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TRANSIENT ANALYSIS OF A SINGLE SERVER QUEUE WITH DISASTERS AND REPAIRS UNDER BERNOULLI WORKING VACATION SCHEDULE

M. LAKSHMI PRIYA, B. JANANI*

Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology,

Chennai 600062, India

Abstract: An $M/M/1$ queueing model with disasters and repairs under Bernoulli working vacation schedule is considered. In this model, after every completion of service the server may take vacation with probability q or the server may render service to the next customer with probability p. By considering the disaster to occur, only when the server is in busy state, the explicit analytical expressions for time dependent probabilities are derived using Laplace transform and generating function technique.

Keywords: disaster; repair; Bernoulli working vacation.

2010 AMS Subject Classification: 90B15.

1. INTRODUCTION

Queues with disasters are extensively discussed by various researchers. As disaster occurs all customers in the system are removed. This type of situations is seen to prevail in the computer networks (where arrival of virus can be considered as disaster), ATM in a bank, manufacturing systems and so on.

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*Corresponding author

E-mail address: jananisrini2009@gmail.com

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Gelenbe [4] was the first to introduce the concept of arrival of negative customers in the queue. For better understanding the reader may refer to Gelenbe [5], Harrison and Pitel [6], Chao[2], Atencia and Bocharov [1], Kumar and Arivudainambi [10], Kumar and Madheswari [11], Yang et al [15].

Yechiali [16] analysed queue with disaster and impatience. Sudesh [13], Dimou and Economou [3] were some of the remarkable papers in queue with disasters and impatience.

Queue with vacations were studied by many researchers since the late 70's. Reader may look in to the survey paper of Ke et al [9] for recent developments in vacation queueing models. But there are only few articles related to queue with disasters and vacations. Queue with disasters and vacations were first introduced by Mytalas and Zazanis [12]. Also reader may refer Ye et al [7], Kalidass et al [8], Suranga Sampath [14], for better understanding of queues with disasters and vacations.

Due to wide spread applications as well as due to flexibility, Bernoulli vacation was analyzed by many researchers. Practically, the server may opt working vacation after every completion of service depending upon his physical condition. More elaborately, a driver can opt long trip or short trip depending upon his physical condition. Motivated by the above example, in this paper we derived transient probabilities of an $M/M/1$ queue with disasters and repairs under Bernoulli working vacation schedule.

The contents of the paper are arranged as follows.

- Section 2 –Description of the model
- Section 3 – Transient Probabilities
- Section 4 - Conclusion and Future scope of the model

2. MODEL DESCRIPTION

A single server queue with disasters and repairs under Bernoulli vacation schedule is considered. Customers are allowed to join the system according to a Poisson process with the rate λ and service takes place exponentially with the rate μ. Whenever the server completes the service to a customer, the server may choose a working vacation with probability q or the server may continue the service to the next waiting customer with probability p. Also the duration of vacation times follow exponential distribution with parameter η. The disaster occurs during the busy period. After the occurrence of disaster all customers in the system are flushed out and system becomes empty. Meanwhile the repair period starts. Both disaster and repair times are exponentially distributed with parameter α and r respectively.

Number of customers in the system and system states are represented by X(t) and $J(t)$ respectively. Mathematically,

$$J(t) = \begin{cases} 1; \textit{server is in busy state} \\ 0; \textit{server is in working vacation state} \\ 2; \textit{server is in repair state} \\ 3; \textit{server being idle} \end{cases}$$

Hence $(\mathcal{J}(t), \chi(t))$ is a Markov process with state space

$$\Omega = \{(0,0) \cup (3,0) \cup (2,0) \cup (j,n); j = 0,1,2,3 \;\; n = 1,2,....\}.$$

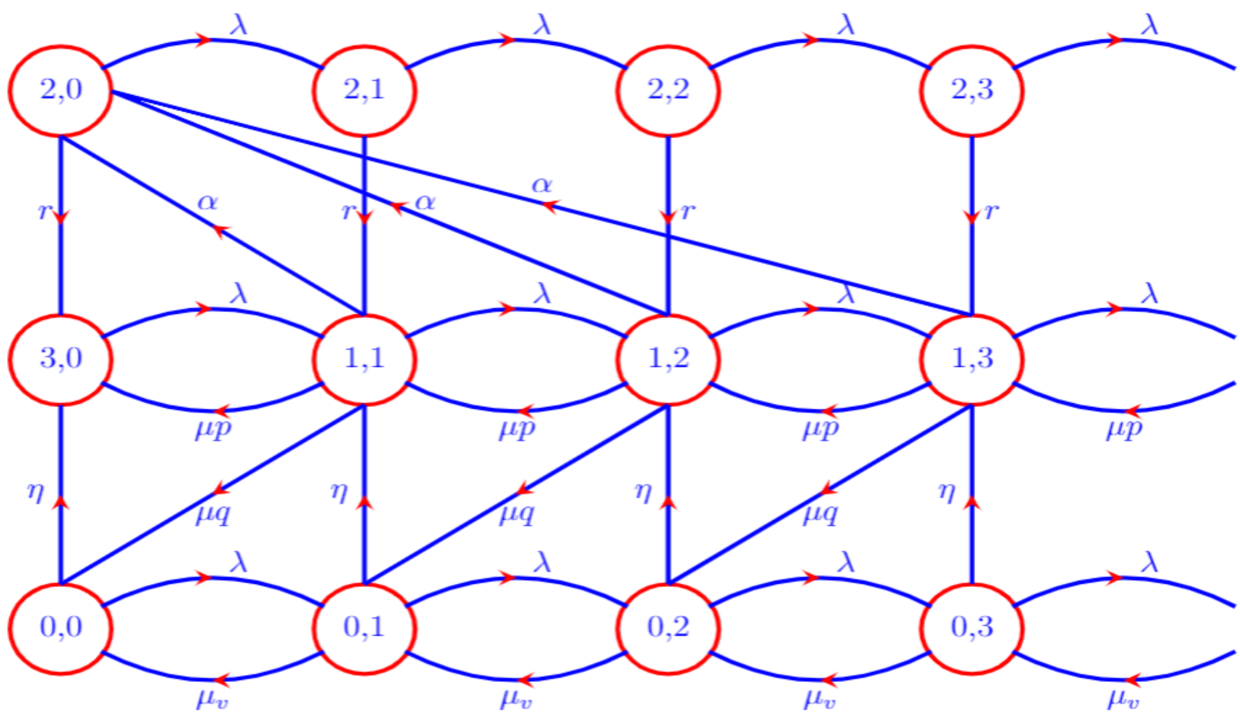

Figure 2.1: State transition diagram of a Single Server Queue with Disasters and Repair under Bernoulli Working Vacation Schedule

Let $P_{j,n}(t)$ denote the time dependent probability for the system to be in state j with n customers at time t. Assume that initially the system is empty and the server is being idle ie., $P_{3,0}(0)=1$. By standard methods, the system of Kolmogorov differential difference equations governing the process are given by

$$P'_{0,0}(t) = -(\lambda+\eta)P_{0,0}(t) + \mu q P_{1,1}(t) + \mu_v P_{0,1}(t), \tag{1}$$

$$P'_{0,n}(t) = -(\lambda+\eta+\mu_v)P_{0,n}(t) + \mu q P_{1,n+1}(t) + \lambda P_{0,n-1}(t) + \mu_v P_{0,n+1}(t), n = 1,2,..., \tag{2}$$

$$P'_{1,1}(t) = -(\lambda+\mu+\alpha)P_{1,1}(t) + \lambda P_{3,0}(t) + \mu p P_{1,2}(t) + \eta P_{0,1}(t) + r P_{2,1}(t), \tag{3}$$

$$P'_{1,n}(t) = -(\lambda+\mu+\alpha)P_{1,n}(t) + \lambda P_{1,n-1}(t) + \mu p P_{1,n+1}(t) + r P_{2,n}(t) + \eta P_{0,n}(t), \quad n = 2,3,..., \tag{4}$$

$$P'_{3,0}(t) = -\lambda P_{3,0}(t) + \mu p P_{1,1}(t) + \eta P_{0,0}(t) + r P_{2,0}(t), \tag{5}$$

$$P'_{2,0}(t) = (\lambda+r)P_{2,0}(t) + \alpha \sum_{n=1}^{\infty} P_{1,n}(t), \tag{6}$$

and

$$P'_{2,n}(t) = -(\lambda+r)P_{2,n}(t) + \lambda P_{2,n-1}(t), n = 1,2,... \tag{7}$$

3. TRANSIENT PROBABILITIES

Define

$$G(z,t) = \sum_{n=1}^{\infty} P_{0,n}(t) z^n.$$

Then,

$$G'(z,t) = \sum_{n=1}^{\infty} P'_{0,n}(t)\, z^n.$$

By substituting equation (2) gives,

$$G'(z,t) - \left(-(\lambda+\eta+\mu_v) + \lambda z + \frac{\mu_v}{z}\right) G(z,t)$$
$$= \lambda z P_{0,0}(t) - \mu_v P_{0,1}(t) + \frac{\mu q}{z} \sum_{n=1}^{\infty} P_{1,n+1}(t) z^{n+1} \quad . \tag{8}$$

Integrating equation (8) with respect to time 't' we get,

$$G(z,t)=\lambda\int_0^t zP_{0,0}(y)\,e^{\left(-(\lambda+\eta+\mu_v)+\lambda z+\frac{\mu_v}{z}\right)(t-y)}dy$$

$$-\mu_v\int_0^t P_{0,1}(y)e^{\left(-(\lambda+\eta+\mu_v)+\lambda z+\frac{\mu_v}{z}\right)(t-y)}dy$$

$$+\mu q\int_0^t\frac{1}{z}\left(\sum_{n=1}^{\infty}\mathrm{P}_{1,n+1}(y)z^{n+1}\right)e^{\left(-(\lambda+\eta+\mu_v)+\lambda z+\frac{\mu_v}{z}\right)(t-y)}dy. \qquad (9)$$

If $\alpha_1=2\sqrt{\lambda\mu_v}$ and $\beta_1=\sqrt{\frac{\lambda}{\mu_v}}$ then

$$e^{\left(\lambda z+\frac{\mu_v}{z}\right)t}=\sum_{n=-\infty}^{\infty}(\beta_1 z)^n I_n(\alpha_1 t).$$

Therefore equation (9) can be written as,

$$G(z,t)=\lambda\int_0^t zP_{0,0}(y)\,e^{-(\lambda+\eta+\mu_v)(t-y)}\sum_{n=-\infty}^{\infty}(\beta_1 z)^n I_n(\alpha_1(t-y))\,dy$$

$$-\mu_v\int_0^t P_{0,1}(y)e^{-(\lambda+\eta+\mu_v)(t-y)}\sum_{n=-\infty}^{\infty}(\beta_1 z)^n I_n(\alpha_1(t-y))\,dy$$

$$+\mu q\int_0^t\frac{1}{z}\left(\sum_{n=1}^{\infty}\mathrm{P}_{1,n+1}(y)z^{n+1}\right)e^{-(\lambda+\eta+\mu_v)(t-y)}\sum_{n=-\infty}^{\infty}(\beta_1 z)^n I_n(\alpha_1(t-y))\,dy$$

$$(10)$$

Equating the coefficient of z^n in equation (10) gives,

$$\mathrm{P}_{0,\mathrm{n}}(t)=\lambda\int_0^t P_{0,0}(y)\,e^{-(\lambda+\eta+\mu_v)(t-y)}\beta_1^{n-1}I_{n-1}\big(\alpha_1(t-y)\big)dy$$

$$-\mu_v\int_0^t P_{0,1}(y)e^{-(\lambda+\eta+\mu_v)(t-y)}\beta_1^{n}I_{n}\big(\alpha_1(t-y)\big)dy$$

$$+\mu q\int_0^t\sum_{m=2}^{\infty}P_{1,m}(y)e^{-(\lambda+\eta+\mu_v)(t-y)}\beta_1^{n-m+1}\,I_{n-m+1}\big(\alpha_1(t-y)\big)dy. \qquad (11)$$

Similarly equating the coefficient of z^{-n} in equation (10) yields,

$$0 = \lambda \int_0^t P_{0,0}(y)\, e^{-(\lambda+\eta+\mu_v)(t-y)} \beta_1^{n-1} I_{n+1}\big(\alpha_1(t-y)\big) dy$$

$$- \mu_v \int_0^t P_{0,1}(y) e^{-(\lambda+\eta+\mu_v)(t-y)} \beta_1^{n} I_n\big(\alpha_1(t-y)\big) dy$$

$$+ \mu q \int_0^t \sum_{m=2}^{\infty} P_{1,m}(y) e^{-(\lambda+\eta+\mu_v)(t-y)} \beta_1^{n-m+1}\, I_{n+m-1}\big(\alpha_1(t-y)\big) dy. \qquad (12)$$

Subtracting equation (12) from (11) gives,

$$\mathrm{P}_{0,\mathrm{n}}(t) = \lambda \int_0^t P_{0,0}(y)\, e^{-(\lambda+\eta+\mu_v)(t-y)} \beta_1^{n-1} \Big(I_{n-1}\big(\alpha_1(t-y)\big) - I_{n+1}\big(\alpha_1(t-y)\big)\Big)\, dy$$

$$+ \mu q \int_0^t \sum_{m=2}^{\infty} P_{1,m}(y) e^{-(\lambda+\eta+\mu_v)(t-y)} \beta_1^{n-m+1} \Big(I_{n-m+1}\big(\alpha_1(t-y)\big)$$

$$- I_{n+m-1}\big(\alpha_1(t-y)\big)\Big)\, dy. \qquad (13)$$

Taking Laplace transform for the equation (13) gives,

$$\widehat{\mathrm{P}}_{0,\mathrm{n}}(s) = \lambda \beta_1^{n-1} \widehat{\mathrm{P}}_{0,0}(s) \left(\frac{P_1 - \sqrt{P_1^2 - \alpha_1^2}}{\alpha_1} \right)^{n-1} \frac{1}{\sqrt{P_1^2 - \alpha_1^2}}$$

$$- \lambda \beta_1^{n-1} \widehat{\mathrm{P}}_{0,0}(s) \left(\frac{P_1 - \sqrt{P_1^2 - \alpha_1^2}}{\alpha_1} \right)^{n+1} \frac{1}{\sqrt{P_1^2 - \alpha_1^2}}$$

$$+ \mu q \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^{n-m+1} \left(\frac{P_1 - \sqrt{P_1^2 - \alpha_1^2}}{\alpha_1} \right)^{n-m+1} \frac{1}{\sqrt{P_1^2 - \alpha_1^2}}$$

$$- \mu q \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^{n-m+1} \left(\frac{P_1 - \sqrt{P_1^2 - \alpha_1^2}}{\alpha_1} \right)^{n+m-1} \frac{1}{\sqrt{P_1^2 - \alpha_1^2}}; n = 1,2 \ldots$$

(14)

where $P_1 = s + \lambda + \eta + \mu_v$.

Substitute $n = 1$ in equation (14) gives,

$$\widehat{\mathrm{P}}_{0,1}(s) = \lambda\widehat{\mathrm{P}}_{0,0}(s)\frac{1}{\sqrt{P_1^2-\alpha_1^2}} - \lambda\widehat{\mathrm{P}}_{0,0}(s)\left(\frac{P_1-\sqrt{P_1^2-\alpha_1^2}}{\alpha_1}\right)^2\frac{1}{\sqrt{P_1^2-\alpha_1^2}} + \mu q\sum_{m=2}^{\infty}\hat{P}_{1,m}(s)\beta_1^{2-m}\left(\left(\frac{P_1-\sqrt{P_1^2-\alpha_1^2}}{\alpha_1}\right)^{2-m} - \left(\frac{P_1-\sqrt{P_1^2-\alpha_1^2}}{\alpha_1}\right)^{m}\right)\frac{1}{\sqrt{P_1^2-\alpha_1^2}}. \quad (15)$$

Taking Laplace transform for the equation (1) we get,

$$(s+\lambda+\eta)\widehat{\mathrm{P}}_{0,0}(s) = \mu q\widehat{\mathrm{P}}_{1,1}(s) + \mu_v\widehat{\mathrm{P}}_{0,1}(s) \quad (16)$$

By substituting equation (15) in equation (16) yields,

$$\widehat{\mathrm{P}}_{0,0}(s)(s+\lambda+\eta)\left\{1-\frac{\lambda\mu_v}{(s+\lambda+\eta)}\left(\frac{1}{\sqrt{P_1^2-\alpha_1^2}}-\widehat{\Omega}_2(s)\right)_v\right\} = \mu q\widehat{\mathrm{P}}_{1,1}(s) + \mu q\sum_{m=2}^{\infty}\hat{P}_{1,m}(s)\beta_1^{2-m}\left(\widehat{\Omega}_{2-m}(s)-\widehat{\Omega}_m(s)\right), \quad (17)$$

where

$$\widehat{\Omega}_i(s) = \left(\frac{P_1-\sqrt{P_1^2-\alpha_1^2}}{\alpha_1}\right)^i\frac{1}{\sqrt{P_1^2-\alpha_1^2}}.$$

On further simplification of equation (17) gives,

$$\widehat{\mathrm{P}}_{0,0}(s) = \frac{1}{s+\lambda+\eta}\sum_{j=0}^{\infty}(F(s))^j\left\{\mu q\widehat{\mathrm{P}}_{1,1}(s) + \mu q\sum_{m=2}^{\infty}\hat{P}_{1,m}(s)\beta_1^{2-m}\left(\widehat{\Omega}_{2-m}(s)-\widehat{\Omega}_m(s)\right)\right\}. \quad (18)$$

Substituting equation (18) in equation (14) gives,

$$\widehat{\mathrm{P}}_{0,\mathrm{n}}(s) = \lambda\beta_1^{n-1}\left(\widehat{\Omega}_{n-1}(s)-\widehat{\Omega}_{n+1}(s)\right)\frac{1}{s+\lambda+\eta}\sum_{j=0}^{\infty}(F(s))^j\left\{\mu q\widehat{\mathrm{P}}_{1,1}(s) + \mu q\sum_{m=2}^{\infty}\hat{P}_{1,m}(s)\beta_1^{2-m}\left(\widehat{\Omega}_{2-m}(s)-\widehat{\Omega}_m(s)\right)\right\} + \mu q\sum_{m=2}^{\infty}\beta_1^{n-m+1}\left(\widehat{\Omega}_{n-m+1}(s)-\widehat{\Omega}_{n+m-1}(s)\right)\widehat{\mathrm{P}}_{1,\mathrm{m}}(s). \quad (19)$$

Equation (19) can also be written as,

$$\widehat{\mathrm{P}}_{0,\mathrm{n}}(s) = \lambda\beta_1^{n-1}\hat{\psi}_{n-1,n+1}(s)\frac{1}{s+\lambda+\eta}\sum_{j=0}^{\infty}\big(F(s)\big)^j\,\mu q\widehat{\mathrm{P}}_{1,1}(s)$$

$$+\lambda\beta_1^{n-1}\hat{\psi}_{n-1,n+1}(s)\frac{1}{s+\lambda+\eta}\sum_{j=0}^{\infty}\big(F(s)\big)^j\,\mu q\sum_{m=2}^{\infty}\hat{P}_{1,m}(s)\beta_1^{2-m}\hat{\psi}_{2-m,m}(s)$$

$$+\mu q\sum_{m=2}^{\infty}\hat{P}_{1,m}(s)\beta_1^{n-m+1}\hat{\psi}_{n-m+1,n+m-1}(s), \qquad (20)$$

where

$$\hat{\psi}_{i-1,i+1}(s) = \widehat{\Omega}_{i-1}(s) - \widehat{\Omega}_{i+1}(s).$$

Taking inverse Laplace transform for the equation (20) gives,

$$P_{0,n}(t) = \lambda\beta_1^{n-1} * \psi_{n-1,n+1}(t) * e^{-(\lambda+\eta)t}$$

$$* \sum_{j=0}^{\infty}(F(t))^{*j} * \left(\mu q P_{1,1}(t) + \mu q\sum_{m=2}^{\infty}\beta_1^{2-m}P_{1,m}(t) * \psi_{2-m,m}(t)\right)$$

$$+\mu q\sum_{m=2}^{\infty}\beta_1^{n-m+1}\psi_{n-m+1,n+m-1}(t) * P_{1,m}(t). \qquad (21)$$

Hence $P_{0,n}(t)$ is expressed explicitly in terms of $P_{1,1}(t)$ and $P_{1,n}(t)$.

Evaluation of $P_{2,0}(t)$

Taking Laplace transform for the equation (6) we get,

$$\widehat{\mathrm{P}}_{2,0}(s) = \frac{\alpha}{s+\lambda+r}\sum_{n=1}^{\infty}\widehat{\mathrm{P}}_{1,\mathrm{n}}(s) \qquad (22)$$

Evaluation of $P_{2,n}(t)$

Taking Laplace transform for the equation (7) we get,

$$\widehat{\mathrm{P}}_{2,\mathrm{n}}(s) = \left(\frac{\lambda}{s+\lambda+r}\right)^n\widehat{\mathrm{P}}_{2,0}(s) \qquad (23)$$

By substituting the equation (22) in equation (23) yields,

$$\widehat{\mathrm{P}}_{2,\mathrm{n}}(s) = \frac{\lambda^n\alpha}{(s+\lambda+r)^{n+1}}\sum_{n=1}^{\infty}\widehat{\mathrm{P}}_{1,\mathrm{n}}(s) \qquad (24)$$

By inverting equation (24) we get,

$$P_{2,n}(t) = \lambda^n\alpha\frac{e^{-(\lambda+r)t}}{n!}t^n * \sum_{n=1}^{\infty}P_{1,n}(t); n = 1,2\ldots \qquad (25)$$

Evaluation of $P_{3,0}(t)$

Taking Laplace transform for the equation (5) we get,

$$\widehat{P}_{3,0}(s) = \frac{1}{s+\lambda} + \frac{\mu p}{s+\lambda}\widehat{P}_{1,1}(s) + \frac{\eta}{s+\lambda}\widehat{P}_{0,0}(s) + \frac{r\alpha}{(s+\lambda)(s+\lambda+r)}\sum_{n=1}^{\infty}\widehat{P}_{1,n}(s) \tag{26}$$

Inverting the equation (26) we get

$$P_{3,0}(t) = e^{-\lambda t} + \mu p e^{-\lambda t} * P_{1,1}(t) + \eta e^{-\lambda t} * P_{0,0}(t) + \alpha\left(e^{-\lambda t} - e^{-(\lambda+r)t}\right) * \sum_{n=1}^{\infty} P_{1,n}(t) \tag{27}$$

Evaluation of $P_{1,n}(t)$

$$H(z,t) = \sum_{n=2}^{\infty} P_{1,n}(t) z^n.$$

Then,

$$H'(z,t) = \sum_{n=2}^{\infty} P'_{1,n}(t)\, z^n.$$

By substituting equation (4) gives,

$$H'(z,t) - \left(-(\lambda+\mu+\alpha) + \lambda z + \frac{\mu p}{z}\right) H(z,t)$$
$$= \lambda z^2 P_{1,1}(t) - \mu p P_{1,2}(t) z + r\sum_{n=2}^{\infty} P_{2,n}(t) z^n + \eta \sum_{n=2}^{\infty} P_{0,n}(t) z^n. \tag{28}$$

Integrating equation (28) with respect to time 't' we get,

$$H(z,t) = \int_0^t \left(\lambda z^2 P_{1,1}(y) - \mu p P_{1,2}(y) z\right) e^{\left(-(\lambda+\mu+\alpha)+\lambda z+\frac{\mu p}{z}\right)(t-y)} dy$$
$$+ r\int_0^t e^{\left(-(\lambda+\mu+\alpha)+\lambda z+\frac{\mu p}{z}\right)(t-y)} \sum_{n=2}^{\infty} P_{2,n}(y) z^n \, dy$$
$$+ \eta \int_0^t \left(\sum_{n=2}^{\infty} P_{0,n}(y) z^n\right) e^{\left(-(\lambda+\mu+\alpha)+\lambda z+\frac{\mu p}{z}\right)(t-y)} dy. \tag{29}$$

If $\alpha = 2\sqrt{\lambda\mu}$ and $\beta = \sqrt{\frac{\lambda}{\mu}}$ then

$$e^{\left(\lambda z+\frac{\mu p}{z}\right)t} = \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha t).$$

Therefore equation (29) can be written as,

$$H(z,t) = \int_0^t \left(\lambda z^2 P_{1,1}(y) - \mu p P_{1,2}(y) z\right) e^{-(\lambda+\mu+\alpha)(t-y)} \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha(t-y))\, dy$$
$$+ r\int_0^t e^{-(\lambda+\mu+\alpha)(t-y)} \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha(t-y)) \sum_{n=2}^{\infty} P_{2,n}(y) z^n\, dy$$
$$+ \eta\int_0^t \left(\sum_{n=2}^{\infty} \mathrm{P}_{0,\mathrm{n}}(y) z^n\right) e^{-(\lambda+\mu+\alpha)(t-y)} \sum_{n=-\infty}^{\infty} (\beta z)^n I_n(\alpha(t-y))\, dy \qquad (30)$$

Equating the coefficient of z^n in equation (30) gives,

$$\mathrm{P}_{1,\mathrm{n}}(t) = \lambda\int_0^t P_{1,1}(y)\, e^{-(\lambda+\mu+\alpha)(t-y)} \beta^{n-2} I_{n-2}\big(\alpha(t-y)\big) dy$$
$$- \mu p\int_0^t P_{1,2}(y) e^{-(\lambda+\mu+\alpha)(t-y)} \beta^{n-1} I_{n-1}\big(\alpha(t-y)\big) dy$$
$$+ r\int_0^t e^{-(\lambda+\mu+\alpha)(t-y)} \sum_{k=2}^{\infty} P_{2,k}(y)\, \beta^{n-k} I_{n-k}\big(\alpha(t-y)\big) dy$$
$$+ \eta\int_0^t \left(\sum_{k=2}^{\infty} \mathrm{P}_{0,\mathrm{k}}(y) \beta^{n-k} I_{n-k}\big(\alpha(t-y)\big)\right) e^{-(\lambda+\mu+\alpha)(t-y)} dy \qquad (31)$$

The equation (31) can also be written as

$$\mathrm{P}_{1,\mathrm{n}}(t) = \lambda\beta^{n-2}\left[P_{1,1}(t) * e^{-(\lambda+\mu+\alpha)t} I_{n-2}(\alpha t)\right] - \mu p\beta^{n-1}\left[P_{1,2}(t) * e^{-(\lambda+\mu+\alpha)t} I_{n-1}(\alpha t)\right]$$
$$+ r\left[\sum_{k=2}^{\infty} P_{2,k}(t) * e^{-(\lambda+\mu+\alpha)t} \beta^{n-k} I_{n-k}(\alpha t)\right]$$
$$+ \eta\left[\sum_{k=2}^{\infty} P_{0,k}(t) * e^{-(\lambda+\mu+\alpha)t} \beta^{n-k} I_{n-k}(\alpha t)\right]; n = 2,3.. \qquad (32)$$

Evaluation of $\mathbf{P}_{1,2}(t)$

Substituting $n = 2$ in the equation (32) yields,

$$\mathrm{P}_{1,2}(t) = \lambda \mathrm{P}_{1,1}(t) * e^{-(\lambda+\mu+\alpha)t} I_0(\alpha t) - \mu p \beta \mathrm{P}_{1,2}(t) * e^{-(\lambda+\mu+\alpha)t} I_1(\alpha t)$$
$$+ r\left[\sum_{k=2}^{\infty} P_{2,k}(t) * e^{-(\lambda+\mu+\alpha)t} \beta^{2-k} I_{2-k}(\alpha t)\right]$$
$$+ \eta\left[\sum_{k=2}^{\infty} P_{0,k}(t) * e^{-(\lambda+\mu+\alpha)t} \beta^{2-k} I_{2-k}(\alpha t)\right] \quad (33)$$

Taking Laplace transform for the equation (33) yields

$$\widehat{\mathrm{P}}_{1,2}(s)\left(1 - \hat{F}_1(s)\right) = \lambda \widehat{\mathrm{P}}_{1,1}(s) \frac{1}{\sqrt{P^2 - \alpha^2}} + \sum_{k=2}^{\infty} \mathrm{r}\widehat{\mathrm{P}}_{2,\mathrm{k}}(s)\hat{\mathrm{u}}_{2-\mathrm{k}}(s) + \sum_{k=2}^{\infty} \eta \widehat{\mathrm{P}}_{0,\mathrm{k}}(s)\hat{\mathrm{u}}_{2-\mathrm{k}}(s) \quad (34)$$

where

$$\hat{F}_1(s) = -\mu p \hat{\mathrm{u}}_1(s),$$

and

$$u_i(s) = \beta^i \left(\frac{P - \sqrt{P^2 - \alpha^2}}{\alpha}\right)^i \frac{1}{\sqrt{P^2 - \alpha^2}} \, .$$

By substituting the equation (20) and (24) in equation (34) yields,

$$\widehat{\mathrm{P}}_{1,2}(s)\left(1 - \hat{F}_1(s)\right)$$
$$= \lambda \widehat{\mathrm{P}}_{1,1}(s) \frac{1}{\sqrt{P^2 - \alpha^2}}$$
$$+ \frac{\alpha}{s + \lambda + r} \sum_{k=2}^{\infty} \left(\frac{\lambda}{s + \lambda + r}\right)^k r \hat{\mathrm{u}}_{2-\mathrm{k}}(s) \left(\sum_{n=1}^{\infty} \widehat{\mathrm{P}}_{1,\mathrm{n}}(s)\right)$$
$$+ \sum_{k=2}^{\infty} \left[\lambda \beta_1^{k-1} \hat{\psi}_{k-1,k+1}(s) \frac{1}{s + \lambda + \eta} \sum_{j=0}^{\infty} \left(F(s)\right)^j \mu q \widehat{\mathrm{P}}_{1,1}(s)\right.$$
$$+ \lambda \beta_1^{k-1} \hat{\psi}_{k-1,k+1}(s) \frac{1}{s + \lambda + \eta} \sum_{j=0}^{\infty} \left(F(s)\right)^j \mu q \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^{2-m} \hat{\psi}_{2-m,m}(s)$$
$$\left. + \mu q \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^{k-m+1} \hat{\psi}_{k-m+1,k+m-1}(s)\right] \eta \hat{\mathrm{u}}_{2-\mathrm{k}}(s)$$

Rewriting the above terms gives,

$$\widehat{\mathrm{P}}_{1,2}(s)$$

$$= \sum_{j=0}^{\infty} \left(\widehat{\mathrm{F}}_1(s)\right)^{\mathrm{j}} \left\{ \widehat{\mathrm{P}}_{1,1}(s) \left[\frac{\lambda}{\sqrt{P^2-\alpha^2}} + \frac{\lambda\mu q\eta}{\beta_1(s+\lambda+\eta)} \sum_{k=2}^{\infty} \beta_1^k \hat{\psi}_{k-1,k+1}(s)\hat{\mathrm{u}}_{2-\mathrm{k}}(s) \sum_{j=0}^{\infty} \big(F(s)\big)^j \right] \right.$$

$$+ \frac{\alpha}{s+\lambda+r} \sum_{k=2}^{\infty} \left(\frac{\lambda}{s+\lambda+r}\right)^k r\hat{\mathrm{u}}_{2-\mathrm{k}}(s) \left(\sum_{n=1}^{\infty} \widehat{\mathrm{P}}_{1,\mathrm{n}}(s) \right)$$

$$+ \frac{\lambda\mu q\eta}{\beta_1(s+\lambda+\eta)} \sum_{k=2}^{\infty} \beta_1^k \hat{\psi}_{k-1,k+1}(s)\hat{\mathrm{u}}_{2-\mathrm{k}}(s) \sum_{j=0}^{\infty} \big(F(s)\big)^j \sum_{m=2}^{\infty} \hat{P}_{1,m}(s)\beta_1^{2-m}\hat{\psi}_{2-m,m}(s)$$

$$\left. + \mu q\eta \sum_{k=2}^{\infty}\sum_{m=2}^{\infty} \hat{P}_{1,m}(s)\beta_1^{k-m+1}\hat{\psi}_{k-m+1,k+m-1}(s)\hat{\mathrm{u}}_{2-\mathrm{k}}(s) \right\} \tag{35}$$

The equation (35) can be written as

$$\widehat{\mathrm{P}}_{1,2}(s)$$

$$= \sum_{j=0}^{\infty} \left(\widehat{\mathrm{F}}_1(s)\right)^{\mathrm{j}} \left[\hat{R}(s)\widehat{\mathrm{P}}_{1,1}(s) + \hat{Q}(s) \sum_{n=1}^{\infty} \widehat{\mathrm{P}}_{1,\mathrm{n}}(s) \right.$$

$$+ \hat{w}(s) \sum_{m=2}^{\infty} \hat{P}_{1,m}(s)\beta_1^{2-m}\hat{\psi}_{2-m,m}(s)$$

$$\left. + \mu q\eta \sum_{k=2}^{\infty}\sum_{m=2}^{\infty} \hat{P}_{1,m}(s)\beta_1^{k-m+1}\hat{\psi}_{k-m+1,k+m-1}(s)\hat{\mathrm{u}}_{2-\mathrm{k}}(s) \right] \tag{36}$$

where

$$\hat{R}(s) = \frac{\lambda}{\sqrt{P^2-\alpha^2}} + \frac{\lambda\mu q\eta}{\beta_1(s+\lambda+\eta)} \sum_{k=2}^{\infty} \beta_1^k \hat{\psi}_{k-1,k+1}(s)\hat{\mathrm{u}}_{2-\mathrm{k}}(s) \sum_{j=0}^{\infty} \big(F(s)\big)^j,$$

$$\hat{Q}(s) = \frac{\alpha}{s+\lambda+r} \sum_{k=2}^{\infty} \left(\frac{\lambda}{s+\lambda+r}\right)^k r\hat{\mathrm{u}}_{2-\mathrm{k}}(s),$$

and

$$\hat{w}(s) = \hat{R}(s) - \frac{\lambda}{\sqrt{P^2-\alpha^2}}.$$

Inverting the equation (36) we get,

$$P_{1,2}(t) = \sum_{j=0}^{\infty} (F_1(t))^{*j} * \left[R(t) * P_{1,1}(t) + Q(t) * \sum_{n=1}^{\infty} P_{1,n}(t) + w(t) * \sum_{m=2}^{\infty} \beta_1^{2-m} P_{1,m}(t) * \psi_{2-m,m}(t) + \mu q \eta \sum_{k=2}^{\infty} \sum_{m=2}^{\infty} P_{1,m}(t) * \beta_1^{k-m+1} \psi_{k-m+1,k+m-1}(t) * u_{2-k}(t) \right] \quad (37)$$

where

$$R(t) = \lambda e^{-(\lambda+\mu+\alpha)t} I_0(\alpha t) + \frac{\mu q \lambda \eta}{\beta_1} e^{-(\lambda+\eta)t} * \sum_{k=2}^{\infty} {\beta_1}^k u_{2-k}(t) * \psi_{k-1,k+1}(t) * \sum_{j=0}^{\infty} (F(t))^{*j},$$

$$Q(t) = \alpha r e^{-(\lambda+r)t} * \sum_{k=2}^{\infty} \lambda^k e^{-(\lambda+r)t} \frac{t^{k-1}}{(k-1)!} * u_{2-k}(t),$$

and

$$w(t) = R(t) - \lambda e^{-(\lambda+\mu+\alpha)t} I_0(\alpha t).$$

Hence $P_{1,2}(t)$ is expressed in terms of $P_{1,1}(t)$ and $P_{1,n}(t)$.

Evaluation of $P_{1,1}(t)$

Taking Laplace transform for the equation (3) we get,

$$\widehat{P}_{1,1}(s) = \frac{\lambda}{s+\lambda+\mu+\alpha} \widehat{P}_{3,0}(s) + \frac{\mu p}{s+\lambda+\mu+\alpha} \widehat{P}_{1,2}(s) + \frac{\eta}{s+\lambda+\mu+\alpha} \widehat{P}_{0,1}(s) + \frac{r}{s+\lambda+\mu+\alpha} \widehat{P}_{2,1}(s) \quad (38)$$

Using the equations (18),(20),(24),(26) and (36) in equation (38) gives,

$$
\begin{aligned}
\widehat{P}_{1,1}(s) = \frac{\lambda}{s+\lambda+\mu+\alpha}\Bigg[\frac{1}{s+\lambda} + \frac{\mu p}{s+\lambda}\widehat{P}_{1,1}(s) \\
+ \frac{\eta}{(s+\lambda)}\frac{1}{(s+\lambda+\eta)}\sum_{j=0}^{\infty}(F(s))^{j}\left\{\mu q\widehat{P}_{1,1}(s) + \mu q\sum_{m=2}^{\infty}\hat{P}_{1,m}(s)\beta_1^{2-m}\hat{\psi}_{2-m,m}(s)\right\} \\
+ \frac{r\alpha}{(s+\lambda)(s+\lambda+r)}\sum_{n=1}^{\infty}\widehat{P}_{1,n}(s)\Bigg] \\
+ \frac{\mu p}{s+\lambda+\mu+\alpha}\Bigg[\sum_{j=0}^{\infty}\left(\widehat{F}_1(s)\right)^{j}\Bigg[\hat{R}(s)\widehat{P}_{1,1}(s) + \hat{Q}(s)\sum_{n=1}^{\infty}\widehat{P}_{1,n}(s) \\
+ \widehat{w}(s)\sum_{m=2}^{\infty}\hat{P}_{1,m}(s)\beta_1^{2-m}\hat{\psi}_{2-m,m}(s) \\
+ \mu q\eta\sum_{k=2}^{\infty}\sum_{m=2}^{\infty}\hat{P}_{1,m}(s)\beta_1^{k-m+1}\hat{\psi}_{k-m+1,k+m-1}(s)\hat{u}_{2-k}(s)\Bigg]\Bigg] \\
+ \frac{\eta}{s+\lambda+\mu+\alpha}\Bigg[\lambda\hat{\psi}_{0,2}(s)\frac{1}{s+\lambda+\eta}\sum_{j=0}^{\infty}\left(F(s)\right)^{j}\mu q\widehat{P}_{1,1}(s) \\
+ \lambda\hat{\psi}_{0,2}(s)\frac{1}{s+\lambda+\eta}\sum_{j=0}^{\infty}\left(F(s)\right)^{j}\mu q\sum_{m=2}^{\infty}\hat{P}_{1,m}(s)\beta_1^{2-m}\hat{\psi}_{2-m,m}(s) \\
+ \mu q\sum_{m=2}^{\infty}\hat{P}_{1,m}(s)\beta_1^{2-m}\hat{\psi}_{2-m,2}(s)\Bigg] + \frac{r}{(s+\lambda+\mu+\alpha)}\frac{\lambda\alpha}{(s+\lambda+r)^2}\sum_{n=1}^{\infty}\widehat{P}_{1,n}(s)
\end{aligned}
$$

Bringing all $\widehat{P}_{1,1}(s)$ to one side, the above equation can be written as

$$\hat{\mathrm{P}}_{1,1}(s)$$

$$= \sum_{j=0}^{\infty} \left(\hat{\mathrm{F}}_2(s)\right)^{\mathrm{j}} \left[\frac{\lambda}{(s+\lambda)(s+\lambda+\mu+\alpha)} \right.$$

$$+ \left[\frac{\lambda}{(s+\lambda+\mu+\alpha)} \frac{r\alpha}{(s+\lambda)(s+\lambda+r)} + \frac{\mu p}{(s+\lambda+\mu+\alpha)} \sum_{j=0}^{\infty} \left(\hat{\mathrm{F}}_1(s)\right)^{\mathrm{j}} \hat{Q}(s) \right.$$

$$\left. + \frac{\lambda}{(s+\lambda+\mu+\alpha)} \frac{r\alpha}{(s+\lambda+r)^2} \right] \sum_{n=2}^{\infty} \hat{\mathrm{P}}_{1,\mathrm{n}}(s)$$

$$+ \left[\frac{\lambda\eta}{(s+\lambda+\mu+\alpha)} \frac{\mu q}{(s+\lambda)(s+\lambda+\eta)} \sum_{j=0}^{\infty} \left(\hat{F}(s)\right)^{j} + \frac{\mu p}{(s+\lambda+\mu+\alpha)} \sum_{j=0}^{\infty} \left(\hat{\mathrm{F}}_1(s)\right)^{\mathrm{j}} \hat{w}(s) \right.$$

$$\left. + \frac{\eta\mu q}{(s+\lambda+\mu+\alpha)} \left(\frac{\lambda}{s+\lambda+\eta} \hat{\psi}_{0,2}(s) \sum_{j=0}^{\infty} \left(\hat{F}(s)\right)^{j} + 1 \right) \right] \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^{2-m} \hat{\psi}_{2-m,m}(s)$$

$$\left. + v \sum_{j=0}^{\infty} \left(\hat{\mathrm{F}}_1(s)\right)^{\mathrm{j}} \sum_{k=2}^{\infty} \sum_{m=2}^{\infty} \hat{P}_{1,m}(s) \beta_1^{k-m+1} \hat{\psi}_{k-m+1,k+m-1}(s)\, \hat{\mathrm{u}}_{2-\mathrm{k}}(s) \right], \qquad (39)$$

where

$$\hat{\mathrm{F}}_2(s) = \frac{\mu p\lambda}{(s+\lambda)(s+\lambda+\mu+\alpha)} + \frac{r\alpha\lambda}{(s+\lambda)(s+\lambda+\mu+\alpha)(s+\lambda+r)}$$

$$+ \frac{\mu p}{(s+\lambda+\mu+\alpha)} \sum_{j=0}^{\infty} \left(\hat{\mathrm{F}}_1(s)\right)^{\mathrm{j}} (\hat{R}(s) + \hat{Q}(s))$$

$$+ \frac{\lambda\eta\mu q}{(s+\lambda+\mu+\alpha)(s+\lambda+\eta)} \hat{\psi}_{0,2}(s) \sum_{j=0}^{\infty} \left(\hat{F}(s)\right)^{j}$$

$$+ \frac{r\lambda\alpha}{(s+\lambda+\mu+\alpha)(s+\lambda+r)^2} + \frac{\eta\mu q}{(s+\lambda)(s+\lambda+\eta)} \sum_{j=0}^{\infty} \left(\hat{F}(s)\right)^{j}.$$

Inverting the equation (39) we get

$$P_{1,1}(t) = \sum_{j=0}^{\infty}\left(F_2(t)\right)^{*j}\Bigg[\lambda e^{-\lambda t} * e^{-(\lambda+\mu+\alpha)t}$$

$$+\Bigg[\lambda e^{-(\lambda+\mu+\alpha)t} * r\alpha e^{-\lambda t} * e^{-(\lambda+r)t} + \mu p e^{-(\lambda+\mu+\alpha)t} * \sum_{j=0}^{\infty}\left(F_1(t)\right)^{*j} * Q(t)$$

$$+ \lambda e^{-(\lambda+\mu+\alpha)t} * r\alpha e^{-(\lambda+r)t} t\Bigg]\sum_{n=2}^{\infty} P_{1,n}(t)$$

$$+\Bigg[\lambda\eta e^{-(\lambda+\mu+\alpha)t} * \mu q e^{-\lambda t}$$

$$* e^{-(\lambda+\eta)t}\sum_{j=0}^{\infty}(F(t))^{*j} + \mu p e^{-(\lambda+\mu+\alpha)t}\sum_{j=0}^{\infty}\left(F_1(t)\right)^{*j} w(t) + \eta\mu q e^{-(\lambda+\mu+\alpha)t}$$

$$*\left(\lambda e^{-(\lambda+\eta)t} * \psi_{0,2}(t) * \sum_{j=0}^{\infty}\left(F(t)\right)^{*j} + \delta(t)\right)\Bigg]$$

$$* \sum_{m=2}^{\infty} P_{1,m}(t)\beta_1^{2-m} * \psi_{2-m,m}(t)$$

$$+ v\sum_{j=0}^{\infty}\left(F_1(t)\right)^{*j} * \sum_{k=2}^{\infty}\sum_{m=2}^{\infty} P_{1,m}(t)\beta_1^{k-m+1} * \psi_{k-m+1,k+m-1}(t)\, u_{2-k}(t)\Bigg],$$

where

$$F_2(t) = \mu p\lambda e^{-\lambda t} * e^{-(\lambda+\mu+\alpha)t} + r\alpha\lambda e^{-\lambda t} * e^{-(\lambda+\mu+\alpha)t} * e^{-(\lambda+r)t} + \mu p e^{-(\lambda+\mu+\alpha)t}$$

$$* \sum_{j=0}^{\infty}\left(F_1(t)\right)^{*j} * (\,R(t) + Q(t)) + \lambda\eta\mu q e^{-(\lambda+\mu+\alpha)t} * e^{-(\lambda+\eta)t} * \psi_{0,2}(t)$$

$$* \sum_{j=0}^{\infty}\left(F(t)\right)^{*j} + r\lambda\alpha e^{-(\lambda+\mu+\alpha)t} * e^{-(\lambda+r)t} t + \eta\mu q e^{-\lambda t} * e^{-(\lambda+\eta)t} * \sum_{j=0}^{\infty}(F(t))^{*j}$$

Hence all probabilities are explicitly expressed in terms of $P_{1,n}(t)$. Therefore by using normalization condition $P_{1,n}(t)$ can be found explicitly.

4. CONCLUSION AND FUTURE SCOPE

Single server queue with system disaster and repair under Bernoulli working vacation schedule is considered. The transient probabilities of the system are derived explicitly. This model can also be extended by allowing disaster to occur in working vacation state.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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