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FURTHER RESULTS ON DISTANCE MAGIC LABELING OF GRAPHS

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Abstract: Let G=G(V, E) be a graph. If for each vertex v, sum of the labeling of the vertices which are at a distance D from v is constant, then such a labeling is said to be D-distance magic labeling and a graph G is said to be D-distance magic graph. In this paper, we study D-distance magic labeling of cycles, complete multipartite graphs, and star graphs. Also we obtain necessary and sufficient conditions for trees, n-star and join of two graphs to admit a distance magic labeling.

Keywords: distance magic labeling; magic constant; complete graph; star graph.

2010 AMS Subject Classification: 05C78.

1. INTRODUCTION

The concept of distance magic labeling of a graph has been motivated by the construction of magic squares. Because of the historical interest in magic squares, in 1963, Sedlacek introduced magic labeling of graph G= G (V, E).

Definition 1.1 [4]: A bijection f from the edge set E to a set of positive integers such that $f(e_i) \neq f(e_j)$ for all distinct $e_i, e_j \in E$ and $\sum_{e\in N_E(x)} f(e)$ is same for every $x \in V$, where

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$N_E(x)$ is the set of edges incident to X.

In 1994, Vilfred and in 2003 Miller et.al separately introduced distance magic labeling.

Definition 1.2 [8]: A distance magic labeling is a bijection $f: V \rightarrow \{1,2, \ldots v\}$ with the property that there is a constant k such that at any vertex x, $\sum_{y \in N(x)} f(y) = k$, where N(x) is the open neighborhood of x.

Later Jinah, introduced variations of distance magic labeling. Here instead of open neighborhood he took closed neighborhood. The concept was independently studied by Simanjuntak, Rodgers and Miller. In particular properties of D-distance magic labeling for a distance set D.

Definition 1.3 [8]: A bijection $f: V \rightarrow \{1,2, \ldots, v\}$ is said to be a D-distance magic labeling (D-DML) if there exists a constant k such that for any vertex x,

$w(x) = \sum_{y \in N_D(x)} f(y) = k \; where \; N_D(x) = \{y \in V / d\,(x, y) \in D\}$. A graph which admits D-DML is called D-distance magic graph (D-DMG).

Definition 1.4 [3]: The join of two graphs G and H having disjoint point set V_1 and V_2 respectively is denoted by G+H and consists of G∪H and all lines joining V_1 with V_2.

For various graph theoretical notations and terminology we refer to F.Harary [3] and D.B.West [11].

2. MAIN RESULTS

Theorem 2.1: A cycle C_4 or a disjoint union of C_4 is (0, 2)-distance magic graph.

Proof: A cycle C_4 is (0, 2) –distance graph (see Figure 1).

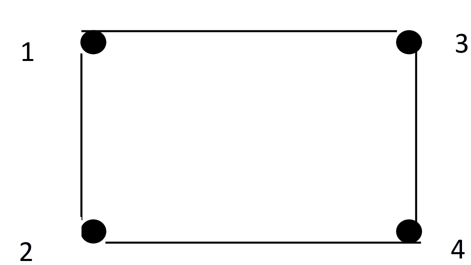

Figure 1: (0, 2) labeling of C_4.

Let G be a disjoint union of k number of $C_4{}'$s with n vertices.

Then n=4k.

Let x_i, y_i, u_i, v_i are the vertices of $i^{th} C_4$.

Let us assume u_i, v_i are at a distance one from x_i, then

$N_{0,2}(x_i) = N_{0,2}(y_i) = \{x_i, y_i\}$, $N_{0,2}(u_i) = N_{0,2}(v_i) = \{u_i, v_i\}$, i= 1, 2,…., k.

Now label the vertices as follows.

$f(x_1) = n, \quad f(y_1) = 1$

$f(u_1) = n - 1, \ f(v_1) = 2$

$f(x_2) = n - 2, \quad f(y_2) = 3$

…

$f(x_k) = 2k + 2, \quad f(y_k) = 2k - 1$

$f(u_k) = 2k + 1, \quad f(v_k) = 2k$

Observe that $w(x_i) = w(y_i) = w(u_i) = w(v_i) = n + 1, \ i = 1, 2 \dots, k.$

With the above labeling one can see that G is a (0, 2)-distance magic graph.

Theorem 2.2: $C_4 + \overline{K_m}$ is (0, 2) –distance magic graph iff $(n + 1) \equiv 0 (mod\ 6)$, where n=m+4, m=1, 2, 4, 5.

Proof: Let a, b, c, d are the 4 vertices of C_4 and d (a, c) =2, d (b, d) =2.

Let $v_1, v_2, \dots, v_n$ are the vertices of $\overline{K_m}$.

Note that $N_{(0,2)}(a) = N_{(0,2)}(c) = \{a, c\}$

$N_{(0,2)}(b) = N_{(0,2)}(d) = \{b, d\}$

And $N_{(0,2)}(v_i) = V, \quad V = \{v_i\}, \ i = 1,2, \dots, m.$

Suppose given graph is (0, 2)-graph, then

w (a)=w(c)=f(a)+f(c)=k

w (b)=w(d)=f(b)+f(d)=k

$w(v_i) = \sum f(v_i) = k, i = 1,2, \dots, m$

$\Rightarrow 3k = \frac{n(n+1)}{2}$

$\Rightarrow k = \frac{n(n+1)}{6}$

If $n(n+1) \ncong 0(mod6)$

Then $6 \nmid n(n+1)$

$\Rightarrow k \notin Z^{+}$, a contradiction.

Further for m=3, 6, 9, …

$n(n+1) \ncong 0(mod6)$

If m=7, 8, 10, 11, 12,…

$n+(n-1) < \frac{n(n+1)}{6}$=k

Therefore it is not possible to label the vertices a, c or b, d such that

$f(a)+f(c)=\frac{n(n+1)}{6}=k$

Or f (b) +f (d) =k

Conversely,

Case1: If m=1

$C_4+\overline{K_1}=W_5$.

(0, 2) labeling of W_5 is given below in Figure 2.

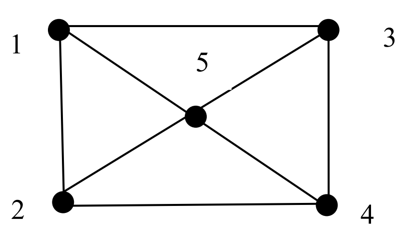

Figure 2: (0, 2) labeling of W_5.

Case 2: When m=2, n=6

Assign f (a) =6, f(c) =1

f (b)=5, f(d)=2

$f(v_1)=3, \qquad f(v_2)=4$

Case3: For m=4, n=8

Assign f (a) =8, f(c) =4

f (b)=7, f (d)=5

$f(v_1) = 1, \quad f(v_2) = 2,$

$f(v_3) = 3, \quad f(v_4) = 6$

Case 4 : For m=5, n=9

Assign f (a) =9, f(c) =6

f (b)=8, f(d)=7

$f(v_1) = 1, f(v_2) = 2, f(v_3) = 3,$

$f(v_4) = 4, f(v_5) = 5$

With the above labeling one can see that $C_4 + \overline{K_m}$ is (0, 2) –distance magic graph under the given conditions.

Theorem 2.3: $C_6 + \overline{K_m}$ is (0, 2) –distance magic graph iff $n(n+1) \equiv 0(mod\ 6)$, where n=m+6, m=2, 3, 5, 6.

Proof: Let a, b, c, d, e, f are the vertices of C_6, where a and b are adjacent vertices.

Vertices e & d are at a distance 2 from a and c &f are at a distance 2 from b.

$v_i, 1 \leq i \leq m$ are vertices of K_m.

Observe that $N_{(0,2)}(a)$ ={a, e, d} $=N_{(0,2)}(e) = N_{(0,2)}(d)$

$N_{(0,2)}(b)$ ={c,f,b}$=N_{(0,2)}(c) = N_{(0,2)}(f)$

$N_{(0,2)}(v_i) = V, V = \{v_i\}, \ i = 1,2, \dots, m.$

Suppose given graph is (0, 2)-graph then

w(a)=w(e)=w(d)=f(a)+f(e)+f(d)=k

w(b)=w(c)=w(f)=f(b)+f(c)+f(f)=k

$w(v_i) = \sum f(v_i) = k, i = 1,2, \dots, m.$

⇒3k=$\frac{n(n+1)}{2}$

⇒k=$\frac{n(n+1)}{6}$

Now, if $n(n+1) \not\equiv 0(mod6)$

Then 6∤n (n+1)

⇒k$\notin Z^+$, a contradiction.

For m=1, 4, 7, 8, 9… either 6∤n (n+1) or 1+2+3+4…..+m < k

Conversely,

Case 1: When m=2.

Assign f (a) = 1, f (e) =3, f (d) =8

f(c) =2, f (f)=4, f(b)=6

$f(v_1) = 5$, $f(v_2) = 7$

Case 2: When m=3.

Assign f (a) =1, f (e) =5, f (d) =9

f (c) =2, f (f)=7, f (b)=6

$f(v_1) = 3$, $f(v_2) = 4$, $f(v_3) = 8$

Case 3: When m=5.

Assign f (a) =9, f (e) =6, f (d) =7

f(c)=11, f(f)=10, f(b)=1

$f(v_1) = 2$, $f(v_2) = 3$, $f(v_3) = 4$, $f(v_4) = 5$, $f(v_5) = 8$

Case 4: When m=6.

Assign f (a) =10, f (e) =9, f (d) =7

f (c)=12, f (f)=8, f (b)=6

$f(v_1) = 1$, $f(v_2) = 2$, $f(v_3) = 3$, $f(v_4) = 4$, $f(v_5) = 5$, $f(v_6) = 11$

With the above labeling one can see that $C_6 + \overline{K_m}$ is (0, 2) –distance magic graph.

Theorem 2.4: A graph K_{n_1,n_2,n_3,n_4} is a (0, 2) –distance magic graph if

i) $n(n+1) \equiv 0 \ (mod \ \ 8)$ where $n = n_1 + n_2 + n_3 + n_4$

ii) $n_2 = n_3 = n_4 = s = even,\ n_1 = s - 1$

Proof: Since $n(n+1) \equiv 0 \ (mod \ \ 8)$

We get, 8k=n (n+1)

k=$\frac{n(n+1)}{8}$

Assign numbers to the vertices of partite sets as follows (see Table 1).

Ist partite set	2nd partite set	3rd partite set	4th partite set
n	n-1	n-2	n-3
n-7	n-6	n-5	n-4
n-8	n-9	n-10	n-11
. . .	. . .	. . .	n-12 . .
-	1	2	3

Table 1: Labeling of vertices of K_{n_1,n_2,n_3,n_4}

With the above labeling we can say that K_{n_1,n_2,n_3,n_4} is a (0, 2) –distance magic graph.

Definition 2.5 [11]: A star is a tree consisting of one vertex adjacent to all the others.

Theorem 2.6: Star graph $K_{1,n-1}$, $n > 3$ are not D-distance magic graph where $D \subset \{0,1,2\}$

Proof: Let V be the vertex set and v be the vertex in V having degree n-1.

Case 1: D= (0, 1)

Proved in [9].

Case 2: D= (0, 2)

$N_D(v) = v$ and $N_D(x) = V - v, for\ every\ x \in V, x \neq v$

But then $w(v) \neq w(x)$

Case 3: D= (1, 2)

$N_D(x) = V - x$, for any $x \in V$ and $N_D(y) = V - y$,$x \neq y \neq v$

Then $w(x) \neq w(y)$

Case 4: D= (1)

$N_D(v) = V - v$ and $N_D(x) = v$ for any $x \neq v$,

$w(v) \neq w(x)$

Case 5: D= (2)

Here $N_D(v) = \emptyset$ but $N_D(x) \neq \emptyset$ for any $x \neq v$.

Implies $w(v) \neq w(x)$

Note: It is to be noted that $K_{1,2} = P_3$ is (0, 2) & (1)-distance graph.

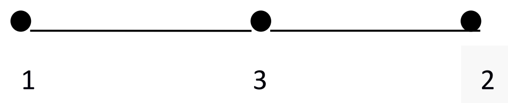

Figure 3: (0, 2) & (1)-DML of P_3.

Theorem 2.7: A tree T with n vertices and diameter 3 is (0, 2)-distance magic iff

i) $\frac{n(n+1)}{2} \cong 0(\text{mod}2)$

ii) for $|N_1| = P_1, |N_2| = P_2,\ P_1 \leq P_2 \quad n + (n-1) + \cdots + n - (P_1 - 1) \geq \frac{n(n+1)}{4}$

where N_i, i=1, 2 is the set such that for any $u, v \in N_i$, i=1,2 , d(u ,v)=2 , $u \neq v$.

Proof: Let V be the vertex set and$v \in V$.

Let N_1 be the set containing v and all other vertices at a distance 2 from v and N_2 be the set containing all the vertices at a distance 1 and 3 from v.

Therefore for any $u \in N_i$,i=1, 2, $N_{0,2}(u) = N_i$ and $N_1 \cap N_2 = \emptyset$ & $N_1 \cup N_2 = V$

Now T is (0, 2)-distance magic graph if $w(x) = w(y) = k$ for any $x \in N_1$ and $y \in N_2$

That implies $2k = \frac{n(n+1)}{2}$

k=$\frac{n(n+1)}{4}$

i) if $\frac{n(n+1)}{2} not \cong 0(\text{mod}2)$ then 4 ∤ n (n+1)

that implies k is not integer, a contradiction.

ii) if $n + (n-1) + \cdots + n - (P_1 - 1) < \frac{n(n+1)}{4}$,$w(x) \neq \frac{n(n+1)}{4} = k$ for $v_i \in N_1$.

Conversely,

Since ii) is true we can easily choose P_1 numbers whose sum=$\frac{n(n+1)}{4}$.

Now, because of (i) there are P_2 numbers and sum of P_2 =$\frac{n(n+1)}{4}$

Hence the proof.

In the following table we have given a (0, 2)-DML of trees of diameter 3 for n=4 to 20. Observe that for n=5, 6, 9, 10, 13,14, 17 &18 condition (i) of above theorem fails.

Number of vertices	Magic constant	P_1	P_2	Labeling of vertices of P_1 and P_2
4	5	2	2	1,4 &2,3
7	14	3 2	4 5	7,6,1 &2,3,4,5 ii) fails
8	18	2 3 4	6 5 4	ii) fails 8,7,3 & 6,5,4,2,1 8,7,2,1&6,5,4,3
11	33	2 3 4 5	9 8 7 6	ii)fails ii)fails 11,10,9,3 & 8,7,6,5,4,2,1 11,10,9,2,1 & 8,7,6,5,43
12	39	2 3 4 5 6	10 9 8 7 6	ii)fails ii)fails 12,11,10,6 & 9,8,7,5,4,3,2,1 12,11,10,5,1& 9,8,7,6,4,3,2 12,11,10,3,2,1 & 9,8,7,6,5,4
15	60	2 3 4 5 6 7	13 12 11 10 9 8	ii)fails ii)fails ii)fails 15,14,13,12,6 &11,10,9,8,7,5,4,3,2,1. 15,14,13,12,1,5 &11,10,9,8,7,6,4,3,2 15,14,13,12,1,2,3 &11, 10,9,8,7,6,5,4.
16	68	2 3 4 5 6 7 8	14 13 12 11 10 9 8	ii)fails ii)fails ii)fails 16,15,14,13,10 &12,11,9,8,7,6,5,4,3,2,1. 16,15,14,13,6,4&12,11,10,9,8,7,5,3,2,1 16,15,14,13,5,3,2 &12,11,10,9,8,7,6,4,1 16,15,14,13,4,3,2,1 &12,11,10,9,8,7,6,5
19	95	2 3 4 5 6 7	17 16 15 14 13 12	ii)fails ii)fails ii)fails ii)fails 19,18,17,16,15,10 &14,13,12,11,9,8,7,6,5,4,3,2,1. 19,18,17,16,15,6,4&

				14,13,12,11,10,9,8,7,5,3,2,1.
		8	11	19,18,17,16,15,5,3,2 &14,13,12,11,10,9,8,7,4,1
		9	10	19,18,17,16,15,4,3,2,1 &14,13,12,11,10,9,8,7,6,5
20	105	2	18	ii)fails
		3	17	ii)fails
		4	16	ii)fails
		5	15	ii)fails
		6	14	20,19,18,17,16,15 &14,13,12,11,10,9,8,7,6,5,4,3,2,1.
		7	13	20,19,18,17,16,14,1 &15,13,12,11,10,9,8,7,6,5,4,3,2.
		8	12	20,19,18,17,16,12,2,1.&15,14,13,11,10,9,8,7,6 ,5,4,3,2
		9	11	20,19,18,17,16,9,3,2,1.& 15,14,13,12,11,10,8,7,6,5,4
		10	10	20,19,18,17,16,5,4,3,2,1 &15,14,13,12,11,10,9,8,7,6

Table 2: (0, 2-DML of trees with diameter 3)

Example 1: (0, 2)-DML of tree with 8 vertices.

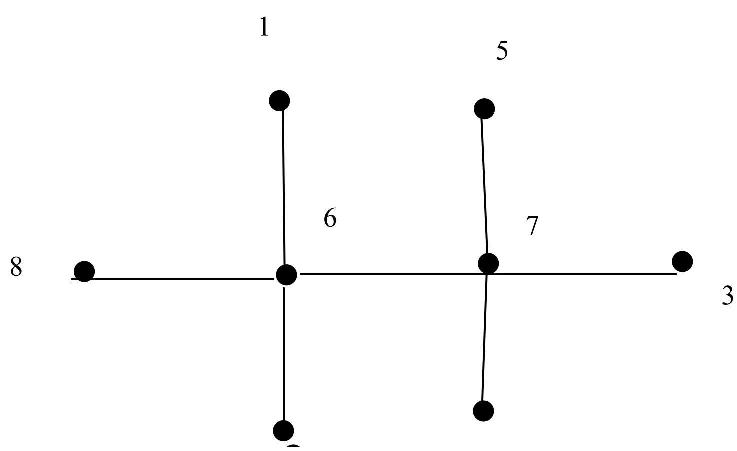

Figure 4: (0, 2)-DML of tree with 8 vertices.

Definition 2.8: The n-star $St(m_1, m_2, \ldots, m_n), 1 < m_1 < m_2 < \cdots < m_n$ is a disjoint union of n stars $K(1, m_1), K(1, m_2), \ldots, K(1, m_n)$.

Theorem 2.9: The n-star $St(m_1, m_2, \dots, m_n), 1 < m_1 < m_2 < \cdots < m_n$ is (0, 1) –distance magic graph iff

i) $p(p+1) \equiv 0 (mod\ 2n)$, p=$m_1 + m_2 + \cdots + m_n$+n

ii) $n + (n-1) + (n-2) \geq k$, k is a magic constant.

iii) n=3, 5.

Proof: Let w(x) = k, $\forall x \in St(m_1, m_2, \dots, m_n)$

Since there are n disjoint set $nk = \frac{p(p+1)}{2}$

k=$\frac{p(p+1)}{2n}$

If $p(p+1) \not\equiv 0\ (mod 2n),\ 2n \nmid p(p+1)$

k is not integer, a contradiction.

ii) n+(n-1)+(n-2) < k, then w(u) cannot be equal to k , $u \in S_{1,2}$

iii) If n=4 then condition i) fails

If n=6, 8,10,….condition i) fails

If n=7,9,11,….condition ii) fails

Conversely, when n=3

We will have $S_{1,2} \cup S_{1,3} \cup S_{1,4}$

Assign $(v_i) = 12,11$ & 3 ,$v_i \in S_{1,2}$, i=1, 2, 3.

$f(v_j) = 10, 9, 5$ & 2 ,$v_j \in S_{1,3}$, j=1, 2,3,4.

$f(v_k) = 1,4,6,7$&8 ,$v_k \in S_{1,4}$, k=1,2,3,4,5.

When n=5

We will have $S_{1,2} \cup S_{1,3} \cup S_{1,4} \cup S_{1,5} \cup S_{1,6}$

Assign $f(v_i) = 25, 24$ &16 , $v_i \in S_{1,2}$, i=1,2,3.

$f(v_j) = 23,22,7$ & 2 ,$v_j \in S_{1,3}$, j=1,2,3,4. 5.

$f(v_k) = 21,20,19,4$&1 ,$v_k \in S_{1,4}$, k=1,2,3,4,

$f(v_l) = 18,17,14,5,3$&8 ,$v_l \in S_{1,5}$, l=1,2,3,4,5,6.

Hence the proof

3. CONCLUSION

In this paper we have studied (0, 2) _distance magic labeling of graphs obtained by some graph operation such as join of two graphs. We also obtained necessary and sufficient conditions for trees, n-stars to admit a distance magic labeling. It is interesting to check D-distance magic labeling of some other classes of graph.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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