c-1)N\lambda_4[\lambda_4(c+v-1)-v\lambda_2^2]} \quad \text{and other covariances vanish.} \tag{2.7}$$

Therefore the conditions (2.2) to (2.7) give a set of conditions for slope rotatability in any general second order response surface design.

3. CONDITIONS FOR MODIFIED SECOND ORDER SLOPE ROTATABLE DESIGNS

A second order response surface design D is said to be modified SOSRD that if the design points are satisfy the conditions (2.2) to (2.6) are met (cf. Hader and Park [5], Victorbabu and Narasimham [18] and further we have Victorbabu [11] suggested the conditions of modified variance and covariances of the estimated parameters are also satisfied.

The usual method of construction of SOSRD is to take combinations with unknown constants, associate a 2^m factorial combinations or suitable fraction of it with factors each at ±1 levels to make the level codes equidistant. All such combinations form a design. Generally SOSRD need at least five levels (suitably coded) at 0, ±1, ±b for all factors ((0,0,…,0) chosen center of the design, unknown level 'b' to be chosen suitably to satisfy slope rotatability). Generation of design points this way ensures satisfaction of all the conditions even though the design points contain unknown levels.

Alternatively by putting some restrictions indicating some relation among $\sum x_{iu}^2, \sum x_{iu}^4$ and $\sum x_{iu}^2 x_{ju}^2$ some equations involving the unknowns are obtained and their solution gives the unknown levels. In SOSRD the restriction used is $V(b_{ij})=4V(b_{ii})$ viz. equation (2.6). Other restrictions are also possible though, it seems, not exploited well. We shall investigate the restriction $(\sum x_{iu}^2)^2=N\sum x_{iu}^2 x_{ju}^2$ i.e., $(N\lambda_2)^2=N(N\lambda_4)$ i.e., $\lambda_2^2=\lambda_4$ to get modified SOSRD. By applying the new restriction in equation (6), we get c=1 or c=5. The non-singularity condition (2.5) leads to c=5. It may be noted $\lambda_2^2=\lambda_4$ and c=5 are equivalent conditions.

The variances and co-variances of the estimated parameters are,

$$V(\hat{b}_0)=\frac{(m+4)\sigma^2}{4N}$$

$$V(\hat{b}_i)=\frac{\sigma^2}{N\sqrt{\lambda_4}}$$

$$V(\hat{b}_{ij})=\frac{\sigma^2}{N\lambda_4}$$

$$V(\hat{b}_{ii})=\frac{\sigma^2}{4N\lambda_4}$$

$Cov(\hat{b}_0,\hat{b}_{ii})=\dfrac{-\sigma^2}{4N\sqrt{\lambda_4}}$ and other co-variances are zero

$$V\left(\frac{\partial\hat{Y}}{\partial X_i}\right)=\left[\frac{\sqrt{\lambda_4}+d^2}{N\lambda_4}\right]\sigma^2 \tag{3.1}$$

Therefore the conditions (2.2) to (2.6) and (3.1) give a set of conditions for modified slope rotatability in any general second order response surface design.

4. MAIN RESULTS

Construction of five level second order rotatable designs using supplementary difference sets (cf. Koukouvinos et al. [15])

Supplementary difference sets: Seberry (1973) defined supplementary difference sets and stated that the parameters $[v,k_1,k_2,...,k_e;\lambda]$ SDS satisfy

$$\lambda(v-1)=\sum_{i=1}^{e}k_i(k_i-1). \tag{4.1}$$

If $k_1=k_2=...=k_e=k$, then $e\text{-}[v;k;\lambda]$ to denote the SDS and equation (4.1) becomes

$$\lambda(v-1)=ek(k-1)$$

Result (i): Let $C_1,C_2,...,C_e$ be 2-subset of Z_v (or any finite abelian group of order v), where $v=n-1=2e+1, C_i=\{i,v-i\}, i=1,2,...,\frac{(v-1)}{2}=1,2,...,e.$ Then the sets $C_1,C_2,...,C_e$ will be an $e\text{-}[v;2;1]$ SDS. Based on these SDS, Koukouvinos et al. [15] constructed SORD in m-factors, constitute of a factorial part with level combinations $(-1,1,0)$ plus a set of $2m$ axial points at a distance b from the origin, following the steps given below.

- First consider an $e\text{-}\{v;2;1\}$, SDS, where $m=\frac{(v-1)}{2}$. Suppose, A is the incidence matrix of the $e\text{-}\{v;2;1\}$, SDS and take the mirror image of A, i.e., replace 0 with 1and 1 with 0.
- Consider the first $\frac{(v-1)}{2}$ columns of A. An array with e rows and e columns, where
- $e=\frac{(v-1)}{2}$, is obtained, whose every column has one zero element and $e-1$ elements equal to 1.
- Superimpose a 2^{e-r} factorial fraction onto the units of each row of the array, while onto the zero elements superimpose $2^{e-r}\times 1$ vector with all elements zero. In this way, a three level design with e factors and $(e-1)\times 2^{e-r}$ runs is obtained.
- Add an axial point $\pm b$ in every column of the design in order to attain the rotatability of the design; b must be equal to $a^{1/4}$, where $a=(2e-5)\times 2^{e-r-1}$.

Further, Koukouvinos et al. [15] stated that, it was convenient to choose to use the smallest fraction of 2^e factorial, so the resulting design has the minimum possible number of runs. However, for more than three factors, it is necessary to use fractions of resolution V in order to attain the rotatability of the design.

Result (ii): Let a supplementary difference set with parameters $e\text{-}\{v;2;1\}$, where $e=\frac{(v-1)}{2}$. Then, Koukouvinos et al. [15] suggested SORD with $m=\frac{(v-1)}{2}$ factors at five levels $(\pm 1,0,\pm b)$ and $N=m2^{t(m)}+n_a$ design points, where $2^{t(m)}$ denotes resolution–V fractional factorial design replicate of 2^m in ± 1 levels, and n_a is number of axial points.

5. PROPOSED METHOD OF CONSTRUCTION OF MODIFIED SOSRD USING SUPPLEMENTARY DIFFERENCE SETS

Following Koukouvinos et al. [15], Mutiso et al. [7], Chiranjeevi and Victorbabu [3] methods of construction of SORD and SOSRD using SDS, here a new method of construction of modified SOSRD using SDS is suggested. Let a supplementary difference set with parameters $e\text{-}\{v;2;1\}$, where $e=\frac{(v-1)}{2}$.

Here, we suggest to construct a SOSRD with $m=\frac{(v-1)}{2}$ factors at five levels $(\pm1,0,\pm b)$ and $N=m2^{t(m)}+2n_a m+n_0$ design points, where $2^{t(m)}$ denotes a resolution-V fractional factorial design replicate of 2^m in ±1 levels, n_a denote axial points, n_0 denote the number of central points and U denotes the combination of the design points generated from different sets of points.

Theorem (5.1): The design points, $\left[1\text{-}\left(v,k,\lambda\right)\right]2^{t(m)}Un_a\left(b,0,0,...,0\right)2^1U\left(n_0\right)$ will give a v- dimensional modified SOSRD using SDS in $N=\frac{\left(2^{t(m)}(e-1)+2n_a b^2\right)^2}{2^{t(m)}(e-2)}$ design points if,

$$b^4=\frac{2^{t(m)-1}(5(e-2)-(e-1))}{n_a}, \tag{5.2}$$

$$n_0=\frac{(2^{t(m)}(e-1)+2n_a b^2)^2}{2^{t(m)}(e-2)}-(m2^{t(m)}+2n_a m) \tag{5.3}$$

Proof: From the design points generated from the SDS simple symmetry conditions of (2.2), (2.3), (2.4), (2.5) and (2.6) are true condition (2.2) is true obviously. Conditions (2.3), (2.4), (2.5) and (2.6) are true as follows.

$$\sum x_{iu}^2 = 2^{t(m)}(e-1)+2b^2 = N\lambda_2, \tag{5.4}$$

$$\sum x_{iu}^4 = 2^{t(m)}(e\text{-}1) + 2b^4 = cN\lambda_4 , \tag{5.5}$$

$$\sum x_{iu}^2 x_{ju}^2 = 2^{t(m)}(e\text{-}2) = N\lambda_4 \tag{5.6}$$

The modified condition $\lambda_2^2 = \lambda_4$, leads to N (Alternatively N may be obtained directly as $N = m2^{t(m)} + 2n_a m + n_0$, where n_0 is given in (5.3). Equation (5.5) and (5.6) leads to b^4 given in equation (5.2).

Example: We illustrate the construction of modified SOSRD for 4- factors with the help of a SDS (v=9, k=2, λ=1). The design points,

$$\left[4\text{-}(9,2,1)\right] 2^{t(m)} \,U\, n_a \left(b,0,0,...,0\right) 2^1 \,U \left(n_0 = 81\right) ,$$

will give a modified SOSRD using SDS in $N = 169$ design points for 4 factors. Here equations (5.4), (5.5) and (5.6) are

$$\sum x_{iu}^2 = 24 + 2n_a b^2 = N\lambda_2 \tag{5.7}$$

$$\sum x_{iu}^4 = 24 + 2n_a b^4 = 5N\lambda_4 \tag{5.8}$$

$$\sum x_{iu}^2 x_{ju}^2 = 16 = N\lambda_4 \tag{5.9}$$

Equations (5.8) and (5.9) leads to $n_a b^4 = 28$, which implies $b^2 = 2$ for $n_a = 7$. From equations (5.7) and (5.9) using the modified condition $(\lambda_2^2 = \lambda_4)$, with $b^2 = 2$ and $n_a = 7$, we get N=169. Equation (5.3) leads to $n_0 = 81$.

The illustrate examples of modified SOSRD using SDS are given bellow in the following table.

Table: A list of modified SOSRD using SDS

m-(v, k, λ)	t(m)	n_a	b^2	n_o	N	$V(\frac{\partial \hat{Y}}{\partial X_i})\sigma^2$
3-(7,2,1)	2	6	1	16	64	$(0.0442+0.125d^2)$
4-(9,2,1)	3	7	2	81	169	$(0.0192+0.0473d^2)$
9-(19,2,1)	4	6	6	105	357	$(0.0050+0.2241\ d^2)$
10-(21,2,1)	4	2	12	88	288	$(0.0052+0.0278\ d^2)$

6. CONCLUSION

In this paper modified second order slope rotatable designs using supplementary difference sets is suggested. Here we may point out that this new method modified second order slope rotatable designs using supplementary difference sets has 288 design points for 10–factors, whereas the corresponding modified SOSRD using BIBD obtained by Victorbabu [12] needs 361 design points. Thus the new method leads to 10-factors modified second order slope rotatable designs using supplementary difference sets in less number of design points then the corresponding modified SOSRD using BIBD.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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