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CHARACTERIZATIONS OF INTRA-REGULAR SEMIRINGS BY (m,n)-INTERIOR IDEALS

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Abstract. We give the concept of an (m,n)-interior ideal of a semiring, and we characterize an intra-regular semiring by (m,n)-interior ideals. In addition, we show that every (m,n)-interior ideal and both m-left ideal and n-right ideal coincide in an intra-regular semiring.

Keywords: (m,n)-quasi-ideal; m-bi-ideal; (m,n)-interior ideal; intra-regular semiring.

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1. INTRODUCTION

The concept of quasi-ideals was introduced for semigroups, cf. [13]. Iseki [5] described some characterizations of quasi-ideals for semirings without a zero element. Later, Donges [3] considered quasi-ideals of semirings with an absorbing zero element and studied some of their properties. Then, Chinram [2] defined a generalization of quasi-ideals of semirings named (m,n)-quasi-ideals and investigated its properties and using their (m,n)-quasi-ideals.

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The concept of regular semirings was introduced by Zeleznekow [15]. Afterward, Shabir, Ali, and Batool [12] presented some properties of quasi-ideals and used them to characterize regular semirings. A generalization of bi-ideals of semirings named m-bi-ideals was introduced by Munir and Shafiq [8]. Moreover, they presented the form of the m-bi-ideal generated by a nonempty subset of semirings.

The purpose of this study is to define (m,n)-interior ideals in semirings. Then, we give some characterizations of intra-regular semirings by m-left ideals, n-right ideals, $\max\{m,n\}$-bi-ideals, (m,n)-quasi-ideals and (m,n)-interior ideals. Moreover, we show that every (m,n)-interior ideal and both m-left ideal and n-right ideal coincide in an intra-regular semiring.

2. PRELIMINARIES

A *semiring* $(S,+,\cdot)$ is a triple consisting of a nonempty set S and two binary operations $+$ and $\cdot$ on S such that $(S,+)$ and $(S,\cdot)$ are semigroups which are connected by the distributive law. From now on, we shall simply write ab instead of $a\cdot b$ for all $a,b\in S$. A nonempty subset T of a semiring S is called a *subsemiring* of S if T is a semiring with respect to the same binary operations of S. A nonempty subset A of a semiring S is called a *left ideal* (resp., *right ideal*) of S if $A+A\subseteq A$ and $SA\subseteq A$ (resp., $AS\subseteq A$). If A is both a left and a right ideal of S, then A is called an *ideal* of S. A semiring S is called *additively commutative* if $a+b=b+a$, for all $a,b\in S$. An element 0 of a semiring S is called *absorbing zero* if $0+x=x=x+0$ and $0x=0=x0$, for all $x\in S$.

Throughout this paper, we assume that every semiring is an additively commutative semiring with absorbing zero and also write S instead of a semiring $(S,+,\cdot)$.

Let A and B be nonempty subsets of S and $a\in S$. Then we denote the following notations:

$$A^n = AA\cdots A \ (n \text{ times}), \text{where } n\in\mathbb{N};$$

$$\Sigma A = \{\sum_{i\in I} a_i \mid a_i\in A \text{ and } I \text{ is a finite subset of } \mathbb{N}\};$$

$$\Sigma AB = \{\sum_{i\in I} a_ib_i \mid a_i\in A, b_i\in B \text{ and } I \text{ is a finite subset of } \mathbb{N}\};$$

$$\Sigma a = \Sigma\{a\}, \text{ for every } a \in S;$$

$$\sum_{i\in\emptyset} a_i = 0, \text{ for every } a_i \in S.$$

Next, we present about some necessary basic properties of a semiring S which occurred in [14] as follows.

Remark 2.1. Let A and B be nonempty subsets of a semiring S. Then the following statements hold:

(i) $A \subseteq \Sigma A$ and $\Sigma(\Sigma A) = \Sigma A$;

(ii) if $A \subseteq B$, then $\Sigma A \subseteq \Sigma B$;

(iii) $A(\Sigma B) \subseteq (\Sigma A)(\Sigma B) \subseteq \Sigma AB$ and $(\Sigma A)B \subseteq (\Sigma A)(\Sigma B) \subseteq \Sigma AB$;

(iv) $\Sigma A(\Sigma B) \subseteq \Sigma AB$ and $\Sigma(\Sigma A)B \subseteq \Sigma AB$;

(v) $\Sigma(A+B) \subseteq \Sigma A + \Sigma B$.

Lemma 2.2. *Let A be a subset of a semiring S. If $A \subseteq \Sigma A^2 + \Sigma SA^2 + \Sigma A^2S + \Sigma SA^2S$, then $A \subseteq \Sigma SA^2S$.*

Proof. Assume that $A \subseteq \Sigma A^2 + \Sigma A^2S + \Sigma SA^2 + \Sigma SA^2S$. Then

$$\begin{aligned}
\Sigma A^2 &\subseteq \Sigma A(\Sigma A^2 + \Sigma A^2S + \Sigma SA^2 + \Sigma SA^2S) \\
&\subseteq \Sigma AA^2 + \Sigma AA^2S + \Sigma ASA^2 + \Sigma ASA^2S \\
&\subseteq \Sigma SA^2 + \Sigma SA^2S + \Sigma SA^2 + \Sigma SA^2S \\
&= \Sigma SA^2 + \Sigma SA^2S, \qquad (1) \\
\Sigma A^2 &\subseteq \Sigma(\Sigma A^2 + \Sigma A^2S + \Sigma SA^2 + \Sigma SA^2S)A \\
&\subseteq \Sigma A^2A + \Sigma A^2SA + \Sigma SA^2A + \Sigma SA^2SA \\
&\subseteq \Sigma A^2S + \Sigma A^2S + \Sigma SA^2S + \Sigma SA^2S \\
&= \Sigma A^2S + \Sigma SA^2S. \qquad (2)
\end{aligned}$$

By (2), we have

$$\begin{aligned} \Sigma SA^2 &\subseteq \Sigma S(\Sigma A^2 S + \Sigma SA^2 S) \\ &\subseteq \Sigma SA^2 S + \Sigma SSA^2 S \\ &\subseteq \Sigma SA^2 S. \qquad (3) \end{aligned}$$

By (1), we have

$$\begin{aligned} \Sigma A^2 S &\subseteq (\Sigma SA^2 + \Sigma SA^2 S)S \\ &\subseteq \Sigma SA^2 S + \Sigma SA^2 SS \\ &\subseteq \Sigma SA^2 S. \qquad (4) \end{aligned}$$

By (1) and (3), we have

$$\begin{aligned} \Sigma A^2 &\subseteq \Sigma SA^2 + \Sigma SA^2 S \\ &\subseteq \Sigma SA^2 S + \Sigma SA^2 S \\ &= \Sigma SA^2 S. \qquad (5) \end{aligned}$$

By (3), (4), (5) and assumption, we have

$$\begin{aligned} A &\subseteq \Sigma A^2 + \Sigma A^2 S + \Sigma SA^2 + \Sigma SA^2 S \\ &\subseteq \Sigma SA^2 S + \Sigma SA^2 S + \Sigma SA^2 S + \Sigma SA^2 S \\ &= \Sigma SA^2 S. \end{aligned}$$

Therefore, $A \subseteq \Sigma SA^2 S$. □

A nonempty subset A of a semiring S is called a *left ideal* (resp., *right ideal*) of S if $A + A \subseteq A$ and $SA \subseteq A$ (resp., $AS \subseteq A$). If A is both a left and a right ideal of S, then A is called an *ideal* of S.

A nonempty subset Q of a semiring S is called a *quasi-ideal* [13] of S if $Q + Q \subseteq Q$ and $(\Sigma SQ) \cap (\Sigma QS) \subseteq Q$. A subsemiring B of a semiring S is called a *bi-ideal* [6] of S if $BSB \subseteq B$.

We note that every left ideal and right ideal of a semiring S is a quasi-ideal, while every quasi-ideal is a bi-ideal of a semiring S. A subsemiring I of a semiring S is called an *interior ideal* [7] of S if $SIS \subseteq I$.

Let $m,n \in \mathbb{N}$. The following definition is a special case of Definition 3.2 in [10]. A subsemiring A of a semiring S is called an *m-left ideal* (resp., *n-right ideal*) [10] of S if $S^m A \subseteq A$ (resp., $AS^n \subseteq A$). A subsemiring Q of a semiring S is called an (m,n)*-quasi-ideal* [2] of S if $(\Sigma S^m Q) \cap (\Sigma Q S^n) \subseteq Q$. A subsemiring B of a semiring S is said to be an *m-bi-ideal* [8] of S if $BS^m B \subseteq B$.

Lemma 2.3. *Every m-left ideal or n-right ideal of a semiring S is an (m,n)-quasi-ideal of S.*

Proof. Assume that Q is an m-left ideal of a semiring S. It is clear that $Q+Q \subseteq Q$. Next, we consider $(\Sigma S^m Q) \cap (\Sigma Q S^n) \subseteq \Sigma S^m Q \subseteq \Sigma Q \subseteq Q$. Hence, Q is an (m,n)-quasi-ideal of S. For the case Q is an n-right ideal, we can prove similar. □

The converse of Lemma 2.3 is not true as show by the following example.

Example 2.4. Let $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a,b,c,d \in \mathbb{N} \cup \{0\} \right\}$. Then S together with the usual addition and multiplication of matrices is a semiring. Let

$$Q = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} \mid x \in \mathbb{N} \cup \{0\} \right\}.$$

It is clear that Q is a subsemiring of S. We consider

$$\Sigma S^3 Q = \left\{ \begin{bmatrix} 0 & x_1 \\ 0 & x_2 \end{bmatrix} \mid x_1, x_2 \in \mathbb{N} \cup \{0\} \right\} \nsubseteq Q,$$

$$\Sigma Q S^2 = \left\{ \begin{bmatrix} 0 & 0 \\ y_1 & y_2 \end{bmatrix} \mid y_1, y_2 \in \mathbb{N} \cup \{0\} \right\} \nsubseteq Q.$$

It follows that

$$(\Sigma S^3 Q) \cap (\Sigma Q S^2) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & x \end{bmatrix} \mid x \in \mathbb{N} \cup \{0\} \right\} = Q.$$

Therefore, Q is a $(3,2)$-quasi-ideal of S, but Q is not a 3-left ideal and 2-right ideal of S.

Lemma 2.5. *Every (m,n)-quasi-ideal of a semiring S is a $\max\{m,n\}$-bi-ideal of S.*

Proof. Assume that B is an (m,n)-quasi-ideal of a semiring S. Then, B is a subsemiring of S. We consider

$$BS^{\max\{m,n\}}B \subseteq BS^mB \subseteq \Sigma BS^mB \subseteq \Sigma S^{m+1}B \subseteq \Sigma S^mB,$$

$$BS^{\max\{m,n\}}B \subseteq BS^nB \subseteq \Sigma BS^nB \subseteq \Sigma BS^{n+1} \subseteq \Sigma BS^n.$$

This implies that $BS^{\max\{m,n\}}B \subseteq (\Sigma S^mB) \cap (\Sigma BS^n) \subseteq B$. Hence, B is a $\max\{m,n\}$-bi-ideal of S. □

The converse of Lemma 2.5 is not true as show by the following example.

Example 2.6. Let $S = \left\{ \begin{bmatrix} 0 & u & v & w \\ 0 & 0 & x & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid u,v,w,x,y,z \in \mathbb{N} \cup \{0\} \right\}$. Then $(S,+,\cdot)$ is a semiring under usual the matrix addition and the matrix multiplication. Let

$$B = \left\{ \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid a,b \in \mathbb{N} \cup \{0\} \right\}.$$

It is not difficult to check that B is a subsemiring of S. Then B is a 2-bi-hyperideal of S, that is, $BS^2B \subseteq B$, see in [8], but B is not a $(2,1)$-quasi-ideal, because

$$(\Sigma S^2B) \cap (\Sigma BS) = \left\{ \begin{bmatrix} 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \mid c \in \mathbb{N} \cup \{0\} \right\} \nsubseteq B.$$

For any nonempty subset A of a semiring S, we denote $L_m(A), R_n(A), Q_{(m,n)}(A)$ and $B_m(A)$ as *the m-left ideal*, *the n-right ideal*, *the* (m,n)*-quasi-ideal* and *the m-bi-ideal of S generated by A*, respectively. If $A = \{a\}$, we define $L_m(a) = L_m(\{a\}), R_n(a) = R_n(\{a\}), Q_{(m,n)}(a) = Q_{(m,n)}(\{a\})$ and $B_m(a) = B_m(\{a\})$. Then we have the following lemma.

Lemma 2.7. [14] *Let A be a nonempty subset of a semiring S. Then the following statements hold:*

(i) $L_m(A) = \Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A$;

(ii) $R_n(A) = \Sigma A + \Sigma A^2 + \cdots + \Sigma A^n + \Sigma A S^n$;

(iii) $Q_{(m,n)}(A) = \Sigma A + \Sigma A^2 + \cdots + \Sigma A^{\max\{m,n\}} + ((\Sigma S^m A) \cap (\Sigma A S^n))$;

(iv) $B_m(A) = \Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+1} + \Sigma A S^m A$.

Corollary 2.8. *Let S be a semiring and $a \in S$. Then the following statements hold:*

(i) $L_m(a) = \Sigma a + \Sigma a^2 + \cdots + \Sigma a^m + \Sigma S^m a$;

(ii) $R_n(a) = \Sigma a + \Sigma a^2 + \cdots + \Sigma a^n + \Sigma a S^n$;

(iii) $Q_{(m,n)}(a) = \Sigma a + \Sigma a^2 + \cdots + \Sigma a^{\max\{m,n\}} + ((\Sigma S^m a) \cap (\Sigma a S^n))$;

(iv) $B_m(a) = \Sigma a + \Sigma a^2 + \cdots + \Sigma a^{m+1} + \Sigma a S^m a$.

3. MAIN RESULTS

In this section, we define the concept of (m,n)-interior ideals in semirings and give characterizations of intra-regular semirings by their (m,n)-interior ideals.

Definition 3.1. [1] Let S be a semiring. An element $a \in S$ is said to be *intra-regular* if $a \in \Sigma S a^2 S$. If every element $a \in S$ is intra-regular, then S is called an *intra-regular semiring*.

We note that S is an intra-regular semiring if and only if $A \subseteq \Sigma S A^2 S$ for any $\emptyset \neq A \subseteq S$.

Definition 3.2. A subsemiring I of a semiring S is said to be an *(m,n)-interior ideal* of S if $S^m I S^n \subseteq I$, where m and n are positive integers.

It is clear that every interior ideal of a semiring S is an (m,n)-interior ideal. In addition, an (m,n)-interior ideal of a semiring S is a (k,l)-interior ideal of S for all $k,l,m,n \in \mathbb{N}$ such that $k \geq m$ and $l \geq n$.

Lemma 3.3. *Every both m-left ideal and n-right ideal of a semiring S is an (m,n)-interior ideal.*

Proof. Assume that I is both an m-left ideal and an n-right ideal of a semiring S. Then, I is a subsemiring of S. Hence, $S^m I S^n \subseteq S^m I \subseteq I$. □

The converse of Lemma 3.3 is not true as show by the following example.

Example 3.4. Let $S=\{a,b,c,d,e\}$. Define two binary operations $+$ and $\cdot$ on S as follows:

$+$	a	b	c	d	e
a	a	b	c	d	e
b	b	b	b	b	b
c	c	b	b	b	b
d	d	b	b	b	b
e	e	b	b	b	b

and

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	b	b	b	b
c	a	b	b	b	b
d	a	b	b	b	c
e	a	b	b	c	c

Then, S is a semiring [11]. Let $I=\{a,b,d\}$. Clearly, I is a subsemiring of S. Next, we consider

$$S^2IS=\{a,b,c\}\{a,b,d\}\{a,b,c,d,e\}=\{a,b\}\{a,b,c,d,e\}=\{a,b\}\subseteq I.$$

Thus, I is a $(2,1)$-interior ideal of S, but it is not a 1-right ideal of S, since $IS=\{a,b,d\}S=\{a,b,c\}\nsubseteq I$.

Let A be a nonempty subset of a semiring S and $m,n\in\mathbb{N}$. we denote the notation $I_{(m,n)}(A)$ to be *the* (m,n)*-interior ideal of S generated by A*. Now, we describe the forms of the (m,n)-interior ideal of a semiring S generated by a nonempty subset A.

Lemma 3.5. *Let A be a nonempty of a semiring S and $m,n\in\mathbb{N}$. Then*

$$I_{(m,n)}(A)=\Sigma A+\Sigma A^2+\cdots+\Sigma A^{m+n}+\Sigma S^mAS^n.$$

Proof. Let $I=\Sigma A+\Sigma A^2+\cdots+\Sigma A^{m+n}+\Sigma S^mAS^n$. Since $0\in S$ and $A\subseteq\Sigma A$, we have $A\subseteq\Sigma A=\Sigma A+0+0+\cdots+0\subseteq I$. It is clear that I is closed under addition, because S is additively commutative. By Remark 2.1, we obtain that

$$\begin{aligned} I^2 &= (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+n} + \Sigma S^m A S^n)^2 \\ &\subseteq \Sigma AA + \Sigma AA^2 + \cdots + \Sigma AA^{m+n} + \Sigma AS^m AS^n \\ &\quad + \cdots + \Sigma A^{m+n}A + \Sigma A^{m+n}A^2 + \cdots + \Sigma A^{m+n}A^{m+n} + \Sigma A^{m+n}S^m AS^n \\ &\quad + \Sigma S^m AS^n A + \Sigma S^m AS^n A^2 + \cdots + \Sigma S^m AS^n A^{m+n} + \Sigma S^m AS^n S^m AS^n \\ &\subseteq \Sigma A^2 + \Sigma A^3 + \cdots + \Sigma A^{m+n} + \Sigma S^m AS^n \subseteq I. \end{aligned}$$

Thus, I is a subsemiring of S. Again, by Remark 2.1, we obtain that

$$\begin{aligned} S^m IS^n &= S^m(\Sigma A + \Sigma A^2 + \Sigma A^3 + \cdots + \Sigma A^{m+n} + \Sigma S^m AS^n)S^n \\ &\subseteq \Sigma S^m AS^n + \Sigma S^m A^2 S^n + \cdots + \Sigma S^m A^{m+n} S^n + \Sigma S^m S^m AS^n S^n \\ &\subseteq \Sigma S^m AS^n \subseteq I. \end{aligned}$$

Hence, I is an (m,n)-interior ideal of S. Next, let K be an (m,n)-interior ideal of S containing A. It follows that $\Sigma A \subseteq \Sigma K \subseteq K, \Sigma A^2 \subseteq \Sigma K^2 \subseteq \Sigma K \subseteq K, \ldots, \Sigma A^{m+n} \subseteq \Sigma K^{m+n} \subseteq \Sigma K \subseteq K$ and $\Sigma S^m AS^n \subseteq \Sigma S^m KS^n \subseteq \Sigma K \subseteq K$. Also, $I = \Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+n} + \Sigma S^m AS^n \subseteq K$. Therefore, I is the (m,n)-interior ideal of S generated by A, that is, $I_{(m,n)}(A) = I = \Sigma A + \Sigma A^2 + \cdots + \Sigma A^{m+n} + \Sigma S^m AS^n$. □

In a particular of Lemma 3.5, if $A = \{a\}$ then we have the following corollary.

Corollary 3.6. *Let S be a semiring and $a \in S$. Then $I_{(m,n)}(a) = \Sigma a + \Sigma a^2 + \cdots + \Sigma a^{m+n} + \Sigma S^m aS^n$.*

Theorem 3.7. *Let S be an intra-regular semiring. Then (m,n)-interior ideals and both m-left ideals and n-right ideals coincide in S.*

Proof. By Lemma 3.3, it is sufficient to show that every (m,n)-interior ideal is both an m-left ideal and an n-right ideal of S. Assume that I is an (m,n)-interior ideal of S. Then, I is a subsemiring of S. Since S is an intra-regular and by Remark 2.1, we have

$$S^m I \subseteq S^m(\Sigma SI^2 S) \subseteq \Sigma S^{m+2} IS \subseteq \cdots \subseteq \Sigma S^{m+2n} IS^n \subseteq \Sigma S^m IS^n \subseteq I.$$

Hence, I is an m-left ideal of S. Similarly, we can show that I is also an n-right ideal of S. □

Theorem 3.8. *A semiring S is intra-regular if and only if $L \cap R \subseteq \Sigma LR$, for every m-left ideal L and n-right ideal R of S.*

Proof. Assume that S is an intra-regular semiring. Let L be an m-left ideal and R be an n-right ideal of S. If $m \geq n$. Let $a \in L \cap R$. By assumption and Remark 2.1, we obtain that

$$a \in \Sigma Sa^2S \subseteq \Sigma S(\Sigma Sa^2S)aS \subseteq \Sigma S^2a^2S^3 \subseteq \cdots \subseteq \Sigma S^m a^2 S^{2m-1}$$

Thus, $a \in \Sigma S^m a^2 S^{2m-1} \subseteq \Sigma S^m a^2 S^n \subseteq \Sigma S^m LRS^n \subseteq \Sigma LR$. Hence, $L \cap R \subseteq \Sigma LR$. For the case $n \geq m$, we can prove similar to the previous case.

Conversely, let A be a nonempty subset of a semiring S. By assumption, $A \subseteq L_m(A) \cap R_n(A) \subseteq \Sigma L_m(A)R_n(A)$. By Lemma 2.7 and Remark 2.1, we obtain that

$$\begin{aligned}
\Sigma L_m(A)R_n(A) &= \Sigma((\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A) \\
&\quad (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^n + \Sigma AS^n)) \\
&\subseteq \Sigma AA + \Sigma AA^2 + \cdots + \Sigma AA^n + \Sigma AAS^n \\
&\quad + \cdots + \Sigma A^m A + \Sigma A^m A^2 + \cdots + \Sigma A^m A^n + \Sigma A^m AS^n \\
&\quad + \cdots + \Sigma S^m AA + \Sigma S^m AA^2 + \cdots + \Sigma S^m AA^n + \Sigma S^m AAS^n \\
&\subseteq \Sigma A^2 + \Sigma SA^2 + \Sigma A^2 S + \Sigma SA^2 S.
\end{aligned}$$

Thus, $A \subseteq \Sigma A^2 + \Sigma SA^2 + \Sigma A^2S + \Sigma SA^2S$. By Lemma 2.2, $A \subseteq \Sigma SA^2S$. Therefore, S is an intra-regular semiring. □

Now, we give characterizations of intra-regular semirings by their (m,n)-interior ideals.

Theorem 3.9. *Let S be a semiring and $k = \max\{m,n\}$, where $m,n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) *S is intra-regular;*

(ii) *$I_{(m,n)}(a) \cap B_k(a) \cap L_m(a) \subseteq \Sigma L_m(a)B_k(a)I_{(m,n)}(a)$, for all $a \in S$;*

(iii) *$I_{(m,n)}(a) \cap Q_{(m,n)}(a) \cap L_m(a) \subseteq \Sigma L_m(a)Q_{(m,n)}(a)I_{(m,n)}(a)$, for all $a \in S$.*

Proof. $(i) \Rightarrow (ii)$ Assume that S is intra-regular. Let $a \in S$. For any $x \in I_{(m,n)}(a) \cap B_k(a) \cap L_m(a)$, we have

$$
\begin{aligned}
x \in \Sigma S x^2 S &\subseteq \Sigma S(\Sigma S x^2 S)(\Sigma S x^2 S)S \subseteq \Sigma S^2 x^2 S^2 x^2 S^2 \\
&\subseteq \Sigma S^2(\Sigma S x^2 S)(\Sigma S x^2 S)S^2(\Sigma S x^2 S)(\Sigma S x^2 S)S^2 \subseteq \Sigma S^3 x^2 S^8 x^2 S^7 \\
&\subseteq \cdots \subseteq \Sigma S^m x^2 S^m x^2 S^n \subseteq \Sigma S^m x^2 S^m x S^n.
\end{aligned}
$$

Thus, $x \in \Sigma S^m x^2 S^m x S^n \subseteq \Sigma S^m L_m(a) B_k(a) S^m I_{(m,n)}(a) S^n \subseteq \Sigma L_m(a) B_k(a) I_{(m,n)}(a)$. Hence, $I_{(m,n)}(a) \cap B_k(a) \cap L_m(a) \subseteq \Sigma L_m(a) B_k(a) I_{(m,n)}(a)$ for all $a \in S$.

$(ii) \Rightarrow (iii)$ Since every (m,n)-quasi-ideal is a k-bi-ideal of S, statements hold.

$(iii) \Rightarrow (i)$ Let $a \in S$. By assumption, $a \in I_{(m,n)}(a) \cap Q_{(m,n)}(a) \cap L_m(a) \subseteq \Sigma L_m(a) Q_{(m,n)}(a) I_{(m,n)}(a)$. By Corollary 2.8, Corollary 3.6 and Remark 2.1, we have

$$
\begin{aligned}
&\Sigma L_m(a) Q_{(m,n)}(a) I_{(m,n)}(a) \\
&\quad = \Sigma(\Sigma a + \Sigma a^2 + \cdots + \Sigma a^m + \Sigma S^m a) \\
&\qquad (\Sigma a + \Sigma a^2 + \cdots + \Sigma a^{\max\{m,n\}} + ((\Sigma S^m a) \cap (\Sigma a S^n))) \\
&\qquad (\Sigma a + \Sigma a^2 + \cdots + \Sigma a^{m+n} + \Sigma S^m a S^n) \\
&\quad \subseteq \Sigma(\Sigma a + \Sigma a^2 + \cdots + \Sigma a^m + \Sigma S^m a) \\
&\qquad (\Sigma a + \Sigma a^2 + \cdots + \Sigma a^{\max\{m,n\}} + \Sigma S^m a) \\
&\qquad (\Sigma a + \Sigma a^2 + \cdots + \Sigma a^{m+n} + \Sigma S^m a S^n) \\
&\quad \subseteq \Sigma S a^2 + \Sigma a^2 S + \Sigma S a^2 S \\
&\quad \subseteq \Sigma a^2 + \Sigma S a^2 + \Sigma a^2 S + \Sigma S a^2 S.
\end{aligned}
$$

It follows that $a \in \Sigma a^2 + \Sigma S a^2 + \Sigma a^2 S + \Sigma S a^2 S$. By Lemma 2.2, $a \in \Sigma S a^2 S$. Therefore, S is intra-regular. $\square$

Theorem 3.10. *Let S be a semiring and $k = \max\{m,n\}$, where $m,n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) *S is intra-regular;*

(ii) *$I_{(m,n)}(a) \cap B_k(a) \cap R_n(a) \subseteq \Sigma I_{(m,n)}(a) B_k(a) R_n(a)$, for all $a \in S$;*

(iii) $I_{(m,n)}(a) \cap Q_{(m,n)}(a) \cap R_n(a) \subseteq \Sigma I_{(m,n)}(a) Q_{(m,n)}(a) R_n(a)$, *for all* $a \in S$.

Proof. The proof is similar to Theorem 3.9. □

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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