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ALWAYS-CONVERGENT HYBRID METHODS FOR FINDING ROOTS OF NONLINEAR EQUATIONS

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Abstract: New hybrid iterative methods for solving nonlinear equations are introduced. These methods combine the well-known Newton and Chebyshev's methods with the always convergent bisection method making them always convergent and faster than the combined methods themselves. Numerical experiments, comparing the new algorithms with others, are performed showing the efficiency of the new proposed methods.

Keywords: nonlinear equations; bisection method; Newton's method; Chebyshev's method.

2010 AMS Subject Classification: 68W25.

1. INTRODUCTION

One of the oldest mathematical problems is finding roots of nonlinear equations. Several numerical methods exist to approximate the solutions of such problems [1-3]. Well known iterative methods include the bisection method, regula-falsi, Newton's and Chebyshev's method.

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Newton's and Chebyshev's methods use an initial guess, which should lie in the neighborhood of the root and also use derivatives [1, 4]. If the initial point was not well-chosen or if the derivative tends to zero, these two methods would diverge or stagnate. The regula-falsi method, one of the oldest bracketing iterative algorithms, finds a simple root of the nonlinear equation by repeated linear interpolation between the current bracketing estimates. For a convex or concave function, the method retains one of the end points of the interval containing the root and converges very slowly. Also it may sometimes diverge [5-6]. The only always-convergent technique is the bisection method which bisects an interval containing the root at each iteration thus rendering this method very slow with a convergence rate of 0.5 [1].

The aim of this paper is to combine the bisection method with Newton's and Chebyshev's methods separately. The merge is done in such a way that when Newton or Chebyshev diverge the bisection is used at the given iteration creating fast convergent hybrid methods.

After this introduction, the algorithms of all the used and developed methods are explained in the second section. The numerical tests are detailed and analyzed in the third section. And the last section is the conclusion.

2. HYBRID METHODS

Consider the nonlinear equation of the type $f(x)=0$. The goal of the iterative method is to approximate the root of this equation. For comparison purposes, eight algorithms will be considered in this work. The two new hybrid methods constructed by combining the bisection method with Newton and Chebyshev's methods separately. Four well-known classical methods: Newton, Chebyshev, bisection and regula-falsi. The "new hybrid iteration method" of Ide [7] which uses Taylor's expansion to create what he calls a hybrid iterative formula. Also, the quadratic iteration formula without derivatives of Wu and Wu [8] is considered.

The algorithms of Newton's, Chebyshev, bisection and regula-falsi are given successively below [4, 9, 10].

Algorithm 2.1 (Newton)

Step1: For a given x_0, find $x_1, x_2, \ldots,$ such that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Step 2: If $(|x_{n+1} - x_n| < \varepsilon \; or \; n > N_{max})$, where ε and N_{max} are the predefined tolerance and maximum number of iterations respectively, then stop

Step 3: *n=n+1* and go to Step 1.

Algorithm 2.2 (Chebyshev)

Step 1: For a given x_0, find $x_1, x_2, \ldots,$ such that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{\left(f(x_n)\right)^2 f''(x_n)}{\left(f'(x_n)\right)^3}$$

Step 2: If $(|x_{n+1} - x_n| < \varepsilon \; or \; n > N_{max})$ then stop

Step 3: *n=n+1* and go to Step 1.

Algorithm 2.3 (Bisection)

Step 1: For a given $[a, b]$with $f(a). f(b) < 0$, find $x_1, x_2, \ldots,$ such that

$$x_{n+1} = \frac{a+b}{2}$$

Step 2: If $(|x_{n+1} - x_n| < \varepsilon \; or \; n > N_{max})$ then stop

Step 3: $f(a). f(x_{n+1}) < 0$, then $b = x_{n+1}$

Else $a = x_{n+1}$

Step 4: *n=n+1* and go to Step 1.

Algorithm 2.4 (Regula-Falsi)

Step 1: For a given $[a, b]$with $f(a). f(b) < 0$, find $x_1, x_2, \ldots,$ such that

$$x_{n+1} = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

Step 2: If $(|x_{n+1} - x_n| < \varepsilon \; or \; n > N_{max})$ then stop

Step 3: If $f(a). f(x_{n+1}) < 0$, then $b = x_{n+1}$

Else $a = x_{n+1}$

Step 4: *n=n+1* and go to Step 1.

Furthermore the hybrid methods, where the bisection method is merged with Chebyshev's or Newton's, combine the convergence of the bisection with the speed of the other one yielding these methods always convergent. The idea of these methods is to perform the bisection method and the other one (either Chebyshev's or Newton's) simultaneously. When the other method diverges or gives a value farther from the root than the center of the considered interval, the bisection method is considered and the center of the interval is used to give the next approximation of the other method. The algorithms of the Hybrid Bisection-Chebyshev, denoted HBC, and of the Hybrid Bisection-Newton, denoted HBN are given below.

Algorithm 2.5 (HBC)

Step 1: For a given $[a, b]$with $f(a).f(b) < 0$, *n=0* and $x_0 = a$, find $x_1, x_2, \ldots,$ such that

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{1}{2}\frac{\left(f(x_n)\right)^2 f''(x_n)}{\left(f'(x_n)\right)^3} \equiv Cheb(x_n) \text{ and } c = \frac{a+b}{2}$$

Step 2: while min($|f(x_{n+1})|$; $|f(c)|$) $> \varepsilon$ then do:

Step 2.1:If x_{n+1} is not a number, or infinite, or $f(x_{n+1})$ is not a number, or infinite, then $x_{n+1} = c$

Step 2.2: If $|f(c)| < |f(x_{n+1})|$

If $f(a).f(c) < 0$

then $b = c$

else $a = c$

end

$c = \frac{a+b}{2}$

$x_{n+1} = Cheb(c)$

Else

If $f(a).f(x_{n+1}) < 0$

then $b = x_{n+1}$

else $a = x_{n+1}$

end

$c = \frac{a+b}{2}$

$x_{n+1} = Cheb(x_n)$

End

Step 3: *n=n+1* and go to Step 2.

Algorithm 2.6 (HBN)

Step 1: For a given $[a, b]$with $f(a).f(b) < 0$, *n=0* and $x_0 = a$, find $x_1, x_2, \ldots,$ such that

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \equiv Newton(x_n)$and $c = \frac{a+b}{2}$

Step 2: while min($|f(x_{n+1})|$; $|f(c)|$) $> \varepsilon$ then do:

Step 2.1: If x_{n+1} is not a number, or infinite, or $f(x_{n+1})$ is not a number, or infinite, then $x_{n+1} = c$

Step 2.2: If $|f(c)| < |f(x_{n+1})|$

If $f(a).f(c) < 0$

then $b = c$

else $a = c$

end

$c = \frac{a+b}{2}$

$x_{n+1} = Newton(c)$

Else

If $f(a).f(x_{n+1}) < 0$

then $b = x_{n+1}$

else $a = x_{n+1}$

end

$c = \frac{a+b}{2}$

$x_{n+1} = Newton(x_n)$

End

Step 3: $n=n+1$ and go to Step 2.

3. NUMERICAL TESTS

Ten numerical examples are considered to test the efficiency of the new hybrid methods. In these tests, the required number of iterations is considered and $\varepsilon = 10^{-15}$ is used with a maximum number of iteration of 2000. Moreover, x_0 is the starting point of Newton, Chebyshev, HBN, HBC and the methods in [7] and [8] whereas $[x_0 ; x_1]$ is the starting interval of Bisection, Regula-Falsi, HBN and HBC. The table below summarizes the results.

No	**Equation**	x_0	x_1	**Bisection**	**Newton**	**Regula-Falsi**	**Chebyshev**	**Wu [8]**	**Ide [7]**	**HBN**	**HBC**
1	$xe^x - 1$	-1	1	50	D	106	D	2000*	2000*	50	50
2	$x^3 - x^2 - x - 1$	1	2	45	D	21	D	D	7	5	4
3	$x^{10} - 1$	0.5	2	52	44	D	70	86	17	6	5
4	$(x-2)^2 - lnx$	2.5	3.5	49	7	17	4	6	5	4	3
5	$e^x - 3x^2$	0.5	1	44	7	10	D	4	6	4	3
6	$e^x - 1 - \cos x$	-0.1	0.7	49	7	11	12	10	9	4	3
7	$e^{x^2+7x-30} - 1$	2.8	3.8	49	17	2000*	D	2000*	10	5	7
8	$\frac{1}{x} - 1$	0.1	2	49	10	2000*	7	2000*	7	6	4
9	$x - 0.8 - 0.2\sin x$	0	1	47	5	6	4	48	D	4	3
10	$\sin\left(\frac{1}{x}\right) - x$	$\sqrt{2}$	0	46	5	D	4	6	6	4	3

*the method did not converge after 2000 iterations; D: the method diverged

Table1: Numerical comparison tests between the eight considered methods

It can be noticed, from the above table, that the two hybrid methods introduced in this paper are always convergent even when Newton's and Chebyshev's method are divergent. This is logical since the bisection method, which is always convergent, is combined with the other two.

Also, these two methods require lesser number of iterations than the methods of Ide [7] and Wu [8] (except for example 5 where HBN and the method of Wu [8] have the same number) since Newton's and Chebyshev's are at least quadratic.

Furthermore, the newly introduced hybrid methods are always better than their classical counterparts i.e. HBC always requires less iterations than bisection and Chebyshev and HBN always requires less iterations than bisection and Newton. This is due to the fact that whenever the solution of Newton's or Chebyshev's iteration falls farther from the root than the center of the considered interval or diverges, the latter takes its place in the next iteration.

Henceforth, the two newly proposed hybrid algorithms perform better than the well-known classical methods and better than the methods in [7-8].

4. CONCLUSION

Two always convergent hybrid methods for solving non-linear equations are proposed. The first is a combination between the bisection and Chebyshev's method and the second between the bisection and Newton's method. These two methods combined the speed of Newton and Chebyshev's methods at one hand with the convergence of the bisection method at the other hand rendering them always convergent and fast. Ten numerical examples, comparing the introduced methods with the bisection, regula-falsi, Newton's, Chebyshev's methods and the methods of Ide [7] and Wu [8], were performed showing that the two new hybrid methods are in fact always convergent and require a lesser number of iterations than the other ones.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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