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DISTANCE RELATED SPECTRUM OF THE ZERO-DIVISOR GRAPH ON THE RING OF INTEGERS MODULO n

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Abstract. For a commutative ring R with non-zero identity, let $Z^*(R)$ denote the set of non-zero zero-divisors of R. The zero-divisor graph of R, denoted by $\Gamma(R)$, is a simple undirected graph with all non-zero zero-divisors as vertices and two distinct vertices $x,y \in Z^*(R)$ are adjacent if and only if $xy = 0$. In this paper, the distance, distance Laplacian and the distance singless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$, for $n = p^3, pq$ are investigated.

Keywords: eigenvalues; distance spectrum; zero-divisor graph; block matrix.

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1. INTRODUCTION

Let $G = (V(G), E(G))$ be a simple, finite, undirected graph where $V(G)$ denotes vertex set and $E(G)$ denotes edge set. The cardinality of $V(G)$ is the order of G. A graph G, is connected if there is a path between any two vertices. For distinct vertices u and v, let $d_G(u,v)$ denote the distance between them, that is the length of a shortest path between u and v. Clearly $d_G(u,u) = 0$ and $d_G(u,v) = \infty$, if there is no path between u and v . If $u \in V(G)$, the open neighborhood of u; denoted by $N_G(u)$ is the set of vertices adjacent to u in G. The cardinality of $N_G(u)$ is

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the degree of u. In a connected graph G, the transmission degree of a vertex v is defined as $Tr(v) = \sum_{u\in V(G)} d_G(u,v)$. $A(G)$, the adjacency matrix of a graph G of order n is a $0-1$ matrix of order $n\times n$ with entries a_{ij} such that $a_{ij} = 1$ if $ij \in E(G)$ and $a_{ij} = 0$ otherwise.

The Laplacian matrix of a graph G is defined as $L(G) = A(G) - Deg(G)$, and signless Laplacian matrix of G is defined as $Q(G) = A(G) + Deg(G)$ where $Deg(G)$ is the diagonal matrix of degree of vertices. Note that $L(G)$ and $Q(G)$ are positive semi definite matrices. The distance matrix of a simple connected graph G of order n is the $n\times n$ symmetric matrix $D = (d_G(u,v))$, the rows and columns labeled by vertices, where $d_G(u,v)$ is the distance between the vertices u and v and diagonal entries are zeroes. The study of distance Laplacian and distance signless Laplacian was initiated in [15] by M. Aouchiche and P. Hansen. For a connected graph G, the distance Laplacian matrix is given by $D^L(G) = Tr(G) - D(G)$ and the distance signless Laplacian matrix is $D^Q(G) = Tr(G) + D(G)$; where $Tr(G)$ is the diagonal matrix of vertex transmission of G.

Let R be a commutative ring with non zero identity. Let $Z^*(R) = Z(R)\setminus(0)$ be the set of non-zero zero-divisors of R. In [11], Beck associated to a commutative ring R its zero-divisor graph $G(R)$ whose vertices are the zero-divisors of R (including 0) and two distinct vertices a and b are adjacent if and only if $ab = 0$ Anderson and Livingston redefined the concept of zero-divisor graph and introduced the subgraph $\Gamma(R)$(of $G(R)$) as zero-divisor graph whose vertices are the non-zero zero-divisor of R and the authors studied the interplay between the ring theoretic properties of a commutative ring and the graph theoretic properties of its zero-divisor graph. In [24], the authors described the structure of $\Gamma(\mathbb{Z}_n)$ as the join of pairwise disjoint induced subgraphs which are regular. In [18] Magi P.M and et. al. has described the analysis of the adjacency matrix and some graph parameters of $\Gamma(\mathbb{Z}_{p^n})$ and described the computation of the eigenvalues of $\Gamma(\mathbb{Z}_{p^n})$ exploring its structure as the generalized join of its induced subgraphs. This paper aims to find the distance, distance Laplacian and distance signless Laplacian of $\Gamma(\mathbb{Z}_n)$, for some values of n.

2. Basic Befinitions and Notations

A graph G is said to be complete if any two distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n. The complement of K_n is a null graph and is denoted by $\overline{K_n}$. A partition $\{V_1, V_2, ..., V_k\}$ of the vertex set of $V(G)$ is said to be an equitable partition, if any two vertices in V_i, have the same number of neighbours in V_j for $1 \leq i \leq j \leq k$. In [22], Sabidussi has defined the generalized join of a family of graphs $\{Y_x\}_{x \in X}$, indexed by $V(X)$, as the graph Z with $V(Z) = \{(x,y) : x \in X, y \in Y_x\}$. and $E(Z) = \{((x,y),(x',y')) : (x,x') \in E(X)$ or else $x = x'$ and $(y,y') \in E(Y_x)\}$. Let G be a finite graph with vertices labeled as $1,2,3,...,n$ and let $H_1, H_2, ..., H_n$ be a family of vertex disjoint graphs. The generalized join of $H_1, H_2, ..., H_n$ denoted by $G[H_1, H_2, ..., H_n]$ is obtained by replacing each vertex i of G by the graph H_i and inserting all or none of the possible edges between H_i and H_j depending on whether or not i and j are adjacent in G. ie, $Z = G[H_1, H_2, ..., H_n]$ is obtained by taking the union of $H_1, H_2, ..., H_n$ and joining each vertex of H_i to all vertices of H_j if and only if $ij \in E(G)$. The basic definitions in graph theory are standard and are from [13,17]. Let M be a square matrix of order $n \times n$ associated to a graph G of order n. The eigenvalues of M are the roots of the characteristic polynomial, $det(xI - M)$ and the spectrum of M is the multi set of all the eigenvalues of M counted with multiplicities. An eigenvalue of a matrix is simple, if its algebraic multiplicity is 1. For a real symmetric matrix, all eigenvalues are real and the algebraic multiplicity of each eigenvalue is same as its geometric multiplicity.

The characteristic polynomial of a graph G is given by $\Phi(G,x) = det(xI - A)$, and the spectrum of G is denoted by $Spec(G)$. Similarly $SpecD(G)$, $SpecD^L(G)$ and $SpecD^Q(G)$ denote the spectrum of G related to the distance, distance Laplacian and the distance signless Laplacian matrix of G respectively. Let us denote $det(xI - D(G)), det(xI - D^L(G))$ and $det(xI - D^Q(G))$ by $\Phi_D(G;x); \Phi_{D^L}(G;x)$ and $\Phi_{D^Q}(G;x)$ respectively. For a connected graph G on n vertices, let $\partial_1 \geq \partial_2 \geq ... \geq \partial_n$ denote the distance spectrum and $\partial_1^L \geq \partial_2^L \geq ... \geq \partial_n^L$ denote the distance Laplacian spectrum and $\partial_1^Q \geq \partial_2^Q \geq ... \geq \partial_n^Q$ denote the distance signless Laplacian spectrum of G. For a connected graph let $\sigma(G)$ denote the *Transmission* of G which is the sum of the distance between the unordered pairs of vertices. Then clearly, $\sigma(G)= \frac{1}{2}\Sigma_{v \in V(G)} Tr(v)$. The greatest distance eigenvalue of a graph G is called the distance spectral radius of G and the distance energy of a connected graph G, is the sum of the absolute values of the distance eigenvalues.

For a natural number n, $\phi(n)$ is the number of positive integer less than n and relatively prime to n. In this paper, J denotes an all-one matrix and O denotes a zero matrix.

3. Preliminaries

Let $\mathbb{Z}_n$ denote the commutative ring of integers modulo n. When n is a prime, $\mathbb{Z}_n$ is an integral domain and has no zero divisors. Thus to avoid triviality, we assume that n is not a prime. we recall that in any finite commutative ring with unity, every non-zero element is either a unit or a zero-divisor. The number of non-zero zero-divisors of $\mathbb{Z}_n$ is $n-\phi(n)-1$. [18].

In [24], Sriparna Chattopadhyay et.al. describe the structure of $\Gamma(\mathbb{Z}_n)$ as the generalised join of its induced subgraphs all of which are regular.

By a proper divisor of n, we mean a positive divisor d such that d/n , $1<d<n$. Let $s(n)$ denote the number of proper divisors of n. Then, $s(n)=\sigma_0(n)-2$, where $\sigma_k(n)$ is the sum of k powers of all divisors of n, including n and 1.

If $n=p_1^{n_1}\cdot p_2^{n_2}\cdots p_r^{n_r}$, where $p_1,p_2,...,p_r$ are distinct primes and $n_1,n_2,...,n_r$ are positive integers,

$$s(n)=\prod_{i=1}^{r}(n_i+1) \quad -2.$$

Let $S(d)=\{k\in\mathbb{Z}_n : gcd(k,n)=d\}$. Clearly $\{S(d_1),S(d_2),...,S(d_{s(n)})\}$ is a collection of pairwise disjoint sets of vertices and is an equitable partition for the vertex set of $\Gamma(\mathbb{Z}_n)$ such that $S(d_i)\cap S(d_j)=\phi, i\neq j$, and any two vertices in $S(d_i)$ have the same number of neighbours in $S(d_j)$ for all divisors d_i,d_j of n. Using elementary number theory, it can be seen that

$|S(d_i)|=\phi(\dfrac{n}{d_i})$, for every $i=1,2,...s(n)$.

Also the subgraph of $\Gamma(\mathbb{Z}_n)$, induced by $S(d_i), i=1,2,...,s(n)$ (denoted by $\Gamma(S(d_i))$) is either $K_{\phi(\frac{n}{d_i})}$ or $\overline{K}_{\phi(\frac{n}{d_i})}$, accordingly as n/d_i^2 or not. For example, in $\Gamma(\mathbb{Z}_{p^3})$, $S(p)$ induces $\overline{K}_{p(p-1)}$ and $S(p^2)$ induces K_{p-1}. In $\Gamma(\mathbb{Z}_{p^2q})$, $S(p),S(q),S(p^2)$ induce $\overline{K}_{(p-1)(q-1)}$, $\overline{K}_{p(p-1)},\overline{K}_{q-1}$ respectively while $S(pq)$ induces K_{p-1}; which is visible from the diagonal blocks $0, J-I$ in the adjacency matrices of respective graphs. It is obvious that $\Gamma(S(d_i))$ is regular for each $i=1,2,...,s(n)$.

Let Υ_n denote the compressed zero-divisor graph which is a simple connected graph associated with $\Gamma(\mathbb{Z}_n)$ with vertices labeled as $1,2,...,s(n)$. See[5]. In [10], the authors designate this graph as proper divisor graph, where as in [16] it is named as projection graph. Then, the vertices i and j are adjacent in Υ_n if and only if n/d_id_j.[24]. Thus the zero divisor graph on the ring of integers modulo n can be portrayed as

$$\mathit{\Gamma}(\mathbb{Z}_n) = \Upsilon_n[\Gamma(S(d_1)), \Gamma(S(d_2)), ..., \Gamma(S(d_{s(n)}))].$$

The fact that the subgraphs $\Gamma(S(d_i)), i = 1, 2, ..., s(n)$, are regular and all the more complete or null graphs; makes the computation of spectrum of $\mathit{\Gamma}(\mathbb{Z}_n)$ interesting, by applying the tools for computing the spectrum of such combinatorial structures like generalized join.

The study of zero-divisor graphs has attracted the attention of researchers since 1988 when I. Beck initiated the study of zero divisor graph in connection with colouring. The characterization of rings based on the graph theoretic properties of the zero-divisor graphs has recently opened the a wide scope of research [1,4,7,19,21,23].

4. DISTANCE SPECTRUM OF $\mathit{\Gamma}(\mathbb{Z}_{pq})$ AND $\mathit{\Gamma}(\mathbb{Z}_{p^3})$ WHERE p AND q ARE DISTINCT PRIMES, $p < q$

The notion of distance and transmission of vertices in a graph is applied in many realms of the physical world including the design of communication networks. A wide survey of distance spectra of graphs can be found in [14]. In analogous with energy of graphs , the concept of distance energy and Laplacian energy of graphs was introduced in [9,12]. In this section, we illustrate the direct computation of the distance spectrum of $\mathit{\Gamma}(\mathbb{Z}_n)$, where $n = p^3, pq$, where p and q are distinct primes, $p < q$. We recall that $\mathit{\Gamma}(\mathbb{Z}_n)$ is a complete graph if and only if n is the square of a prime and a complete bipartite graph if and only if n is the product of two distinct primes or $n = 8$. We make use of the equitable partition of the vertex set of $\mathit{\Gamma}(\mathbb{Z}_{p^3})$ and its combinatorial structure as the Υ_n-join of the induced subgraphs, where Υ_n is the compressed zero divisor graph of $\mathit{\Gamma}(\mathbb{Z}_n)$, as described in section 3.

The following lemmas are used to find the characteristic polynomial of the block matrices in the proof of the following theorems.

Lemma 4.1. [8] Let M,N,P,Q be matrices and let M be invertible. Let $S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}$. Then,

$$detS = detM.\, det(Q - PM^{-1}N).$$

$(Q - PM^{-1}N)$ is called the Schur complement of M in S.

Definition 4.1. [3] Let $\mathbf{1}_n$ denote an all-one vector. The coronal of a matrix A, denoted by $\Gamma_A(x)$, is defined as the sum of the entries of the matrix $(xI - A)^{-1}$. That is,

$$\Gamma_A(x) = (\mathbf{1}_n)^T.(xI - A)^{-1}.\mathbf{1}_n$$

Lemma 4.2. [25] Let A be an $n \times n$ matrix such that each row sum is a constant t. Then the coronal of A is given by

$$\Gamma_A(x) = \frac{n}{x - t}$$

.

Lemma 4.3. [26] Let A be an $n \times n$ matrix and $J_{n\times n}$ denote an all one matrix. Then,

$$det(xI_n - A - \alpha J_{n\times n}) = (1 - \alpha\Gamma_A(x)).det(xI_n - A),$$

where α is any real number.

Theorem 4.1. For any two distinct primes p and q, the distance spectrum of $\Gamma(\mathbb{Z}_{pq})$ is given by

$SpecD(\Gamma(\mathbb{Z}_{pq}))$ =

$$\begin{pmatrix} -2 & p+q-4+\sqrt{p^2+q^2-pq-(p+q)+1} & p+q-4-\sqrt{p^2+q^2-pq-(p+q)+1} \\ p+q-4 & 1 & 1 \end{pmatrix}$$

Proof. Let $G = \Gamma(\mathbb{Z}_{pq})$.The only proper divisors of pq are p and q and the number of non-zero zero-divisors of $\mathbb{Z}_{pq}$ is $p+q-2$. Labeling these $p+q-2$ vertices of G properly, it can be seen that,

$$S(p) = \{k_1 p : k_1 = 1,2,...,q-1\},$$

$$S(q) = \{k_2 q : k-2 = 1,2,...,p-1\}.$$

And these two sets of vertices of G, from an equitable partition of $V(G)$. Then clearly $|S(p)| = q-1$ and $|S(q)| = p-1$. While labeling the vertices, let those from $S(p)$ be arranged first and then $S(q)$. Also $S(p)$ and $S(q)$ induce null graphs of order $q-1$ and $p-1$ respectively. Thus $G = \Gamma(\mathbb{Z}_{pq})$ is the K_2- join of $\overline{K}_{q-1}$ and $\overline{K}_{p-1}$.
That is,

$$\Gamma(\mathbb{Z}_{pq}) = \overline{K}_{q-1} \vee \overline{K}_{p-1}.$$

Clearly the distance between any two vertices of $S(q)$ as well as $S(p)$ in G, is 2. Also the distance between a vertex in $S(q)$ and a vertex in $S(p)$ is 1 in G. Thus, if J denotes an all-one matrix, the distance matrix of G is given by

$$D(G) = \left[\begin{array}{c|c} 2(J-I)_{(q-1)\times(q-1)} & J_{(q-1)\times(p-1)} \\ \hline J_{(p-1)\times(q-1)} & 2(J-I)_{(p-1)\times(p-1)} \end{array}\right] =$$

$$\left[\begin{array}{c|c} A & J \\ \hline J^T & B \end{array}\right]$$

where $A = 2(J-I)_{(q-1)\times(q-1)}$, $B = 2(J-I)_{(p-1)\times(p-1)}$,
Using Lemma:4.1 it can be easily seen that,

$$\phi_D(G;x) = det(xI-A).det\left[(xI-B) - J^T(xI-A)^{-1}J\right]$$

Using Lemma:4.2 and Lemma:4.3,

$$\phi_D(G;x) = det(xI-A)\cdot det(xI-B)\cdot(1-\Gamma_A(x).\Gamma_B(x)) \tag{1}$$

Since A and B are symmetric matrices with constant row sums $2(q-2)$ and $2(p-2)$ respectively it follows from that,

$$\Gamma_A(x) = \frac{q-1}{x-2(p-2)}$$

and

$$\Gamma_B(x) = \frac{p-1}{x-2(q-2)}.$$

Since $A = 2(J-I)$, where J and I are all-one matrix of size $q-1$ and identity matrix of size $q-1$ respectively, again by Lemma:4.3,

$$det(xI-A) = det(xI-2J+2I) = (x+2)^{q-2}.(x-2(q-2)).$$

Similarly, since $B = 2(J-I)$ is of size $p-1$,

$$det(xI-B) = det(xI-2J+2I) = (x+2)^{p-2}.(x-2(p-2)).$$

Thus from equation(1), it follows that
$\phi_D(G;x) = (x+2)^{q-2}\cdot(x+2)^{p-2}.Q(x)$, where $Q(x) = x^2-2x(p+q-4)+3pq-7(p+q)+15$.
Thus $SpecD(\Gamma(\mathbb{Z}_{pq})) =$

$$\begin{pmatrix} -2 & p+q-4+\sqrt{p^2+q^2-pq-(p+q)+1} & p+q-4-\sqrt{p^2+q^2-pq-(p+q)+1} \\ p+q-4 & 1 & 1 \end{pmatrix}$$

□

Note that by Perron Frobenius theorm, the largest eigen value is positive and simple. It is obvious that the number of distinct distance eigenvalues of $\Gamma(\mathbb{Z}_{pq})$ is 3 and we have the following corollary.

Corollary 4.1. The distance spectral radius of $\Gamma(\mathbb{Z}_{pq})$, when p and q are distinct primes , $p<q$ is $p+q-4+\sqrt{p^2+q^2-pq-(p+q)+1}$.

Let $\partial_1 \geq \partial_2 \geq ... \geq \partial_n$ be the distance eigenvalues of a simple connected graph G, the the distance energy of G is given by,
$\mathscr{E}(G) = \Sigma_{i=1}^{n}|\partial_i|$ .The following corollary gives a lower bound for the distance energy of $\Gamma(\mathbb{Z}_{pq})$.

Corollary 4.2. For any two distinct prime p,q, $\mathscr{E}(\Gamma(\mathbb{Z}_{pq})) \geq 2(p+q-4)$.

Theorem 4.2. For any prime $p \neq 2$, the distance spectrum of $\Gamma(\mathbb{Z}_{p^3})$ is given by
$SpecD(\Gamma(\mathbb{Z}_{p^3}))$ =

$$\begin{pmatrix} -2 & -1 & \frac{2p^2-p-4+\sqrt{4p^4-8p^3+p^2+4p}}{2} & \frac{2p^2-p-4-\sqrt{4p^4-8p^3+p^2+4p}}{2} \\ p^2-p-1 & p-2 & 1 & 1 \end{pmatrix}$$

Proof. Let $G = \Gamma(\mathbb{Z}_{p^3})$. Then as described in Theorem 4.1, it can be seen that the number of non-zero zero-divisors of $\mathbb{Z}_{p^3}$ is p^2-1. These p^2-1 vertices of G are partitioned (equitable partition) as follows.

$S(p) = \{k_1 p : k_1 = 1, 2, ... p^2 - 1, \text{where } p \nmid k_1 \}$.
$S(p^2) = \{k_2 p^2 : k_2 = 1, 2, ... p - 1, \text{where } p \nmid k_2\}$.

Clearly $S(p)$ induces a null graph of order $p(p-1)$ and $S(p^2)$ induces a complete graph on $p-1$ vertices, as described in section:3. Thus $\Gamma(\mathbb{Z}_{p^3}) = \overline{K}_{p(p-1)} \vee K_{p-1}$. Recall that the distance matrix of a complete graph is the same as its adjacency matrix. Thus the distance matrix of G is given as follows.

$$D(G) = \left[\begin{array}{c|c} 2(J-I)_{p(p-1)\times p(p-1)} & J_{p(p-1)\times(p-1)} \\ \hline J_{(p-1)\times p(p-1)} & (J-I)_{(p-1)\times(p-1)} \end{array}\right] =$$

$$\left[\begin{array}{c|c} A & J \\ \hline J^T & B \end{array}\right]$$

where $A = 2(J-I)_{p(p-1)\times p(p-1)}$, $B = (J-I)_{(p-1)\times(p-1)}$,
Proceeding as in the previous theorem, we get,

$$SpecD(\Gamma(\mathbb{Z}_{p^3})) = \begin{pmatrix} -2 & -1 & \frac{2p^2-p-4+\sqrt{4p^4-8p^3+p^2+4p}}{2} & \frac{2p^2-p-4-\sqrt{4p^4-8p^3+p^2+4p}}{2} \\ p^2-p-1 & p-2 & 1 & 1 \end{pmatrix}$$

$\square$

Corollary 4.3. The distance spectral radius of $\Gamma(\mathbb{Z}_{p^3})$, is, $\frac{2p^2-p-4+\sqrt{4p^4-8p^3+p^2+4p}}{2}$.

Corollary 4.4. For any prime p, $\mathscr{E}(\Gamma(\mathbb{Z}_{p^3})) \geq 3(p^2-p-6)$.

5. DISTANCE LAPLACIAN SPECTRUM FOR $\Gamma(\mathbb{Z}_{pq})$, AND $\Gamma(\mathbb{Z}_{p^3})$, WHERE p AND q ARE DISTINCT PRIMES, $p < q$.

In this section, we compute the Laplacian spectrum of $\Gamma(\mathbb{Z}_{pq})$, and $\Gamma(\mathbb{Z}_{p^3})$, where p and q are distinct primes, $p < q$. Each row sum of $D^L(G)$ is zero and for a connected graph G of order n, 0 is a simple eigenvalue of $D^L(G)$ with $\mathbf{1}_n$ as the corresponding eigen vector.[15]

Theorem 5.1. For any two distinct primes p and q, the distance Laplacian spectrum of $\Gamma(\mathbb{Z}_{pq})$ is given by,

$$SpecD^L(\Gamma(\mathbb{Z}_{pq})) = \begin{pmatrix} 0 & p+q-2 & p+2q-3 & 2p+q-3 \\ 1 & 1 & q-2 & p-2 \end{pmatrix}$$

Proof. Let $G=\Gamma(\mathbb{Z}_{pq})$. Then $G=\overline{K}_{q-1}\vee\overline{K}_{p-1}$. Also $S(p)=\{k_1p : k_1=1,2,...,q-1\}$ and $S(q)=\{k_2q : k-2=1,2,...,p-1\}$ form an equitable partition of $V(G)$. Thus for any vertex $v\in S(p)$,

$$\begin{aligned} Tr(v) &= \sum_{u\in V(G)} d_G(u,v) \\ &= \sum_{u\in S(p)} d_G(u,v)+\sum_{u\in S(q)} d_G(u,v) \\ &= 2(q-2)+(p-1)=p+2q-5. \end{aligned}$$

Similarly for any vertex $w\in S(q)$,

$$\begin{aligned} Tr(w) &= \sum_{u\in V(G)} d_G(u,w) \\ &= \sum_{u\in S(p)} d_G(u,w)+\sum_{u\in S(q)} d_G(u,w) \\ &= (q-1)+2(p-2)=2p+q-5. \end{aligned}$$

Thus the Transmission matrix of G is given by

$$Tr(G)=\left[\begin{array}{c|c} (p+2q-5)I_{(q-1)\times(q-1)} & O_{(q-1)\times(p-1)} \\ \hline O_{(p-1)\times(q-1)} & (2p+q-5)I_{(p-1)\times(p-1)} \end{array}\right] \tag{2}$$

And the distance matrix of G is given by,

$$D(G)=\left[\begin{array}{c|c} 2(J-I)_{(q-1)\times(q-1)} & J_{(q-1)\times(p-1)} \\ \hline J_{(p-1)\times(q-1)} & 2(J-I)_{(p-1)\times(p-1)} \end{array}\right] \tag{3}$$

Thus the distance Laplacian matrix of G is given by

$$D^L(G)=\left[\begin{array}{c|c} (p+2q-5)I-2(J-I)_{(q-1)\times(q-1)} & -J_{(q-1)\times(p-1)} \\ \hline -J_{(p-1)\times(q-1)} & (2p+q-5)I-2(J-I)_{(p-1)\times(p-1)} \end{array}\right]=$$

$$\left[\begin{array}{c|c} A & -J \\ \hline -J^T & B \end{array}\right]$$

where $A=(p+2q-3)I-2J_{(q-1)\times(q-1)}$, $\quad B=(2p+q-3)I-2J_{(p-1)\times(p-1)}$,
Proceeding as in the previous section, we get

$$SpecD^{L}(\Gamma(\mathbb{Z}_{pq}))=\begin{pmatrix} 0 & p+q-2 & p+2q-3 & 2p+q-3 \\ 1 & 1 & q-2 & p-2 \end{pmatrix}$$

□

Theorem 5.2. For any prime $p\neq 2$, the distance Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^3})$ is given by
$SpecD^{L}(\Gamma(\mathbb{Z}_{p^3}))$ = $\begin{pmatrix} 0 & p^2-1 & 2p^2-p-1 \\ 1 & p-1 & p^2-p-1 \end{pmatrix}$.

Proof. Let $G=\Gamma(\mathbb{Z}_{p^3})$. $S(p)=\{k_1p : k_1=1,2,...p^2-1,\text{where } p\nmid k_1\ \}$.
$S(p^2)=\{k_2p^2 : k_2=1,2,...p-1,\text{where } p\nmid k_2\}$ form an equitable partition for the vertex set $V(G)$. Also $\Gamma(\mathbb{Z}_{p^3})=\overline{K}_{p(p-1)}\vee K_{p-1}$. The transmission degree of any vertex in $S(p)$ in G, is $2p^2-p-3$ and of any vertex in $S(p^2)$ is p^2-2. Thus the transmission matrix the distance matrix and of G are given as follows.

$$Tr(G)=\left[\begin{array}{c|c} (2p^2-p-3)I_{p(p-1)\times p(p-1)} & O_{p(p-1)\times(p-1)} \\ \hline O_{(p-1)\times p(p-1)} & (p^2-2)I_{(p-1)\times(p-1)} \end{array}\right] \tag{4}$$

$$D(G)=\left[\begin{array}{c|c} 2(J-I)_{p(p-1)\times p(p-1)} & J_{p(p-1)\times(p-1)} \\ \hline J_{(p-1)\times p(p-1)} & (J-I)_{(p-1)\times(p-1)} \end{array}\right] \tag{5}$$

Thus it can be easily seen that the distance Laplacian matrix of G is

$$D^{L}(G)=\left[\begin{array}{c|c} (2p^2-p-1)I-2J_{p(p-1)\times p(p-1)} & -J_{p(p-1)\times(p-1)} \\ \hline -J_{(p-1)\times p(p-1)} & (p^2-1)I-J_{(p-1)\times(p-1)} \end{array}\right]$$

A similar computation shows that ,

$$\Phi_{D^L}(G;x)=\left[x-(2p^2-p-1)\right]^{p^2-p-1}.\left[x-(p^2-1)\right]^{p-2}.Q(x),$$

where $Q(x)=x\left(x-(p^2-1)\right)$. Thus ,

$$SpecD^{L}(\Gamma(\mathbb{Z}_{p^3}))=\begin{pmatrix} 0 & p^2-1 & 2p^2-p-1 \\ 1 & p-1 & p^2-p-1 \end{pmatrix}$$

□

6. Distance Signless Laplacian Spectrum for $\Gamma(\mathbb{Z}_{pq})$, and $\Gamma(\mathbb{Z}_{p^3})$, where p and q Are Distinct Primes, $p < q$.

In this section we compute the distance signless Laplacian of $\Gamma(\mathbb{Z}_{pq})$, and $\Gamma(\mathbb{Z}_{p^3})$, where p and q are distinct primes, $p < q$. Note that if G is a connected graph, $D^Q(G)$ is a real, symmetric, nonnegative , irreducible and positive semi definite matrix. Thus all eigenvalues of $D^Q(G)$ are real and nonnegative and also by the Perron Frobenius theorem, the largest eigenvalue of $D^Q(G)$, called the distance signless Laplacian spectral radius of G, denoted by $\partial^Q(G)$ is positive and simple.

Theorem 6.1. For any two distinct primes p and q, the distance signless Laplacian spectrum of $\Gamma(\mathbb{Z}_{pq})$ is given by
$SpecD^Q(\Gamma(\mathbb{Z}_{pq}))=$

$$\begin{pmatrix} p+2q-7 & 2p+q-7 & \frac{5(p+q)-18+\sqrt{9(p-q)^2+4(p-1)(q-1)}}{2} & \frac{5(p+q)-18-\sqrt{9(p-q)^2+4(p-1)(q-1)}}{2} \\ q-2 & p-2 & 1 & 1 \end{pmatrix}$$

Proof. let $G=\Gamma(\mathbb{Z}_{pq})$. Then as in the previous section, we count the transmission degree of each vertex $v \in V(G)$ and see that,

$$Tr(G)=\left[\begin{array}{c|c} (p+2q-5)I_{(q-1)\times(q-1)} & O_{(q-1)\times(p-1)} \\ \hline O_{(p-1)\times(q-1)} & (2p+q-5)I_{(p-1)\times(p-1)} \end{array}\right]$$

And the distance matrix of G is given by,

$$D(G)=\left[\begin{array}{c|c} 2(J-I)_{(q-1)\times(q-1)} & J_{(q-1)\times(p-1)} \\ \hline J_{(p-1)\times(q-1)} & 2(J-I)_{(p-1)\times(p-1)} \end{array}\right]$$

Thus the distance signless Laplacian matrix of G is given by

$$D^Q(G)=\left[\begin{array}{c|c} (p+2q-5)I+2(J-I)_{(q-1)\times(q-1)} & J_{(q-1)\times(p-1)} \\ \hline J_{(p-1)\times(q-1)} & (2p+q-5)I+2(J-I)_{(p-1)\times(p-1)} \end{array}\right]=$$

$$\left[\begin{array}{c|c} A & J \\ \hline J^T & B \end{array}\right]$$

where $A = (p+2q-7)I + 2J_{(q-1)\times(q-1)}$, $\quad B = (2p+q-7)I + 2J_{(p-1)\times(p-1)}$,
Thus computing the characteristic polynomial of $D^Q(G)$, we get ,
$\Phi_{D^Q}(G;x) = (x-(p+2q-7))^{q-2}.(x-(2p+q-7))^{p-2}.Q(x)$, where
$Q(x) = x^2 - (5p+5q-18)x + 4[(p-1)(p-2)+(q-1)(q-2)+4(p-2)(q-2)]$. Thus we have, $SpecD^Q(\Gamma(\mathbb{Z}_{pq}))$ =

$$\begin{pmatrix} p+2q-7 & 2p+q-7 & \frac{5(p+q)-18+\sqrt{9(p-q)^2+4(p-1)(q-1)}}{2} & \frac{5(p+q)-18-\sqrt{9(p-q)^2+4(p-1)(q-1)}}{2} \\ q-2 & p-2 & 1 & 1 \end{pmatrix}$$

□

Corollary 6.1. Let $G = \Gamma(\mathbb{Z}_{pq})$). Then, the distance signless Laplacian spectral radius, $\partial^Q(G) = \dfrac{5(p+q)-18+\sqrt{9(p-q)^2+4(p-1)(q-1)}}{2}$.

Theorem 6.2. For any prime $p \neq 2$, the distance signless Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^3})$ is given by
$SpecD^Q(\Gamma(\mathbb{Z}_{p^3})) =$

$$\begin{pmatrix} 2p^2-p-5 & p^2-3 & \frac{5p^2-2p-9+\sqrt{9p^4-20p^3+2p^2+12p+1}}{2} & \frac{5p^2-2p-9-\sqrt{9p^4-20p^3+2p^2+12p+1}}{2} \\ p^2-p-1 & p-2 & 1 & 1 \end{pmatrix}$$

Proof. Let $G = \Gamma(\mathbb{Z}_{p^3})$. $\Gamma(\mathbb{Z}_{p^3}) = \overline{K}_{p(p-1)} \vee K_{p-1}$. The transmission degree of any vertex in $S(p)$ in G, is $2p^2-p-3$ and of any vertex in $S(p^2)$ is p^2-2. The transmission matrix and the distance matrix and of G are described in the previous section, in equations (4) and (5). Thus it can be easily seen that the distance Laplacian matrix of G is

$$D^Q(G) = \left[\begin{array}{c|c} (2p^2-p-5)I + 2J_{p(p-1)\times p(p-1)} & J_{p(p-1)\times(p-1)} \\ \hline J_{(p-1)\times p(p-1)} & (p^2-3)I + J_{(p-1)\times(p-1)} \end{array}\right]$$

A similar computation shows that ,

$$\Phi_{D^Q}(G;x) = \left[x-(2p^2-p-5)\right]^{p^2-p-1}.\left[x-(p^2-3)\right]^{p-2}.Q(x),$$

where $Q(x) = x^2-(5p^2-2p-9)x+(4p^4-22p^2+6p+20)$. Thus ,
$SpecD^Q(\Gamma(\mathbb{Z}_{p^3}))$ =

$$\begin{pmatrix} 2p^2-p-5 & p^2-3 & \frac{5p^2-2p-9+\sqrt{9p^4-20p^3+2p^2+12p+1}}{2} & \frac{5p^2-2p-9-\sqrt{9p^4-20p^3+2p^2+12p+1}}{2} \\ p^2-p-1 & p-2 & 1 & 1 \end{pmatrix}$$

□

Corollary 6.2. Let $G = \Gamma(\mathbb{Z}_{p^3}))$. Then, the distance signless Laplacian spectral radius, $\partial^Q(G) = \dfrac{5p^2-2p-9+\sqrt{9p^4-20p^3+2p^2+12p+1}}{2}$.

CONCLUSION

The notion of distance and transmission of vertices of graphs is applied in many realms of the physical world including the design of communication networks. The special combinatorial structure as well as the typical block structure of the distance related matrices of the graphs discussed in this paper plays a major role to motivate us to inculcate the tools of matrices in the computation of the spectrum.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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