GENERALIZED PADOVAN AND POLYNOMIAL SEQUENCES

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Abstract: By generalizing the well-known classical Padovan Sequence using Quadratic, Cubic and general polynomial sequences as coefficients we arrive at various limiting ratios in this paper. Few illustrations are provided to justify the results arrived regarding the limiting ratios. The idea of limiting ratio will provide the asymptotic behavior of the terms of the sequence. In this sense, the results obtained in this paper will provide more information about the behavior of Padovan sequence in combination with general polynomial sequences.

Keywords: generalized Padovan sequence; quadratic sequence; cubic sequence; general polynomial sequence; recurrence relation; limiting ratio.

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1. INTRODUCTION

The classic Padovan sequence was named after British polymath Richard Padovan and it was subsequently popularized by Ian Stewart through his books. It is well known that the ratio of successive terms of Padovan sequence approach a number called Plastic number given by 1.32471 approximately. In this paper, by considering quadratic, cubic and general polynomial sequences we will construct certain recurrence relations and obtain interesting results regarding

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the limiting ratios of such relations. This will provide a new insight of the behavior of class of Padovan sequences in more general form.

2. LIMITING RATIO

The ratio of $(n + 1)$th term to the nth term of any sequence as $n \to \infty$ is defined as Limiting Ratio of that sequence (2.1). We denote the limiting ratio by λ.

Thus, if $\{s_n\}_{n=1}^{\infty}$ is a sequence of real numbers then $\lambda = \lim \dfrac{s_{n+1}}{s_n}$ as $n \to \infty$ (2.2). Using the definition of limiting ratio, for any natural number r we have

$$\lim \frac{s_{n+r}}{s_n} = \lim \left(\frac{s_{n+r}}{s_{n+r-1}} \times \frac{s_{n+r-1}}{s_{n+r-2}} \times \cdots \times \frac{s_{n+1}}{s_n} \right) = \lambda \times \lambda \times \cdots \times \lambda = \lambda^r \ (2.3)$$

3. QUADRATIC SEQUENCE

The sequence defined by $\left\{ Ak^2 + Bk + C \right\}_{k=1}^{\infty}$ (3.1) is called the Quadratic sequence. Here A, B, C are some real numbers such that A is non-zero.

4. CUBIC SEQUENCE

The sequence defined by $\left\{ Pk^3 + Qk^2 + Rk + S \right\}_{k=1}^{\infty}$ (4.1) is called the Cubic sequence. Here P, Q, R, S are some real numbers such that P is non-zero.

5. GENERALIZED PADOVAN AND QUADRATIC SEQUENCES

Let $P(1) = 1, P(2) = 1, P(3) = k$ where k is a positive real number. Let us define the generalized Padovan sequence using Quadratic sequence as coefficients through the recurrence relation

$$P(n+3) = k^4 P(n+1) + \left(Ak^2 + Bk + C \right) P(n), \ n \geq 1 \ (5.1).$$

Now from (5.1), we get $\dfrac{P(n+3)}{P(n)} = k^4 \dfrac{P(n+1)}{P(n)} + \left(Ak^2 + Bk + C \right)$ (5.2)

If λ is the limiting ratio of the generalized Padovan sequence and Quadratic sequence as defined in (5.1), then from (2.3) we get $\lambda^3 = k^4\lambda + \left(Ak^2 + Bk + C\right)$ (5.3).

From (5.3) we get $\lambda^2 = k^4 + \dfrac{Ak^2 + Bk + C}{\lambda}$ (5.4). If the limiting ratio λ exist, then from (5.4) we must have $\lambda = O(k^2)$. In particular, if $\lambda = k^2$ then $\dfrac{Ak^2 + Bk + C}{\lambda} = \dfrac{Ak^2 + Bk + C}{k^2} \to A$ as $k \to \infty$.

Hence as $k \to \infty$ from (5.4) we have $\lambda^2 = k^4 + A$ from which $\lambda = \sqrt{k^4 + A}$.

Thus, as $k \to \infty$, the limiting ratio of generalized Padovan sequence and Quadratic sequence as defined in (5.1) is $\sqrt{k^4 + A}$ (5.5). Note that this value doesn't depend of the constants B and C.

5.1 ILLUSTRATIONS

5.1.1 Let us consider central polygonal numbers whose kth term is given by $s_k = \dfrac{k^2 + k + 2}{2}, k \geq 0$ (5.6). The terms of this sequence are 1, 2, 4, 7, 11, 16, 22, 29, . . . We observe that these numbers are one plus the triangular numbers. These numbers describe the maximum number of pieces that can be made in a pancake or pizza (or any circular object) with a given number of straight cuts. In this sense, this sequence is also sometimes informally referred as Lazy caterer's sequence.

By definition (3.1), we notice that the Lazy caterer's sequence is a Quadratic sequence with $A = \dfrac{1}{2}, B = \dfrac{1}{2}, C = 1$. If we now consider the generalized Padovan sequence along with Lazy caterer's sequence as defined in (5.1) then from (5.3) we find that the limiting ratio λ is the positive root of $\lambda^3 - k^4\lambda - \left(\dfrac{k^2 + k + 2}{2}\right) = 0$ (5.7). In particular if $k = 10$ then equation (5.7) becomes $\lambda^3 - 10000\lambda - 56 = 0$ (5.8). Using Newton Raphson method, we find that the positive root of equation (5.8) is approximately 100.00279. Thus the limiting ratio of (5.7) for $k = 10$ is approximately 100.00279.

From (5.5) the limiting ratio must be $\sqrt{k^4 + A} = \sqrt{10000.5} = 100.002499$ approximately. We notice that this value agree with the actual computation up to first three decimal places. We also notice that for small values of A, the limiting ratio is very close to k^2 (5.9). In our case since, $k = 10$ and $A = 0.5$ is very small, the limiting ratio is very close to $k^2 = 100$.

5.1.2 Let the Quadratic sequence be defined by $s_k = 1000k^2 - 81k - 1729$. Then according to (5.3) the limiting ratio is the positive root of $\lambda^3 - k^4\lambda - \left(1000k^2 - 81k - 1729\right) = 0$ (5.10). In particular if $k = 12$ then equation (5.10) becomes $\lambda^3 - 20736\lambda - 141299 = 0$ (5.11). Using Newton Raphson method, we find that the positive root of equation (5.11) is approximately 147.29325. Thus the limiting ratio of (5.10) for $k = 12$ is approximately 147.29325.

From (5.5) the limiting ratio must be $\sqrt{k^4 + A} = \sqrt{21736} = 147.43133$ approximately. Thus, we see this value agree with the actual computation verifying our result obtained in (5.5). Upon choosing k as much higher value in say thousands or millions, we would get more accurate answers for the limiting ratios.

6. GENERALIZED PADOVAN AND CUBIC SEQUENCES

Let $P(1) = 1, P(2) = 1, P(3) = k$ where k is a positive real number. Let us define the generalized Padovan sequence using Cubic sequence as coefficients through the recurrence relation
$$P(n+3) = k^6 P(n+1) + \left(Pk^3 + Qk^2 + Rk + S\right) P(n), \ n \geq 1 \ (6.1).$$

From (6,1), we get $\dfrac{P(n+3)}{P(n)} = k^6 \dfrac{P(n+1)}{P(n)} + \left(Pk^3 + Qk^2 + Rk + S\right)$ (6.2).

If λ is the limiting ratio of the generalized Padovan sequence and Cubic sequence as defined in (6.1), then from (2.3) we get $\lambda^3 = k^6\lambda + \left(Pk^3 + Qk^2 + Rk + S\right)$ (6.3)

From (6.3), we get $\lambda^2 = k^6 + \dfrac{Pk^3 + Qk^2 + Rk + S}{\lambda}$ (6.4). If the limiting ratio λ has to exist, then we should have $\lambda = O(k^3)$.

In particular, if $\lambda = k^3$ then $\dfrac{Pk^3 + Qk^2 + Rk + S}{\lambda} = \dfrac{Pk^3 + Qk^2 + Rk + S}{k^3} \to P$. Thus as $k \to \infty$

equation (6.4) can be written as $\lambda^2 = k^6 + P$ from which $\lambda = \sqrt{k^6 + P}$.

Thus, as $k \to \infty$, the limiting ratio of generalized Padovan sequence and Cubic sequence as

defined in (6.1) is $\sqrt{k^6 + P}$ (6.5). Note that this value doesn't depend of the constants Q, R and

S.

6.1 Illustrations

6.1.1 Let us consider Tetrahedral numbers whose kth term is given by

$s_k = \dfrac{k(k+1)(k+2)}{6} = \dfrac{k^3 + 3k^2 + 2k}{6}, k \geq 0$ (6.6). The terms of this sequence are 0, 1, 4, 10, 20,

35, 56, 84, 120, ... These numbers geometrically represent pyramid shapes with triangular base.

Hence they are also called as Triangular Pyramidal Numbers.

By definition (4.1), we notice that Tetrahedral numbers sequence is a cubic sequence with

$P = \dfrac{1}{6}, Q = \dfrac{1}{2}, R = \dfrac{1}{3}, S = 0$. If we consider generalized Padovan sequence along with Tetrahedral

numbers sequence then from (6.3), we find the limiting ratio is the positive root of the equation

$\lambda^3 - k^6 \lambda - \left(\dfrac{k^3 + 3k^2 + 2k}{6} \right) = 0$ (6.7). In particular if $k = 5$, then equation (6.7) becomes

$\lambda^3 - 15625\lambda - 35 = 0$ (6.8). Using Newton Raphson method, we find that the positive root of

equation (6.8) is approximately 125.00111. Thus the limiting ratio of (6.7) for $k = 5$ is

approximately 125.00111.

From (6.5) the limiting ratio must be $\sqrt{k^6 + P} = \sqrt{15625.166667} = 125.00066$ approximately. We

notice that this value agree with the actual computation up to first two decimal places. We also

notice that for small values of P, the limiting ratio is very close to k^3 (6.9). In our case since,

$k = 5$ and $P = 0.1666$ is very small, the limiting ratio is very close to $k^3 = 125$.

6.1.2 Let the Cubic sequence be defined by $s_k = 1000k^3 - 50k^2 - 6174k - 3435$. Then according

to (6.3) the limiting ratio is the positive root of

$\lambda^3 - k^6\lambda - \left(1000k^3 - 50k^2 - 6174k - 3435\right) = 0$ (6.10). In particular, if we choose $k = 12$ then

(6.10) becomes $\lambda^3 - 2985984\lambda - 1643277 = 0$ (6.11). Using Newton Raphson method, we find

that the positive root of equation (6.11) is approximately 1728.27509. Thus the limiting ratio of

(6.10) for $k = 6$ is approximately 1728.27509.

From (6.5) the limiting ratio must be $\sqrt{k^6 + P} = \sqrt{2986984} = 1728.28932$ approximately. Thus,

we see this value agree with the actual computation verifying our result obtained in (6.5). Upon

choosing k as much higher value in say thousands or millions, we would get more accurate

answers for the limiting ratios.

7. GENERALIZED PADOVAN AND GENERAL POLYNOMIAL SEQUENCES

Let $s_k = a_0 k^n + a_1 k^{n-1} + a_2 k^{n-2} + \cdots + a_{n-1} k + a_n$, $a_0 \neq 0, n \geq 1$ be the general polynomial sequence

of degree n. We get Quadratic and Cubic sequences if $n = 2, 3$ respectively.

Let $P(1) = 1, P(2) = 1, P(3) = k$ where k is a positive real number. Let us define the generalized

Padovan sequence using general polynomial sequence s_k as coefficients through the recurrence

relation $P(n+3) = k^{2n} P(n+1) + \left(a_0 k^n + a_1 k^{n-1} + a_2 k^{n-2} + \cdots + a_{n-1} k + a_n\right) P(n)$, $n \geq 1$ (7.1)

From (7.1), we get $\dfrac{P(n+3)}{P(n)} = k^{2n} \dfrac{P(n+1)}{P(n)} + \left(a_0 k^n + a_1 k^{n-1} + a_2 k^{n-2} + \cdots + a_{n-1} k + a_n\right)$ (7.2)

If λ is the limiting ratio of (7.1), then from (2.3), we get

$\lambda^3 = k^{2n}\lambda + \left(a_0 k^n + a_1 k^{n-1} + a_2 k^{n-2} + \cdots + a_{n-1} k + a_n\right)$ (7.3).

We can express (7.3) as $\lambda^2 = k^{2n} + \dfrac{a_0 k^n + a_1 k^{n-1} + a_2 k^{n-2} + \cdots + a_{n-1} k + a_n}{\lambda}$ (7.4)

If the limiting ratio λ has to exist, then from (7.4), we should have $\lambda = O(k^n)$. In particular, if

$\lambda = k^n$ then $\dfrac{a_0 k^n + a_1 k^{n-1} + a_2 k^{n-2} + \cdots + a_{n-1} k + a_n}{\lambda} = \dfrac{a_0 k^n + a_1 k^{n-1} + a_2 k^{n-2} + \cdots + a_{n-1} k + a_n}{k^n} \to a_0$ as

$k \to \infty$. Thus in the case when $k \to \infty$, from (7.4), we have $\lambda^2 = k^{2n} + a_0$ from which

$$\lambda = \sqrt{k^{2n} + a_0} \, .$$

Thus, as $k \to \infty$, the limiting ratio of generalized Padovan sequence and general polynomial sequence as defined in (7.1) is $\sqrt{k^{2n} + a_0}$ (7.5). The result obtained in (7.5) gives the more general case and we notice that the limiting ratios depends only on k and a_0. We also notice that if the constant term a_0 of the polynomial sequence is small then from (7.5), we see that the limiting ratio will be very close to k^n.

8. CONCLUSION

By considering Quadratic, Cubic and General polynomial sequence of nth degree, and constructing specific recurrence relations for each of them given by equations (5.1), (6.1), (7.1) we have obtained limiting ratios provided by the equations (5.5), (6.5), (7.5) respectively. The limiting ratios obtained for Quadratic and Cubic sequences are verified by two suitable illustrations in each case. In all the cases we see that if the constant term of the polynomial sequence of nth degree is a small number then the limiting ratio is approximately k^n giving a nice relationship between the degree of the polynomial sequence and the limiting ratio. This paper thus establishes the existence of limiting ratios of general Padovan sequence combined with general polynomial sequences of nth degree.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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