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ON FUZZY UPPER AND LOWER α-ℓ-CONTINUITY AND THEIR DECOMPOSITION

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Abstract. The main purpose of this paper is to introduce and study the concepts of fuzzy upper and lower α-ℓ-continuous (resp. β-ℓ-continuous, semi-ℓ-continuous and pre-ℓ-continuous) multifunctions via fuzzy ideals. Several properties of these multifunctions along with their mutual relationships are established with the help of examples. Also, we give the decomposition of fuzzy upper (resp. lower) α-ℓ-continuity and the decomposition of fuzzy upper (resp. lower) ℓ-continuity [15]. Later, we introduce new types of r-fuzzy compactness in a fuzzy ideal topological space (X,τ,ℓ) based on the sense of Šostak.

Keywords: fuzzy ideal topological space; fuzzy upper and lower α-ℓ-continuous (resp. β-ℓ-continuous, semi-ℓ-continuous and pre-ℓ-continuous) multifunctions; compactness.

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1. INTRODUCTION AND PRELIMINARIES

The theory of fuzzy sets provides a framework for mathematical modeling of those real world situations, which involve an element of uncertainty, imprecision, or vagueness in their description. Since its inception thirty years ago by Zadeh [18], this theory has found wide applications

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in engineering, economics, information sciences, medicine, etc.; for details the reader is referred to [9, 19]. A fuzzy multifunction is a fuzzy set valued function [4, 10, 16, 17]. Fuzzy multifunctions arise in many applications, for instance, the budget multifunction occurs in economic theory, noncooperative games, artificial intelligence, and decision theory. The biggest difference between fuzzy functions and fuzzy multifunctions has to do with the definition of an inverse image. For a fuzzy multifunction there are two types of inverses. These two definitions of the inverse then leads to two definitions of continuity. Ramadan and Abd El-latif [12], introduced and studied the concepts of fuzzy upper and lower almost continuous, weakly continuous and almost weakly continuous multifunctions where the domain of these functions is a classical topological space with their values as arbitrary fuzzy sets in fuzzy topological space. Taha and Abbas [14], introduced and studied the concepts of fuzzy upper and lower α-continuous (resp. β-continuous, semi-continuous and pre-continuous) multifunctions between two fuzzy topological spaces (X,τ) and (Y,η); for more details the reader is referred to [1-3, 6-7, 14-15].

In our theoretical work, we introduce the notions of fuzzy upper and lower α-ℓ-continuous (resp. β-ℓ-continuous, semi-ℓ-continuous and pre-ℓ-continuous) multifunctions via fuzzy ideals and study their various properties. Also, we discuss the relations of these multifunctions with each other with the help of examples. Moreover, we give the decomposition of fuzzy upper (resp. lower) α-ℓ-continuity and the decomposition of fuzzy upper (resp. lower) ℓ-continuity. Later, we introduce new types of r-fuzzy compactness in a fuzzy ideal topological space (X,τ,ℓ) based on the sense of Šostak. Throughout this paper, X refers to an initial universe. The family of all fuzzy sets in X is denoted by I^X and for $\lambda \in I^X$, $\lambda^c(x) = 1-\lambda(x)$ for all $x \in X$ (where $I=[0,1]$ and $I_\circ=(0,1]$). For $t \in I$, $\underline{t}(x)=t$ for all $x \in X$. The fuzzy difference between two fuzzy sets [14] $\lambda,\mu \in I^X$ defined as follows:

$$\lambda \overline{\wedge} \mu = \begin{cases} \underline{0}, & \text{if} \quad \lambda \leq \mu, \\ \lambda \wedge \mu^c, & \text{otherwise.} \end{cases}$$

All other notations are standard notations of fuzzy set theory. Now, we recall that a fuzzy idea ℓ on X [11], is a map $\ell : I^X \longrightarrow I$ that satisfies the following conditions: (i) $\forall\, \lambda,\mu \in I^X$ and $\lambda \leq \mu$ $\Rightarrow \ell(\mu) \leq \ell(\lambda)$. (ii) $\forall\, \lambda,\mu \in I^X \Rightarrow \ell(\lambda \vee \mu) \geq \ell(\lambda) \wedge \ell(\mu)$. Also, ℓ is called proper if $\ell(\underline{1}) = 0$ and there exists $\mu \in I^X$ such that $\ell(\mu) > 0$. The simplest fuzzy ideals on X, ℓ_0 and ℓ_1 defined as follows:

$$\ell_0(\lambda)=\begin{cases}1, & \text{if} \quad \lambda=\underline{0},\\ 0, & \text{otherwise}\end{cases} \qquad \text{and} \qquad \ell_1(\lambda)=1\ \forall\ \lambda\in I^X.$$

If ℓ^1 and ℓ^2 are fuzzy ideals on X, we say that ℓ^1 is finer than ℓ^2 (ℓ^2 is coarser than ℓ^1), denoted by $\ell^2\leq\ell^1$, iff $\ell^2(\lambda)\leq\ell^1(\lambda)\ \forall\ \lambda\in I^X$.

Let (X,τ) be a fuzzy topological space in Šostak sense [13], the closure and the interior of any fuzzy set $\lambda\in I^X$denoted by $C_\tau(\lambda,r)$ and $I_\tau(\lambda,r)$. Let (X,τ,ℓ) be a fuzzy ideal topological space, $\lambda\in I^X$ and $r\in I_\circ$, then the r-fuzzy local function [14] λ^*_r of λ defined as follows: $\lambda^*_r=\bigwedge\{\mu\in I^X\ :\ \ell(\lambda\ \overline{\wedge}\ \mu)\geq r,\ \tau(\mu^c)\geq r\}$. If we take $\ell=\ell_0$, for each $\lambda\in I^X$ we have $\lambda^*_r=\bigwedge\{\mu\in I^X\ :\ \lambda\leq\mu,\ \tau(\mu^c)\geq r\}=C_\tau(\lambda,r)$. Also, if we take $\ell=\ell_1$ (resp. $\ell(\lambda)\geq r$), for each $\lambda\in I^X$ we have $\lambda^*_r=\underline{0}$. Moreover, we define an operator $Cl^*:I^X\times I_\circ\to I^X$ as follows: $Cl^*(\lambda,r)=\lambda\ \vee\ \lambda^*_r$. Let (X,τ,ℓ) be a fuzzy ideal topological space and $r\in I_\circ$, then $\lambda\in I^X$ is said to be r-fuzzy ℓ-open [15] iff $\lambda\leq I_\tau(\lambda^*_r,r)$.

A mapping $F:X\multimap Y$ is called a fuzzy multifunction [5] iff $F(x)\in I^Y$ for each $x\in X$. The degree of membership of y in $F(x)$ is denoted by $F(x)(y)=G_F(x,y)$ for any $(x,y)\in X\times Y$. Also, F is Normalized iff for each $x\in X$, there exists $y_0\in Y$ such that $G_F(x,y_0)=1$ and F is Crisp iff $G_F(x,y)=1$ for each $x\in X$ and $y\in Y$. The upper inverse $F^u(\mu)$, the lower inverse $F^l(\mu)$ of $\mu\in I^Y$ and the image $F(\lambda)$ of $\lambda\in I^X$ are defined as follows: $F^u(\mu)(x)=\bigwedge_{y\in Y}[G^c_F(x,y)\vee\mu(y)]$, $F^l(\mu)(x)=\bigvee_{y\in Y}[G_F(x,y)\wedge\mu(y)]$ and $F(\lambda)(y)=\bigvee_{x\in X}[G_F(x,y)\wedge\lambda(x)]$. All definitions and properties of image, lower and upper are found in [1]. A fuzzy multifunction $F:(X,\tau,\ell)\multimap(Y,\eta)$ is called fuzzy upper (resp. lower) ℓ-continuous [15] iff $F^u(\mu)$ (resp. $F^l(\mu)$) is r-fuzzy ℓ-open for each $\mu\in I^Y$ with $\eta(\mu)\geq r$, $r\in I_o$.

2. SEVERAL TYPES OF FUZZY α-ℓ-CONTINUOUS MULTIFUNCTIONS

Definition 2.1. A fuzzy multifunction $F:(X,\tau,\ell)\multimap(Y,\eta)$ is called:

(1) Fuzzy upper α-ℓ-continuous (resp. β-ℓ-continuous, semi-ℓ-continuous and pre-ℓ-continuous) at a fuzzy point $x_t\in dom(F)$ iff $x_t\in F^u(\mu)$ for each $\mu\in I^Y$ and $\eta(\mu)\geq r$ there exists $\lambda\in I^X$ with $\lambda\leq I_\tau(Cl^*(I_\tau(\lambda,r),r),r)$ (resp. $\lambda\leq C_\tau(I_\tau(Cl^*(\lambda,r),r),r)$, $\lambda\leq Cl^*(I_\tau(\lambda,r),r)$ and $\lambda\leq I_\tau(Cl^*(\lambda,r),r)$) and $x_t\in\lambda$ such that $\lambda\wedge dom(F)\leq F^u(\mu)$.

(2) Fuzzy lower α-ℓ-continuous (resp. β-ℓ-continuous, semi-ℓ-continuous and pre-ℓ-continuous) at a fuzzy point $x_t \in dom(F)$ iff $x_t \in F^l(\mu)$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in I^X$ with $\lambda \leq I_\tau(Cl^*(I_\tau(\lambda,r),r),r)$ (resp. $\lambda \leq C_\tau(I_\tau(Cl^*(\lambda,r),r),r)$, $\lambda \leq Cl^*(I_\tau(\lambda,r),r)$ and $\lambda \leq I_\tau(Cl^*(\lambda,r),r)$) and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$.

(3) Fuzzy upper α-ℓ -continuous (lower α-ℓ-continuous, upper β-ℓ-continuous, lower β-ℓ-continuous, upper semi-ℓ-continuous, lower semi-ℓ-continuous, upper pre-ℓ-continuous and lower pre-ℓ-continuous) iff it is fuzzy upper α-ℓ-continuous (lower α-ℓ-continuous, upper β-ℓ-continuous, lower β-ℓ-continuous, upper semi-ℓ-continuo-us, lower semi-ℓ-continuous, upper pre-ℓ-continuous and lower pre-ℓ-continuous) at every $x_t \in dom(F)$.

The following implications hold:

$$\begin{array}{ccccc} \text{semi-continuity} & \Rightarrow & \alpha\text{-}\ell\text{-continuity} & \Rightarrow & \text{semi-}\ell\text{-continuity} \\ & & \Downarrow & & \Downarrow \\ \ell\text{-continuity} & \Rightarrow & \text{pre-}\ell\text{-continuity} & \Rightarrow & \beta\text{-}\ell\text{-continuity.} \end{array}$$

In general the converses are not true.

Example 2.2. Let $X = \{x_1,x_2,x_3,x_4\}$, $Y = \{y_1,y_2,y_3,y_4\}$ and $F : X \multimap Y$ be a fuzzy multifunction defined by $G_F(x_1, y_1) = 0.8$, $G_F(x_1,y_2) = 0.0$, $G_F(x_1,y_3) = 0.3$, $G_F(x_1,y_4) = 0.0$, $G_F(x_2,y_1) = 1.0$, $G_F(x_2,y_2) = 1.0$, $G_F(x_2,y_3) = 1.0$, $G_F(x_2,y_4) = 0.0$, $G_F(x_3,y_1) = 0.0$, $G_F(x_3,y_2) = 0.0$, $G_F(x_3,y_3) = 0.4$, $G_F(x_3,y_4) = 0.0$, $G_F(x_4,y_1) = 0.0$, $G_F(x_4,y_2) = 0.0$, $G_F(x_4,y_3) = 0.7$, $G_F(x_4,y_4) = 1.0$. Define $\mu_1,\mu_2 \in I^X$ and $\mu_3 \in I^Y$ as follows: $\mu_1 = \{\frac{x_1}{0.9},\frac{x_2}{0.9},\frac{x_3}{0.5},\frac{x_4}{0.5}\}$, $\mu_2 = \{\frac{x_1}{0.9},\frac{x_2}{0.9},\frac{x_3}{0.9},\frac{x_4}{0.5}\}$ and $\mu_3 = \{\frac{y_1}{0.9},\frac{y_2}{0.9},\frac{y_3}{0.9},\frac{y_4}{0.5}\}$. Define $\tau,\ell : I^X \longrightarrow I$ and $\eta : I^Y \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_1, \\ 0, & \text{otherwise,} \end{cases} \qquad \ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{1}{2}, & \text{if } \underline{0} < \nu < \underline{0.3}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\eta(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2}, & \text{if } \mu = \mu_3, \\ 0, & \text{otherwise.} \end{cases}$$

Then, $F : (X,\tau,\ell) \multimap (Y,\eta)$ is fuzzy upper α-ℓ-continuous but it is not fuzzy upper semi-continuous [1].

Remark 2.3. **(i)** If F is normalized, F is fuzzy upper α-ℓ-continuous (resp. β-ℓ-continuous, semi-ℓ-continuous and pre-ℓ-continuous) at a fuzzy point $x_t \in dom(F)$ iff $x_t \in F^u(\mu)$ for each $\mu \in I^Y$ and $\eta(\mu) \geq r$ there exists $\lambda \in I^X$ with $\lambda \leq I_\tau(Cl^*(I_\tau(\lambda,r),r),r)$ (resp. $\lambda \leq C_\tau(I_\tau(Cl^*(\lambda,r),r),r)$, $\lambda \leq Cl^*(I_\tau(\lambda,r),r)$ and $\lambda \leq I_\tau(Cl^*(\lambda,r),r))$ and $x_t \in \lambda$ such that $\lambda \leq F^u(\mu)$. **(ii)** Fuzzy upper (resp. lower) semi-ℓ-continuity and fuzzy upper (resp. lower) pre-ℓ-continuity are independent notions as shown by Example 2.4 and Example 2.5. **(iii)** Fuzzy upper (resp. lower) α-ℓ-continuity (resp. β-ℓ-continuity and pre-ℓ-continuity) $\Longrightarrow$ fuzzy upper (resp. lower) α-continuity (resp. β-continuity and precontinuity) [14]. In general the converses are not true as shown by Example 2.6, Example 2.7 and Example 2.8. **(iv)** Fuzzy upper (resp. lower) α-ℓ_0-continuous (resp. β-ℓ_0-continuous and pre-ℓ_0-continuous) $\Longleftrightarrow$ fuzzy upper (resp. lower) α-continuous (resp. β-continuous and pre-continuous).

Example 2.4. Let $X = \{x_1,x_2\}$, $Y = \{y_1,y_2,y_3\}$ and $F : X \multimap Y$ be a fuzzy multifunction defined by $G_F(x_1,y_1) = 0.1$, $G_F(x_1,y_2) = 1.0$, $G_F(x_1,y_3) = 0.3$, $G_F(x_2,y_1) = 0.5$, $G_F(x_2,y_2) = 0.1$ and $G_F(x_2,y_3) = 1.0$. Define $\tau,\ell : I^X \longrightarrow I$ and $\eta : I^Y \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \underline{0.2}, \\ \frac{3}{4}, & \text{if } \lambda = \underline{0.8}, \\ 0, & \text{otherwise,} \end{cases} \qquad \ell(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{1}{2}, & \text{if } \underline{0} < \nu < \underline{0.4}, \\ 0, & \text{otherwise,} \end{cases}$$

$$\eta(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0},\underline{1}\}, \\ \frac{1}{2}, & \text{if } \mu = \underline{0.5}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, (1) $F : (X,\tau,\ell) \multimap (Y,\eta)$ is fuzzy upper (resp. lower) pre-ℓ-continuous but it is not fuzzy upper (resp. lower) semi-ℓ-continuous.

(2) $F : (X,\tau,\ell) \multimap (Y,\eta)$ is fuzzy upper (resp. lower) β-ℓ-continuous but it is not fuzzy upper (resp. lower) semi-ℓ-continuous.

(3) $F : (X,\tau,\ell) \multimap (Y,\eta)$ is fuzzy upper (resp. lower) pre-ℓ-continuous but it is not fuzzy upper (resp. lower) α-ℓ-continuous.

Example 2.5. Let $X = \{x_1,x_2,x_3\}$, $Y = \{y_1,y_2,y_3\}$ and $F : X \multimap Y$ be fuzzy multifunction defined by $G_F(x_1,y_1) = 0.8, G_F(x_1,y_2) = 0.3, G_F(x_1,y_3) = 0.3,$ $G_F(x_2,y_1) = 0.6,$ $G_F(x_2,y_2) =$

0.1, $G_F(x_2,y_3)=0.4$, $G_F(x_3,y_1)=0.1$, $G_F(x_3,y_2)=0.2$, $G_F(x_3,y_3)=1.0$. Define $\mu_1,\mu_2\in I^X$ and $\mu_3\in I^Y$ as follows: $\mu_1=\{\frac{x_1}{0.3},\frac{x_2}{0.4},\frac{x_3}{0.8}\}$, $\mu_2=\{\frac{x_1}{0.2},\frac{x_2}{0.3},\frac{x_3}{0.2}\}$ and $\mu_3=\{\frac{y_1}{0.3},\frac{y_2}{0.4},\frac{y_3}{0.8}\}$. Define $\tau,\ell : I^X\longrightarrow I$ and $\eta : I^Y\to I$ as follows:

$$\tau(\lambda)=\begin{cases}1, & \text{if} \quad \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2}, & \text{if} \quad \lambda=\mu_2,\\ 0, & \text{otherwise},\end{cases}\qquad \ell(\nu)=\begin{cases}1, & \text{if} \quad \nu=\underline{0},\\ \frac{2}{3}, & \text{if} \quad \underline{0}<\nu<\underline{0.2},\\ 0, & \text{otherwise},\end{cases}$$

$$\eta(\mu)=\begin{cases}1, & \text{if} \quad \mu\in\{\underline{0},\underline{1}\},\\ \frac{3}{4}, & \text{if} \quad \mu=\mu_3,\\ 0, & \text{otherwise}.\end{cases}$$

Then, (1) $F:(X,\tau,\ell)\multimap(Y,\eta)$ is fuzzy upper (resp. lower) semi-ℓ-continuous but it is neither fuzzy upper (resp. lower) pre-ℓ-continuous nor fuzzy upper (resp. lower) α-ℓ-continuous.

(2) $F:(X,\tau,\ell)\multimap(Y,\eta)$ is fuzzy upper (resp. lower) β-ℓ-continuous but it is not fuzzy upper (resp. lower) pre-ℓ-continuous.

Example 2.6. Let $X=\{x_1,x_2\}$, $Y=\{y_1,y_2,y_3\}$ and $F:X\multimap Y$ be a fuzzy multifunction defined by $G_F(x_1,y_1)=0.1$, $G_F(x_1,y_2)=1.0$, $G_F(x_1,y_3)=0.3$, $G_F(x_2,y_1)=0.5$, $G_F(x_2,y_2)=0.1$ and $G_F(x_2,y_3)=1.0$. Define $\tau,\ell : I^X\longrightarrow I$ and $\eta : I^Y\to I$ as follows:

$$\tau(\lambda)=\begin{cases}1, & \text{if} \quad \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2}, & \text{if} \quad \lambda=\underline{0.3},\\ \frac{3}{4}, & \text{if} \quad \lambda=\underline{0.7},\\ 0, & \text{otherwise},\end{cases}\qquad \ell(\nu)=\begin{cases}1, & \text{if} \quad \nu=\underline{0},\\ \frac{3}{4}, & \text{if} \quad \underline{0}<\nu\le\underline{0.6},\\ 0, & \text{otherwise},\end{cases}$$

$$\eta(\mu)=\begin{cases}1, & \text{if} \quad \mu\in\{\underline{0},\underline{1}\},\\ \frac{1}{2}, & \text{if} \quad \mu=\underline{0.6},\\ 0, & \text{otherwise}.\end{cases}$$

Then, $F:(X,\tau,\ell)\multimap(Y,\eta)$ is fuzzy upper (resp. lower) precontinuous but it is not fuzzy upper (resp. lower) pre-ℓ-continuous.

Example 2.7. Let $X=\{x_1,x_2,x_3,x_4\}$, $Y=\{y_1,y_2,y_3,y_4\}$ and $F:X\multimap Y$ be a fuzzy multifunction defined by $G_F(x_1,y_1)=0.8$, $G_F(x_1,y_2)=0.0$, $G_F(x_1,y_3)=0.3$, $G_F(x_1,y_4)=0.0$, $G_F(x_2,y_1)=1.0$, $G_F(x_2,y_2)=1.0$, $G_F(x_2,y_3)=1.0$, $G_F(x_2,y_4)=0.0$, $G_F(x_3,y_1)=0.0$,

$G_F(x_3,y_2)=0.0,\ G_F(x_3,y_3)=0.4,\ G_F(x_3,y_4)=0.0,\ G_F(x_4,y_1)=0.0,\ G_F(x_4,y_2)=0.0,$ $G_F(x_4,y_3)=0.7,\ G_F(x_4,y_4)=1.0$. Define $\mu_1,\mu_2\in I^X$ and $\mu_3\in I^Y$ as follows:

$\mu_1=\{\frac{x_1}{0.9},\frac{x_2}{0.9},\frac{x_3}{0.5},\frac{x_4}{0.5}\}$, $\mu_2=\{\frac{x_1}{0.9},\frac{x_2}{0.9},\frac{x_3}{0.9},\frac{x_4}{0.5}\}$ and $\mu_3=\{\frac{y_1}{0.9},\frac{y_2}{0.9},\frac{y_3}{0.9},\frac{y_4}{0.5}\}$. Define $\tau,\ell:I^X\longrightarrow I$ and $\eta:I^Y\to I$ as follows:

$$\tau(\lambda)=\begin{cases}1, & \text{if}\quad \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{3}, & \text{if}\quad \lambda=\mu_1,\\ 0, & \text{otherwise,}\end{cases}\qquad \ell(\nu)=\begin{cases}1, & \text{if}\quad \nu=\underline{0},\\ \frac{2}{3}, & \text{if}\quad \underline{0}<\nu\leq\underline{0.9},\\ 0, & \text{otherwise,}\end{cases}$$

$$\eta(\mu)=\begin{cases}1, & \text{if}\quad \mu\in\{\underline{0},\underline{1}\},\\ \frac{1}{3}, & \text{if}\quad \mu=\mu_3,\\ 0, & \text{otherwise.}\end{cases}$$

Then, $F:(X,\tau,\ell)\multimap(Y,\eta)$ is fuzzy upper α-continuous but it is not fuzzy upper α-ℓ-continuous.

Example 2.8. Let $X=\{x_1,x_2\}$, $Y=\{y_1,y_2,y_3\}$ and $F:X\multimap Y$ be a fuzzy multifunction defined by $G_F(x_1,y_1)=0.1$, $G_F(x_1,y_2)=1.0, G_F(x_1,y_3)=0.3$, $G_F(x_2,y_1)=0.5$, $G_F(x_2,y_2)=0.1$ and $G_F(x_2,y_3)=1.0$. Define $\tau,\ell:I^X\longrightarrow I$ and $\eta:I^Y\to I$ as follows:

$$\tau(\lambda)=\begin{cases}1, & \text{if}\quad \lambda\in\{\underline{0},\underline{1}\},\\ \frac{1}{2}, & \text{if}\quad \lambda=\underline{0.2},\\ \frac{3}{4}, & \text{if}\quad \lambda=\underline{0.8},\\ 0, & \text{otherwise,}\end{cases}\qquad \ell(\nu)=\begin{cases}1, & \text{if}\quad \nu=\underline{0},\\ \frac{1}{2}, & \text{if}\quad \underline{0}<\nu\leq\underline{0.8},\\ 0, & \text{otherwise,}\end{cases}$$

$$\eta(\mu)=\begin{cases}1, & \text{if}\quad \mu\in\{\underline{0},\underline{1}\},\\ \frac{1}{2}, & \text{if}\quad \mu=\underline{0.5},\\ 0, & \text{otherwise.}\end{cases}$$

Then, $F:(X,\tau,\ell)\multimap(Y,\eta)$ is fuzzy upper (resp. lower) β-continuous but it is not fuzzy upper (resp. lower) β-ℓ-continuous.

Theorem 2.9 For a fuzzy multifunction $F:(X,\tau,\ell)\multimap(Y,\eta)$, $\mu\in I^Y$ and $r\in I_o$ the following statements are equivalent:

(1) F is fuzzy lower α-ℓ-continuous.

(2) $F^l(\mu)\leq I_\tau(Cl^*(I_\tau(F^l(\mu),r),r),r)$, if $\eta(\mu)\geq r$.

(3) $C_\tau(int^*(C_\tau(F^u(\mu),r),r),r) \leq F^u(\mu)$, if $\eta(\mu^c) \geq r$.

(4) $F^l(I_\eta(\mu,r)) \leq I_\tau(Cl^*(I_\tau(F^l(I_\eta(\mu,r)),r),r),r)$.

Proof. (1) $\Rightarrow$ (2) Let $x_t \in dom(F)$, $\mu \in I^Y$ with $\eta(\mu) \geq r$ and $x_t \in F^l(\mu)$. Then there exists $\lambda \in I^X$ with $\lambda \leq I_\tau(Cl^*(I_\tau(\lambda,r),r),r)$ and $x_t \in \lambda$ such that $\lambda \leq F^l(\mu)$. Thus, $x_t \in \lambda \leq I_\tau(Cl^*(I_\tau(F^l(\mu),r),r),r)$ and $F^l(\mu) \leq I_\tau(Cl^*(I_\tau(F^l(\mu),r),r),r)$.

(2) $\Rightarrow$ (3) Let $\mu \in I^Y$ with $\eta(\mu^c) \geq r$.

Then by (2), $(F^u(\mu))^c = F^l(\mu^c) \leq I_\tau(Cl^*(I_\tau(F^l(\mu^c),r),r),r) = (C_\tau(int^*(C_\tau(F^u(\mu),r),r),r))^c$. Thus, $F^u(\mu) \geq C_\tau(int^*(C_\tau(F^u(\mu),r),r),r)$.

(3) $\Rightarrow$ (4) Since $C_\tau(int^*(C_\tau(F^u(C_\eta(\mu,r)),r),r),r) \leq F^u(C_\eta(\mu,r))$ for each $\mu \in I^Y$. By the straighforward calculations, $F^l(I_\eta(\mu,r)) \leq I_\tau(Cl^*(I_\tau(F^l(I_\eta(\mu,r)),r),r),r)$.

(4) $\Rightarrow$ (1) Let $x_t \in dom(F)$, $\mu \in I^Y$ with $\eta(\mu) \geq r$ and $x_t \in F^l(\mu)$. Then by (4) and $\mu = I_\eta(\mu,r)$, $x_t \in F^l(\mu) \leq I_\tau(Cl^*(I_\tau(F^l(\mu),r),r),r)$. Thus, F is fuzzy lower α-ℓ-continuous.

The following theorems are similarly proved as in Theorem 2.9.

Theorem 2.10 For a fuzzy multifunction $F : (X,\tau,\ell) \multimap (Y,\eta)$, $\mu \in I^Y$ and $r \in I_o$ the following statements are equivalent:

(1) F is fuzzy lower β-ℓ-continuous.

(2) $F^l(\mu) \leq C_\tau(I_\tau(Cl^*(F^l(\mu),r),r),r)$, if $\eta(\mu) \geq r$.

(3) $I_\tau(C_\tau(int^*(F^u(\mu),r),r),r) \leq F^u(\mu)$, if $\eta(\mu^c) \geq r$.

(4) $F^l(I_\eta(\mu,r)) \leq C_\tau(I_\tau(Cl^*(F^l(I_\eta(\mu,r)),r),r),r)$.

Theorem 2.11 For a fuzzy multifunction $F : (X,\tau,\ell) \multimap (Y,\eta)$, $\mu \in I^Y$ and $r \in I_o$ the following statements are equivalent:

(1) F is fuzzy lower semi-ℓ-continuous.

(2) $F^l(\mu) \leq Cl^*(I_\tau(F^l(\mu),r),r)$, if $\eta(\mu) \geq r$.

(3) $int^*(C_\tau(F^u(\mu),r),r) \leq F^u(\mu)$, if $\eta(\mu^c) \geq r$.

(4) $F^l(I_\eta(\mu,r)) \leq Cl^*(I_\tau(F^l(I_\eta(\mu,r)),r),r)$.

Theorem 2.12 For a fuzzy multifunction $F : (X,\tau,\ell) \multimap (Y,\eta)$, $\mu \in I^Y$ and $r \in I_o$ the following statements are equivalent:

(1) F is fuzzy lower pre-ℓ-continuous.

(2) $F^l(\mu) \leq I_\tau(Cl^*(F^l(\mu),r),r)$, if $\eta(\mu) \geq r$.

(3) $C_\tau(int^*(F^u(\mu),r),r) \le F^u(\mu)$, if $\eta(\mu^c) \ge r$.

(4) $F^l(I_\eta(\mu,r)) \le I_\tau(Cl^*(F^l(I_\eta(\mu,r)),r),r)$.

Theorem 2.13 For a normalized fuzzy multifunction $F : (X,\tau,\ell) \multimap (Y,\eta)$, $\mu \in I^Y$ and $r \in I_o$ the following statements are equivalent:

(1) F is fuzzy upper α-ℓ-continuous.

(2) $F^u(\mu) \le I_\tau(Cl^*(I_\tau(F^u(\mu),r),r),r)$, if $\eta(\mu) \ge r$.

(3) $C_\tau(int^*(C_\tau(F^l(\mu),r),r),r) \le F^l(\mu)$, if $\eta(\mu^c) \ge r$.

(4) $F^u(I_\eta(\mu,r)) \le I_\tau(Cl^*(I_\tau(F^u(I_\eta(\mu,r)),r),r),r)$.

Theorem 2.14 For a normalized fuzzy multifunction $F : (X,\tau,\ell) \multimap (Y,\eta)$, $\mu \in I^Y$ and $r \in I_o$ the following statements are equivalent:

(1) F is fuzzy upper β-ℓ-continuous.

(2) $F^u(\mu) \le C_\tau(I_\tau(Cl^*(F^u(\mu),r),r),r)$, if $\eta(\mu) \ge r$.

(3) $I_\tau(C_\tau(int^*(F^l(\mu),r),r),r) \le F^l(\mu)$, if $\eta(\mu^c) \ge r$.

(4) $F^u(I_\eta(\mu,r)) \le C_\tau(I_\tau(Cl^*(F^u(I_\eta(\mu,r)),r),r),r)$.

Theorem 2.15 For a normalized fuzzy multifunction $F : (X,\tau,\ell) \multimap (Y,\eta)$, $\mu \in I^Y$ and $r \in I_o$ the following statements are equivalent:

(1) F is fuzzy upper semi-ℓ-continuous.

(2) $F^u(\mu) \le Cl^*(I_\tau(F^u(\mu),r),r)$, if $\eta(\mu) \ge r$.

(3) $int^*(C_\tau(F^l(\mu),r),r) \le F^l(\mu)$, if $\eta(\mu^c) \ge r$.

(4) $F^u(I_\eta(\mu,r)) \le Cl^*(I_\tau(F^u(I_\eta(\mu,r)),r),r)$.

Theorem 2.16 For a normalized fuzzy multifunction $F : (X,\tau,\ell) \multimap (Y,\eta)$, $\mu \in I^Y$ and $r \in I_o$ the following statements are equivalent:

(1) F is fuzzy upper pre-ℓ-continuous.

(2) $F^u(\mu) \le I_\tau(Cl^*(F^u(\mu),r),r)$, if $\eta(\mu) \ge r$.

(3) $C_\tau(int^*(F^l(\mu),r),r) \le F^l(\mu)$, if $\eta(\mu^c) \ge r$.

(4) $F^u(I_\eta(\mu,r)) \le I_\tau(Cl^*(F^u(I_\eta(\mu,r)),r),r)$.

3. DECOMPOSITION OF FUZZY α-ℓ-CONTINUOUS MULTIFUNCTIONS

The following results give the decomposition of fuzzy upper (resp. lower) α-ℓ-continuity and the decomposition of fuzzy upper (resp. lower) ℓ-continuity.

Theorem 3.1 A fuzzy multifunction (resp. normalized fuzzy multifunction) $F:(X,\tau,\ell)\multimap(Y,\eta)$ is fuzzy lower (resp. upper) α-ℓ-continuous iff it is both fuzzy lower (resp. upper) semi-ℓ-continuous and fuzzy lower (upper) pre-ℓ-continuous.

Proof. ($\Rightarrow$) Let F be fuzzy upper α-ℓ-continuous, then

$$F^u(\mu)\leq I_\tau(Cl^*(I_\tau(F^u(\mu),r),r),r)\leq Cl^*(I_\tau(F^u(\mu),r),r)$$

for every $\mu\in I^Y$ with $\eta(\mu)\geq r$ and $r\in I_o$. This shows that F is fuzzy upper semi-ℓ-continuous. Moreover, $F^u(\mu)\leq I_\tau(Cl^*(I_\tau(F^u(\mu),r),r),r)=I_\tau(I_\tau(F^u(\mu),r)\vee[I_\tau(F^u(\mu),r)]^*_r,r)\leq I_\tau(F^u(\mu)\vee[F^u(\mu)]^*_r,r)=I_\tau(Cl^*(F^u(\mu),r),r)$. Therefore, F is fuzzy upper pre-ℓ-continuous.

($\Leftarrow$) Let F be fuzzy upper pre-ℓ-continuous and fuzzy upper semi-ℓ-continuous. Then, for every $\mu\in I^Y$ with $\eta(\mu)\geq r$ and $r\in I_o$

$$F^u(\mu)\leq I_\tau(Cl^*(F^u(\mu),r),r)\leq I_\tau(Cl^*(Cl^*(I_\tau(F^u(\mu),r),r),r),r)=I_\tau(Cl^*(I_\tau(F^u(\mu),r),r),r).$$

Thus, F is fuzzy upper α-ℓ-continuous. Other case is similarly proved.

Definition 3.2 A fuzzy multifunction $F:(X,\tau,\ell)\multimap(Y,\eta)$ is called fuzzy upper (resp. lower) $*$-ℓ-continuous iff $F^u(\mu)\leq(F^u(\mu))^*_r$ (resp. $F^l(\mu)\leq(F^l(\mu))^*_r$) for every $\mu\in I^Y$ with $\eta(\mu)\geq r$ and $r\in I_o$.

Theorem 3.3 A fuzzy multifunction (resp. normalized fuzzy multifunction) $F:(X,\tau,\ell)\multimap(Y,\eta)$ is fuzzy lower (resp. upper) ℓ-continuous iff it is both fuzzy lower (resp. upper) pre-ℓ-continuous and fuzzy lower (resp. upper) $*$-ℓ-continuous.

Proof. ($\Rightarrow$) Let F be fuzzy upper ℓ-continuous, then

$$F^u(\mu)\leq I_\tau([F^u(\mu)]^*_r,r)\leq[F^u(\mu)]^*_r$$

for every $\mu\in I^Y$ with $\eta(\mu)\geq r$ and $r\in I_o$. This shows that F is fuzzy upper $*$-ℓ-continuous. Moreover, $F^u(\mu)\leq I_\tau([F^u(\mu)]^*_r,r)\leq I_\tau(F^u(\mu)\vee[F^u(\mu)]^*_r,r)=I_\tau(Cl^*(F^u(\mu),r),r)$. Therefore, F is fuzzy upper pre-ℓ-continuous.

($\Leftarrow$) Let F be fuzzy upper pre-ℓ-continuous and fuzzy upper $*$-ℓ-continuous,

$$F^u(\mu)\leq I_\tau(Cl^*(F^u(\mu),r),r)=I_\tau(F^u(\mu)\vee[F^u(\mu)]^*_r,r)=I_\tau([F^u(\mu)]^*_r,r).$$

Thus, F is fuzzy upper ℓ-continuous. Other case is similarly proved.

Remark 3.4 Fuzzy upper (resp. lower) pre-ℓ-continuity and fuzzy upper (resp. lower) $*$-ℓ-continuity are independent notions as shown by Example 3.5.

Example 3.5. Let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2, y_3\}$ and $F : X \multimap Y$ be fuzzy multifunction defined by $G_F(x_1,y_1) = 0.5, G_F(x_1,y_2) = 0.3, G_F(x_1,y_3) = 0.3$, $G_F(x_2,y_1) = 0.5$, $G_F(x_2,y_2) = 0.7$, $G_F(x_2,y_3) = 0.4$, $G_F(x_3,y_1) = 0.1$, $G_F(x_3,y_2) = 0.4$, $G_F(x_3,y_3) = 1.0$. Define $\mu_1, \mu_2 \in I^X$ and $\mu_3, \mu_4 \in I^Y$ as follows: $\mu_1 = \{\frac{x_1}{0.5}, \frac{x_2}{0.4}, \frac{x_3}{0.6}\}$, $\mu_2 = \{\frac{x_1}{0.5}, \frac{x_2}{0.3}, \frac{x_3}{0.4}\}$, $\mu_3 = \{\frac{y_1}{0.5}, \frac{y_2}{0.3}, \frac{y_3}{0.4}\}$ and $\mu_4 = \{\frac{y_1}{0.5}, \frac{y_2}{0.4}, \frac{y_3}{0.6}\}$. Define $\tau, \ell^1, \ell^2 : I^X \longrightarrow I$ and $\eta_1, \eta_2 : I^Y \to I$ as follows:

$$\tau(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \lambda = \mu_1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\ell^1(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{1}{2}, & \text{if } \underline{0} < \nu < \underline{0.3}, \\ 0, & \text{otherwise}, \end{cases} \qquad \ell^2(\nu) = \begin{cases} 1, & \text{if } \nu = \underline{0}, \\ \frac{1}{2}, & \text{if } \underline{0} < \nu \leq \underline{0.6}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\eta_1(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \mu = \mu_3, \\ 0, & \text{otherwise}, \end{cases} \qquad \eta_2(\mu) = \begin{cases} 1, & \text{if } \mu \in \{\underline{0}, \underline{1}\}, \\ \frac{1}{2}, & \text{if } \mu = \mu_4, \\ 0, & \text{otherwise}. \end{cases}$$

Then, (1) $F : (X, \tau, \ell^1) \multimap (Y, \eta_1)$ is fuzzy upper $*$-continuous but it is not fuzzy upper pre-ℓ-continuous. (2) $F : (X, \tau, \ell^2) \multimap (Y, \eta_2)$ is fuzzy upper pre-ℓ-continuous but it is neither fuzzy upper $*$-continuous nor fuzzy upper ℓ-continuous.

Lemma 3.6 If a fuzzy multifunction (resp. normalized fuzzy multifunction) $F : (X, \tau, \ell) \multimap (Y, \eta, \ell')$ is fuzzy lower (resp. upper) pre-ℓ-continuous and $Cl^*(F^l(\mu), r) \leq F^l(Cl^*(\mu, r))$ (resp. $Cl^*(F^u(\mu), r) \leq F^u(Cl^*(\mu, r))$) for every $\mu \in I^Y$ with $\eta(\mu) \geq r$, $r \in I_o$, then F is fuzzy lower (resp. upper) weakly ℓ'-continuous.

Corollary 3.7 (1) Let $F : (X, \tau, \ell) \multimap (Y, \eta)$ and $H : (Y, \eta) \multimap (Z, \gamma)$ be two fuzzy multifunctions. Then $H \circ F$ is fuzzy lower α-ℓ-continuous (resp. β-ℓ-continuous, semi-ℓ-continuous and pre-ℓ-continuous) if F is fuzzy lower α-ℓ-continuous (resp. β-ℓ-continuous, semi-ℓ-continuous and pre-ℓ-continuous) and H is fuzzy lower semi-continuous. **(2)** Let $F : (X, \tau, \ell) \multimap (Y, \eta)$ and

$H:(Y,\eta)\multimap(Z,\gamma)$ be two normalized fuzzy multifunctions. Then $H\circ F$ is fuzzy upper α-ℓ-continuous (resp. β-ℓ-continuous, semi-ℓ-continuous and pre-ℓ-continuous) if F is fuzzy upper α-ℓ-continuous (resp. β-ℓ-continuous, semi-ℓ-continuous and pre-ℓ-continuous) and H is fuzzy upper semi-continuous.

4. New types of r-Fuzzy Compactness via Fuzzy Ideals

Definition 4.1. Let (X,τ,ℓ) be a fuzzy ideal topological space and $r\in I_\circ$. Then $\lambda\in I^X$ is called r-fuzzy $*$-compact (resp. r-fuzzy almost $*$-compact and r-fuzzy nearly $*$-compact) iff for every family $\{\mu_i\in I^X\mid\tau^*(\mu_i)\geq r\}_{i\in\Gamma}$ such that $\lambda\leq\bigvee_{i\in\Gamma}\mu_i$, there exists a finite subset $\Gamma_\circ$ of Γ such that $\lambda\leq\bigvee_{i\in\Gamma_\circ}\mu_i$ (resp. $\lambda\leq\bigvee_{i\in\Gamma_\circ}Cl^*(\mu_i,r)$ and $\lambda\leq\bigvee_{i\in\Gamma_\circ}I_\tau(Cl^*(\mu_i,r),r)$).

Corollary 4.2 Let (X,τ,ℓ) be a fuzzy ideal topological space and $r\in I_\circ$.

(1) Every r-fuzzy $*$-compact set is r-fuzzy compact [8].

(2) Every r-fuzzy $*$-compact set is r-fuzzy almost $*$-compact.

(3) Every r-fuzzy almost $*$-compact set is r-fuzzy almost compact [8].

(4) Every r-fuzzy nearly $*$-compact set is r-fuzzy almost $*$-compact.

(5) Every r-fuzzy nearly $*$-compact set is r-fuzzy nearly compact [8].

Corollary 4.3 Let (X,τ,ℓ) be a fuzzy ideal topological space, $\lambda\in I^X$ and $r\in I_\circ$. If we take $\ell=\ell_0$, we have

(1) r-fuzzy $*$-compact and r-fuzzy compact are equivalent.

(2) r-fuzzy almost $*$-compact and r-fuzzy almost compact are equivalent.

(3) r-fuzzy nearly $*$-compact and r-fuzzy nearly compact are equivalent.

Theorem 4.4 Let $F:(X,\tau,\ell)\multimap(Y,\eta)$ be a crisp fuzzy upper semi-continuous multifunction and compact-valued. Then if $\lambda\in I^X$ is r-fuzzy $*$-compact, $F(\lambda)$ is r-fuzzy compact .

Proof. Let $\{\mu_i\in I^Y\mid\eta(\mu_i)\geq r\}_{i\in\Gamma}$ with $F(\lambda)\leq\bigvee_{i\in\Gamma}\mu_i$. Since $\lambda=\bigvee_{x_t\in\lambda}x_t$,

$$F(\lambda)=F(\bigvee_{x_t\in\lambda}x_t)=\bigvee_{x_t\in\lambda}F(x_t)\leq\bigvee_{i\in\Gamma}\mu_i.$$

It follows that for each $x_t\in\lambda$, $F(x_t)\leq\bigvee_{i\in\Gamma}\mu_i$. Since F is compact-valued, then there exists finite subset Γ_{x_t} of Γ such that $F(x_t)\leq\bigvee_{n\in\Gamma_{x_t}}\mu_n=\mu_{x_t}$. By Proposition 2.1.10 [14], we have

$x_t \leq F^u(F(x_t)) \leq F^u(\mu_{x_t})$ and

$$\lambda = \bigvee_{x_t \in \lambda} x_t \leq \bigvee_{x_t \in \lambda} F^u(\mu_{x_t}).$$

Since $\eta(\mu_{x_t}) \geq r$ then from Theorem 2.2.3 [14], we have $\tau(F^u(\mu_{x_t})) \geq \eta(\mu_{x_t}) \geq r$. Hence $\{F^u(\mu_{x_t}) :\ \tau^*(F^u(\mu_{x_t})) \geq r,\ x_t \in \lambda\}$ is a family covering the set λ. Since λ is r-fuzzy $*$-compact, then there exists finite index set N of Γ_{x_t} such that $\lambda \leq \bigvee_{n\in N} F^u(\mu_{x_{(t_n)}})$. From Propositions 2.1.6 and 2.1.10 [14],

$$F(\lambda) \leq F(\bigvee_{n\in N} F^u(\mu_{x_{(t_n)}})) = \bigvee_{n\in N} F(F^u(\mu_{x_{(t_n)}})) \leq \bigvee_{n\in N} \mu_{x_{(t_n)}}.$$

Then $F(\lambda)$ is r-fuzzy compact.

The following theorems are similarly proved as in Theorem 4.4.

Theorem 4.5 Let $F : (X,\tau,\ell) \multimap (Y,\eta)$ be a crisp fuzzy upper almost continuous multifunction and compact-valued. Then if $\lambda \in I^X$ is r-fuzzy $*$-compact, $F(\lambda)$ is r-fuzzy nearly compact .

Theorem 4.6 Let $F : (X,\tau,\ell) \multimap (Y,\eta)$ be a crisp fuzzy upper weakly continuous multifunction and compact-valued. Then if $\lambda \in I^X$ is r-fuzzy $*$-compact, $F(\lambda)$ is r-fuzzy almost compact .

5. CONCLUSION

In the present work, we have continued to study the continuity of fuzzy multifunctions via fuzzy ideals. We introduce the notions of fuzzy upper and lower α-ℓ-continuous (resp. β-ℓ-continuous, semi-ℓ-continuous and pre-ℓ-continuous) multifunctions via fuzzy ideals and study their various properties. Also, we discuss the relations of these multifunctions with each other with the help of examples. Next, we give the decomposition of fuzzy upper (resp. lower) α-ℓ-continuity and the decomposition of fuzzy upper (resp. lower) ℓ-continuity. Later, we introduce new types of r-fuzzy compactness in a fuzzy ideal topological space (X,τ,ℓ) based on the sense of Šostak. We hope that the findings in this paper will help researcher enhance and promote the further study on continuity of fuzzy multifunctions to carry out a general framework for their applications in practical life.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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