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COMMON FIXED POINTS FOR A PAIR OF SELFMAPS IN EXTENDED RECTANGULAR B-METRIC SPACES FOR ALMOST SUZUKI NONLINEAR TYPE CONTRACTIONS

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Abstract: In this paper, we define almost Suzuki generalized rational type contraction and almost Suzuki Ciric type contraction for a pair of selfmaps in extended rectangular B-metric spaces and prove the existence of common fixed points for these contractions. The presented results are supported by examples. Our results improve and generalize the many results in the literature.

Keywords: extended rectangular B- metric spaces; almost Suzuki generalized rational type contractions; almost Suzuki Ciric type contractions; common fixed points.

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1. INTRODUCTION

The notions of the metric spaces were generalized by many authors in several ways. Out of all, we start with generalized metric space. Branciari [2] introduced the concept of generalized metric space (rectangular space) and obtained Banach contraction mapping principle in the

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setting of generalized metric spaces.

Definition 1.1. [2] Let A be a non-empty set. A mapping $\upsilon : A \times A \to [0, \infty)$ is said to be a *rectangular metric* on A if, υ satisfies the following (for all $\mu, \eta \in A$ and all distinct $r, s \in A \setminus \{\mu, \eta\}$):

(1) $\upsilon(\mu, \eta) = 0$ if and only if $\mu = \eta$

(2) $\upsilon(\mu, \eta) = \upsilon(\eta, \mu)$,

(3) $\upsilon(\mu, \eta) \leq \upsilon(\mu, r) + \upsilon(r, s) + \upsilon(s, \eta)$.

Then the pair (A, υ) is said to be a rectangular metric space.

George *et. al.*, [4] introduced the notion of rectangular B-metric space as a generalization of rectangular metric space and they presented some fixed point results for certain contraction mappings.

Definition 1.2.[4] Let A be a non-empty set with the parameter $B \geq 1$. A mapping $\upsilon_b : A \times A \to \mathbb{R}^+$ is said to be a *rectangular B-metric* on A if there exists $B \geq 1$ such that υ_b satisfies the following (for all $\mu, \eta \in A$ and all distinct $r, s \in A \setminus \{\mu, \eta\}$):

(1) $\upsilon_b(\mu, \eta) = 0$ if and only if $\mu = \eta$,

(2) $\upsilon_b(\mu, \eta) = \upsilon_b(\eta, \mu)$,

(3) $\upsilon_b(\mu, \eta) \leq B[\upsilon_b(\mu, r) + \upsilon_b(r, s) + \upsilon_b(s, \eta)]$.

Then the pair (A, υ_b) is said to be a rectangular B-metric space.

Recently, new type of generalized B-metric space namely extended B-metric space is introduced Kamran *et. al.*, [5].

Definition 1.3 [5]. Let A be a non-empty set and $\zeta: A\times A \to [1, \infty)$. A mapping $\sigma_\zeta: A\times A \to \mathbb{R}^+$ is said to be an *extended B-metric on* A, if σ_ζ satisfies the following for all μ, η and $q \in A$

(1) $\sigma_\zeta(\mu, \eta) = 0$ if and only if $\mu = \eta$

(2) $\sigma_\zeta(\mu, \eta) = \sigma_\zeta(\eta, \mu)$,

(3) $\sigma_\zeta(\mu, \eta) \leq \zeta(\mu, \eta)[\sigma_\zeta(\mu, q) + \sigma_\zeta(q, \eta)]$.

Then the pair (A, σ_ζ) is said to be an extended B-metric space.

Influenced by the concepts of rectangular B-metric spaces and extended B-metric spaces, Asim, Imdad and Radenovic[1] introduced the concept of extended rectangular B-metric spaces and obtained fixed points.

Definition 1.4 [1]. Let N be a non-empty set and $\varsigma : N\times N\rightarrow [1,\infty)$. A mapping $\Omega_\varsigma : NXN \rightarrow R^+$ is said to be an *extended rectangular B- metric* on N if Ω_ς satisfies the following (for all $\mu, \eta \in N$ and all distinct $p, q \in N \setminus \{\mu, \eta\}$) :

(1) $\Omega_\varsigma(\mu, \eta) = 0$ if and only if $\mu = \eta$,

(2) $\Omega_\varsigma(\mu,\eta) = \Omega_\varsigma(\eta,\mu)$,

(3) $\Omega_\varsigma(\mu, \eta) \le \varsigma(\mu, \eta)[\Omega_\varsigma(\mu, p) + \Omega_\varsigma(p, q) + \Omega_\varsigma(q, \eta)]$.

Then the pair $(\aleph, \Omega_\varsigma)$ is said to be an extended rectangular B-metric space.

Definition 1.5. [1] Let (N, Ω_ς) be an extended rectangular B-metric space with ς. Then:

(1) A sequence $\{u_n\}$ in N is said to be Ω_ς *convergent to u* in N if $\lim_{n\to\infty} \Omega\varsigma(u_n, u) = 0$. In this case we write $\lim_{n\to\infty} u_n = u$.

(2) A sequence $\{u_n\}$ in N is said to be Ω_ς *Cauchy* if $\lim_{n,m\to\infty} \Omega\varsigma(u_n, u_m) = 0$.

(3) (N, Ω_ς) is said to be a Ω_ς *complete* if every Ω_ς Cauchy in (N, Ω_ς) is Ω_ς convergent to some point in N.

Lemma 1.6[1]. Let (N, Ω_ς) be an extended rectangular B-metric space with ς, the sequence $\{u_n\}$ in N such that $u_n\neq u_m$, whenever $n\neq m$, then $\{u_n\}$ converges at most one point.

The authors Asim, Imdad and Radenovic[3] proved the analog of Banach contraction type principle in the context of extended rectangular B-metric space.

Theorem 1.7[1]. Let (N,Ω_ς) be a extended rectangular B-metric space with the function ς and $T : N \rightarrow N$. Suppose that :

(i) for all $\mu, \eta \in N$, there exists $\lambda \in [0,1)$, $\Omega_\varsigma(T\mu, T\eta) \le \lambda\Omega_\varsigma(\mu, \eta)$,

(ii) $\lim_{n,m\to\infty} \Omega\varsigma(u_n, u_m) < \frac{1}{\lambda}$

(iii) (N, Ω_ς) is T - orbitally complete

(iv) T is orbitally continuous.

Then T has a unique fixed point in N.

Lemma 1.8 [6]. Let (N, Ω_ς) be an extended rectangular B-metric space and if there exists $t \in [0, 1)$ such that for $u_0 \in N$, the sequence $\{u_n\}$ satisfies $\lim_{n,m\to\infty} \Omega\varsigma(u_n, u_m) < \frac{1}{t}$ and $0< \Omega\varsigma(u_n, u_{n+1}) \leq t\, \Omega\varsigma(u_n, u_{n-1})$ then for any $n \epsilon \mathbb{N}$ the sequence $\{u_n\}$ is a Ω_ς Cauchy sequence in N.

In 2009, Suzuki [8] proved certain remarkable results to improve the results of Banach and Edelstein [3].

Theorem 1.9 [8]. Let (N, d) be a compact metric space and T: N →N be a mapping. Assume that

$$d(\mu,T\mu) < d(\mu, \eta) \Rightarrow d(T\mu, T\eta) < d(\mu,\eta) \text{ for all distinct } \mu, \eta \in N.$$

Then T has a unique fixed point in N.

Recently, general rational type contractive conditions was employed by Olatinwo and Ishola [7] in complete metric spaces to generalize many results.

Let (N, d) be a complete metric space and T : N → N be a continuous map such that there exist α, p, q, r, s , t $\in \mathbb{R}^+$ and $\beta \in [0, 1)$ such that $\forall\ \mu, \eta \in N$,

$$d(T\mu, T\eta) \leq \frac{\alpha[p + d(\mu,T\mu)][d(\eta ,T \eta)]^q\ [d(\eta ,T\mu)]^r}{sd(\mu,T\eta)+td(\eta,T\mu)+d(\mu,\eta)} + \beta d(\mu, \eta)$$

with $sd(\mu,T\eta) + td(\eta,T\mu) + d(\mu,\eta > 0$.

Motivated by the works of Suzuki[8] and Olatino and Ishola [7], in this paper, we define almost Suzuki generalized rational type contraction and almost Suzuki Ciric type contraction for a pair of selfmaps in extended rectangular B-metric spaces and obtained common fixed points. The presented results are supported through examples. Our results improve and generalize many results in the literature.

2. ALMOST SUZUKI GENERALIZED RATIONAL TYPE CONTRACTIONS

Definition 2.1. Let (N, Ω_ς) be an extended rectangular B-metric space with function ς: N× N→ [1, ∞). Let S and T be two selfmaps on N. We say that the pair (S, T) is almost Suzuki generalized rational type contraction if for any α, p, q, r, s, $\zeta, \lambda \in \mathbb{R}^+$ and $\beta \in [0, 1)$ such that $\forall\ \mu, \eta \in N$, (2.1) $\frac{1}{2} min\{\ \Omega\varsigma(\mu, S\mu), \Omega\varsigma(\eta, T\eta)\} \leq \Omega\varsigma(\mu, \eta) \Rightarrow$

$$\Omega\varsigma\ (S\mu, T\eta) \leq \frac{\alpha[\ P+\ \Omega\varsigma(\mu\ S\mu)]\Omega\varsigma(\mu,T\ \eta))^q\ \Omega\varsigma(\eta,T\eta)^r\ \Omega\varsigma(\eta,S\mu)^s}{\zeta\Omega\varsigma(\eta,S\mu)+\lambda\Omega\varsigma(\mu,T\eta)+\Omega\varsigma(\mu,\eta)} + \beta\ \Omega\varsigma\ (\mu, \eta\ )$$

with $\zeta\Omega\varsigma(\eta, S\mu) + \lambda\Omega\varsigma(\mu, T\eta) + \Omega\varsigma(\mu, \eta\ ) > 0$.

Theorem 2.2. Let (N, Ω_ς) be an extended rectangular B-metric space with function ς : N× N→ [1, ∞). Let S and T be two selfmaps on N such that the pair (S, T) is almost Suzuki generalized rational type contraction. Further, assume that :

(2.2) For each $x_0 \in N, the\ sequence\ \{x_n\}$ in N is such that $\lim_{m,n\to\infty} \theta(x_{n\,,}\, x_m) < \frac{1}{\beta}$.

(2.3) Either S or T is continuous.

Then S and T have a unique common fixed point in N.

Proof. Let $x_o \epsilon N$, we can construct a sequence in $\{x_n\}$ in N such that

(2.4) $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for all n in $\mathbb{N}$.

Suppose that $x_{n_0} = x_{n_0+1}$ for some in $\mathbb{N}$ then k= x_{n_0} is a fixed point of S. To show that k is a common fixed point, we consider the following two cases.

If n_0=2m, we have

(2.5) $x_{2m} = x_{2m+1} \Rightarrow x_{2m} = Sx_{2m}$, x_{2m} is a fixed point of S.

We now assert that $Sx_{2m} = Tx_{2m+1}$.

Suppose that $\Omega\varsigma(Sx_{2m}, Tx_{2m+1}) > 0$.

Since $\frac{1}{2}\min\{\ \Omega\varsigma(x_{2m+1}, Tx_{2m+1}),\ \ \Omega\varsigma(x_{2m}, x_{2m+1})\} = 0 \leq \Omega\varsigma(x_{2m}, x_{2m+1}) = 0 \Rightarrow$ from (2.1), we have

(2.6)

$$\Omega\varsigma(Sx_{2m}, Tx_{2m+1}\ )$$
$$\leq \frac{\alpha[p + \Omega\varsigma(x_{2m}, Sx_{2m})]\Omega\varsigma\big(x_{2m}, Tx_{2m+1}\big)^q \Omega\varsigma(x_{2m+1}, Tx_{2m+1})^r\ \Omega\varsigma\big(x_{2m+1,}Sx_{2m}\big)^s}{\zeta\Omega\varsigma\big(x_{2m+1\,,}\, Sx_{2m}\big) + \lambda\Omega\varsigma(x_{2m},\ Tx_{2m+1}) + \Omega\varsigma(x_{2m},\ x_{2m+1})}$$
$$+ \beta\Omega\varsigma(x_{2m}, x_{2m+1}\ ).$$

Thus, from (2.4) and (2.5), we get $\Omega\varsigma(Sx_{2m}, Tx_{2m+1}\ ) \leq 0 \Rightarrow Sx_{2m} = Tx_{2m+1}$.

Hence $x_{2m} = Sx_{2m} = x_{2m+1} = Tx_{2m+1}$.

Thus, k= x_{n_0} is a common fixed point of S and T.

Similarly, when n_0=2m+1, then following on the same steps of above case we get k=x_{n_0} is a fixed point of S and T.

Hence without loss of generality suppose that $x_n \neq x_{n+1}$ for all n $\in\mathbb{N}$. We now show that N $\{x_n\}$ is a Cauchy sequence in N.

Case (a): Assume that n is even, using the pair (S, T) is almost Suzuki generalized rational type contraction, we have

½ min$\{\Omega\varsigma(x_{2n}, Sx_{2n}), \Omega\varsigma(x_{2n+1}, Tx_{2n+1})\} \leq \Omega\varsigma(x_{2n}, x_{2n+1})$ ⇒from (2.1) and (2.4), it follows that

$$\Omega\varsigma(Sx_{2n}, Tx_{2n+1}) \leq \frac{\alpha[p+\Omega\varsigma(x_{2n}, Sx_{2n})]\,\Omega\varsigma(x_{2n}, Tx_{2n+1})^q\,\Omega\varsigma(x_{2n+1}, Tx_{2n+1})^r\,\Omega\varsigma(x_{2n+1}, Sx_{2n})^s}{\zeta\Omega\varsigma(x_{2n+1}, Sx_{2n})+\lambda\Omega\varsigma(x_{2n}, Tx_{2n+1})+\Omega\varsigma(x_{2n}, x_{2n+1})} + \beta\Omega\varsigma(x_{2n}, x_{2n+1}) = \beta\Omega\varsigma(x_{2n}, x_{2n+1}).$$

Thus,

(2.7) $\quad \Omega\varsigma(x_{2n+1}, x_{2n+2}) \leq \beta\Omega\varsigma(x_{2n}, x_{2n+1})$.

Case (b): Assume that n is odd, then

½ min$\{\Omega\varsigma(x_{2n}, Sx_{2n}), \Omega\varsigma(x_{2n-1}, Tx_{2n-1})\} \leq \Omega\varsigma(x_{2n}, x_{2n-1})$ ⇒ from (2.1) and (2.4), it follows that

$$\Omega\varsigma(Sx_{2n}, Tx_{2n-1}) \leq \frac{\alpha[p+\Omega\varsigma(x_{2n}, Sx_{2n})]\,\Omega\varsigma(x_{2n}, Tx_{2n-1})^q\,\Omega\varsigma(x_{2n-1}, Tx_{2n-1})^r\,\Omega\varsigma(x_{2n-1}, Sx_{2n})^s}{\zeta\Omega\varsigma(x_{2n-1}, Sx_{2n})+\lambda\Omega\varsigma(x_{2n}, Tx_{2n-1})+\Omega\varsigma(x_{2n}, x_{2n-1})} + \beta\Omega\varsigma(x_{2n}, x_{2n-1}) = \beta\Omega\varsigma(x_{2n}, x_{2n-1}).$$

Thus,

(2.8) $\quad \Omega\varsigma(x_{2n+1}, x_{2n}) \leq \beta\Omega\varsigma(x_{2n}, x_{2n-1})$.

Hence from Case (a) and Case (b), we conclude that

(2.9) $\quad \Omega\varsigma(x_n, x_{n+1}) \leq \beta\Omega\varsigma(x_n, x_{n-1})$ for all n in $\mathbb{N}$.

Thus, in view of Lemma 1.8 and condition (2.2) of our assumption, it follows that $\{x_n\}$ is a Cauchy sequence in N. Since (N, $\Omega\varsigma$) is complete, there exists u in N such that

(2.10) $\quad \lim_{n\to\infty} x_n = u$.

We now show that u is a common fixed point of S and T.

First we suppose that S is continuous on N, then we have

(2.11) $\quad Su= \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} x_{2n+1} = u.$

Next we assert that Tu=u.

Using condition (2.1) and (2.11), we have

$½ \min\{\Omega\varsigma(u, Su), \Omega\varsigma(Tu, u)\} = 0 \le \Omega\varsigma(u, u) = 0$

$\Rightarrow \Omega\varsigma\ (Su, Tu) \le \frac{\alpha[p + \Omega\varsigma(u,Su)]\ \Omega\varsigma(u,Tu)^q\ \Omega\varsigma(u,Tu)^r\ \Omega\varsigma(u,Su)^s}{\zeta\Omega\varsigma(u,Su)+\lambda\Omega\varsigma(u,Tu)+\Omega\varsigma(u,u)} + \beta\, \Omega\varsigma\ (u, u).$

Thus, $\Omega\varsigma\ (Su, Tu) \le 0 \Rightarrow u=Tu$. Hence

(2.12) $\quad u=Su=Tu$.

We now show that S and T have a unique common fixed point in N. Indeed, let u and v be two common fixed points of S and T. Therefore

(2.13) $\quad u=Su=Tu \quad$ and $v=Sv=Tv$.

Now, $1/2\min\{\ \Omega\varsigma\ (Su, u),\ \Omega\varsigma\ (v, Tv)\}= 0\le d(u,v) \Rightarrow$ from (2.1) and (2.13), we have

$\Omega\varsigma\ (Su, Tv) \le \frac{\alpha[p + \Omega\varsigma(u,Su)]\ \Omega\varsigma(u,Tv)^q\ \Omega\varsigma(v,Tv)^r\ \Omega\varsigma(v,Su)^s}{\zeta\Omega\varsigma(v,Su)+\lambda\Omega\varsigma(u,Tv)+\Omega\varsigma(u,v)} + \beta\, \Omega\varsigma\ (u, v)$

$\Rightarrow \Omega\varsigma\ (Su, Tv) \le 0 \Rightarrow u=Su=Tv=v.$

Therefore u is a unique common fixed point of S and T.

By choosing S=T, we have the following corollary.

Corollary 2.3. Let (N, Ω_ς) be an extended rectangular B-metric space with function $\varsigma : N\times N\to [1, \infty)$.

Let T be a selfmap on N satisfying the following conditions:

(i) if for any α, p, q, r, s, $\zeta, \lambda \in \mathbb{R}^+$ and $\beta \in [0, 1)$ such that $\forall\ \mu, \eta \in N$,

(2.14) $\quad \frac{1}{2}min\{\ \Omega\varsigma(\mu, T\mu), \Omega\varsigma(\eta, T\eta)\} \le \Omega\varsigma(\mu, \eta) \Rightarrow$

$\Omega\varsigma\ (T\mu, T\eta) \le \alpha\ \frac{[p + \Omega\varsigma(\mu,T\mu)]\ \Omega\varsigma(\mu,T\ \eta))^q\ \Omega\varsigma(\eta,T\eta)^r\ \Omega\varsigma(\eta,T\mu)^s}{\zeta\Omega\varsigma(\eta,T\mu)+\lambda(\mu,T\eta)+\Omega\varsigma(\mu,\eta)} + \beta\, \Omega\varsigma\ (\mu, \eta)$

with $\zeta\Omega\varsigma(\eta, T\mu) + \lambda(\mu, T\eta) + \Omega\varsigma(\mu, \eta) > 0$.

(2.15) For any sequence $\{x_n\}$ in N is such that $\lim_{m,n\to\infty} \theta(x_n, x_m) < \frac{1}{\beta}$.

(2.16) T is continuous.

Then T has a unique fixed point in N.

By choosing α=0, in Theorem 2.1, we have the following Corollary.

Corollary 2.4. Let (N, Ω_ς) be an extended rectangular B-metric space with function ς : N× N→ [1, ∞).

Let T be a selfmap on N satisfying the following conditions:

(2.17) if for any β ∈ [0, 1) such that ∀ μ, η ∈ N,

$$\frac{1}{2} min\{ \Omega\varsigma(\mu,S\mu),\Omega\varsigma(\eta,T\eta)\} \leq \Omega\varsigma(\mu,\eta) \Rightarrow \Omega\varsigma (S\mu, T\eta) \leq \beta\, \Omega\varsigma (\mu, \eta).$$

(2.18) For any sequence $\{x_n\}$ in N is such that $\lim_{m,n\to\infty} \theta(x_n, x_m) < \frac{1}{\beta}$.

(2.19) Either S or T is continuous

Then S and T have a unique fixed point in N.

By choosing p=r=1 , q=0,s=0, $\zeta\Omega\varsigma(\eta,S\mu) + \lambda(\mu,T\eta) = 1$, we have the following corollary.

Corollary 2.5. Let (N, Ω_ς) be an extended rectangular B-metric space with function ς : N× N→ [1, ∞).

Let S and T be two selfmaps on N such that : if for any α∈ $\mathbb{R}^+$ and β ∈ [0, 1) such that ∀ μ, η ∈ N,

(2.20) $\frac{1}{2} min\{ \Omega\varsigma(\mu,S\mu),\Omega\varsigma(\eta,T\eta)\} \leq \Omega\varsigma(\mu,\eta) \Rightarrow$

$$\Omega\varsigma (S\mu, T\eta) \leq \alpha \frac{[1 + \Omega\varsigma(\mu,S\mu)]\Omega\varsigma(\eta,T\eta)}{1+\Omega\varsigma(\mu,\eta)} + \beta\, \Omega\varsigma (\mu, \eta)$$

(2.21) For any sequence $\{x_n\}$ in N, is such that $\lim_{m,n\to\infty} \theta(x_n, x_m) < \frac{1}{\beta}$.

(2.22) Either S or T is continuous.

Then S and T has a unique fixed point in N.

By choosing p=0, q=0,s=0, $\zeta = 1 , r = 1 \; and \; \lambda = 1$, we have the following corollary.

Corollary 2.6. Let (N, Ω_ς) be an extended rectangular B-metric space with function ς : N× N→ [1, ∞). Let S and T be two selfmaps on N satisfying:

(2.23) $if\ for\ any\ \beta \in [0,1) such\ that\ \forall\ \mu,\eta \in N,$

$$\frac{1}{2} min\{ \Omega\varsigma(\mu,S\mu),\Omega\varsigma(\eta,T\eta)\} \leq \Omega\varsigma(\mu,\eta) \Rightarrow$$

$\Omega\varsigma$ (Sμ, Tη) $\leq$ α $\frac{\Omega\varsigma(\mu,S\mu)\Omega\varsigma(\eta,T\eta)}{\Omega\varsigma(\eta,S\mu)+\Omega\varsigma(\mu,T\eta)+\Omega\varsigma(\mu,\eta)}$ + β $\Omega\varsigma$ (μ, η)

with $\Omega\varsigma(\eta, S\mu) + \Omega\varsigma(\mu, T\eta) + \Omega\varsigma(\mu, \eta) > 0$.

(2.24) For any sequence $\{x_n\}$ in N is such that $\lim_{m,n\to\infty} \theta(x_n, x_m) < \frac{1}{\beta}$.

(2.25) Either S or T is continuous

Then S and T has a unique fixed point in N.

Example 2.7. Let R=$[0,\frac{1}{5}]$ and S=$(\frac{1}{5}, 1]$ and N= RUS. We define the function ς : N× N→ $[1, \infty)$ by $\varsigma(\mu, \eta)= \mu+\eta +1$ and we define $\Omega\varsigma: N \to N$ by

$$\Omega_\varsigma(\mu,\eta) = \begin{cases} 0 & if\ \mu=\eta \\ 2 & if\ \mu\in R \\ 1 & if\ \mu\in S \\ (\mu+\eta)^2 & otherwise \end{cases}$$

Clearly, $\Omega\varsigma$ is an extended rectangular B- metric space with respect to ς.

We define S and T on N by

Sμ = $\frac{1}{4}$ for all $\mu \in N$ and T$\mu = \begin{cases} \frac{1}{4}\ if\ \mu \in R \\ \frac{\mu+1}{5}\ if\ \mu \in S. \end{cases}$

When $x_0 \in R$, we have $x_n = \frac{1}{4}$, so $\lim_{m,n\to\infty} \varsigma(x_n, x_m) = \frac{1}{4} + \frac{1}{4} + 1 = \frac{3}{2} < \frac{1}{\beta}$, when $\beta = \frac{5}{8}$ and when $x_0 \in S, we\ have\ x_n = \frac{1}{4}$, so $\lim_{m,n\to\infty} \varsigma(x_n, x_m) = \frac{1}{4} + \frac{1}{4} + 1 = \frac{3}{2} < \frac{1}{\beta}, when\ \beta = \frac{5}{8}$.

We now verify the inequality (2.1) with α=4.7, p= 4, q= 1, r=1, s= $1, \zeta = 1\ \lambda = 1$.

Case(i): When $\mu, \eta \in R$, without loss of generality assume that $\mu \neq \eta$,

$\frac{1}{2}\ min\{\ \Omega\varsigma(\mu, S\mu), \Omega\varsigma(\eta, T\eta)\} = \frac{1}{2} min\{\ \Omega\varsigma\left(\mu, \frac{1}{4}\right), \Omega\varsigma\left(\eta, \frac{1}{4}\right)\} = \frac{1}{2}\Omega\varsigma\left(\mu, \frac{1}{4}\right) \leq \Omega\varsigma(\mu, \eta) = 2$

⇒from (2.1), we have $\Omega\varsigma$ (Sμ, Tη)=0, so the inequality (2.1) holds.

Case(ii) Let $\mu, \eta \in S$.

Without loss of generality suppose that $\mu \neq \eta$.

$\frac{1}{2}\ min\{\ \Omega\varsigma(\mu, S\mu), \Omega\varsigma(\eta, T\eta)\} = \frac{1}{2} min\{1,1\} = \frac{1}{2} \leq \Omega\varsigma(\mu, \eta)=1 \Rightarrow$ from (2.1) , we have

$1 = \Omega\varsigma$ (Sμ, Tη) $\leq$ $4.7\frac{[4+\Omega\varsigma(\mu,\frac{1}{4})]\ \Omega\varsigma(\eta,\frac{\eta+1}{5})\Omega\varsigma\ (\mu,\frac{\eta+1}{5})\Omega\varsigma\ (\eta,\frac{1}{4})}{\Omega\varsigma\ (\mu,\frac{\eta+1}{5})+\Omega\varsigma(\eta,\frac{1}{4})+1}$+$\beta$1=4.7$\frac{[4+1]}{3}$+ $\frac{5}{8}$

$1 \leq 8.291$, so that the inequality (2.1) holds.

Case(iii) : When $\mu \in S$, $\eta \in R$.

$\frac{1}{2}\ min\{\ \Omega\varsigma(\mu, S\mu), \Omega\varsigma(\eta, T\eta)\} = \frac{1}{2}min\{(\mu+\frac{1}{4})^2, \Omega\varsigma\ (\eta, \frac{1}{4})\} = \frac{1}{2}(\mu+\frac{1}{4})^2\ \leq (\mu+\eta)^2 =$

$\Omega\varsigma(\mu, \eta)$ ⇒from (2.1),

we have $\Omega\varsigma$ (Sμ, Tη)=0, the inequality (2.1) holds.

Case(iv): When $\mu \in R$, $\eta \in S$

$\frac{1}{2}\ min\{\ \Omega\varsigma(\mu, S\mu), \Omega\varsigma(\eta, T\eta)\} = \frac{1}{2}min\{\ \Omega\varsigma(\mu, \frac{1}{4}), \Omega\varsigma\ (\eta, \frac{\eta+1}{5})\} = \frac{1}{2}(\mu+\frac{1}{4})^2\ \leq$

$(\mu+\eta)^2 = \Omega\varsigma(\mu, \eta) \Rightarrow\ from\ (2.1), we\ have,$

$1 = \Omega\varsigma\ (S\mu, T\eta) \leq 4.7\frac{[4+(\mu+\frac{1}{4})^2](\mu+\frac{\eta+1}{5})^2}{(\mu+\frac{\eta+1}{5})^2+(\mu+\eta)^2+1} + \beta(\mu+\eta)^2$

the minimum value of RHS attains at $\mu = 0, \eta \to \frac{1}{5}$ in this case we have

$1 \leq 1.027$, so inequality (2.1) holds.

Thus S and T satisfies all the conditions of Theorem 2.1. Also "1/4" is the unique common fixed point of S and T.

We observe that inequality (2.17) fails to hold at $\mu = \frac{1}{4}$, η =1 for any β, since

$\frac{1}{2}\ min\{\ \Omega\varsigma(\mu, S\mu), \Omega\varsigma(\eta, T\eta)\} = 0 \leq 1 =\ \Omega\varsigma(\mu, \eta) \Rightarrow$

$1 = \Omega\varsigma\ (S\mu, T\eta) < \beta\ \Omega\varsigma(\frac{1}{4}, 1) < \beta 1.$

This inequality forces $\beta > 1\ which\ contradicts\ the\ fact\ that\ \beta < 1.$

Thus Theorem 2.1 generalizes Corollary 2.4.

3. ALMOST SUZUKI CIRIC TYPE CONTRACTION

Definition 3.1. Let (N, Ω_ς) be an extended rectangular B-metric space with function ς: N× N→ [1, ∞). Let S and T be two selfmaps on N. We say that the pair (S, T) is *almost Suzuki Ciric type contraction* if for any

(3.1) β ∈ [0, 1) and L ≥0 such that ∀ μ, η ∈ N,

$$\frac{1}{2}min\{\ \Omega\varsigma(\mu, S\mu), \Omega\varsigma(\eta, T\eta)\} \leq \Omega\varsigma(\mu, \eta) \Rightarrow$$

$\Omega\varsigma$ $(S\mu, T\eta) \leq \beta \max\{ \Omega\varsigma(\mu,\eta), \Omega\varsigma(\mu,S\mu), \Omega\varsigma(\eta,T\eta)\} + L\, \Omega\varsigma(\mu,T\eta)\Omega\varsigma(\eta,S\mu)$

Theorem 3.2. Let (N, Ω_ς) be an extended rectangular B-metric space with function ς : N× N→ [1, ∞). Let S and T be two selfmaps on N such that the pair (S, T) is almost Suzuki Ciric type contraction. Further, assume that:

(3.2) For any sequence $\{x_n\}$ in N is such that $\lim_{m,n\to\infty} \theta(x_{n}, x_m) < \frac{1}{\beta}$.

(3.3) Either S or T is continuous.

Then S and T have a common fixed point in N.

Proof. Let $x_o \epsilon N$, we can construct a sequence $\{x_n\}$ in N such that

(3.4) $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for all n in $\mathbb{N}$.

Suppose that $x_{n_0} = x_{n_0+1}$ for some n_0 in $\mathbb{N}$ then k= x_{n_0} is a fixed point of S. To show that k is a common fixed point, we consider the following two cases.

If n_0=2m, we have

(3.5) $x_{2m} = x_{2m+1} \Rightarrow x_{2m} = Sx_{2m}$, x_{2m} is a fixed point of S.

We now assert that $Sx_{2m} = Tx_{2m+1}$.

Suppose that $\Omega\varsigma(Sx_{2m}, Tx_{2m+1}) > 0$.

Since $\frac{1}{2}$ $\min\{ \Omega\varsigma(x_{2m+1}, Tx_{2m+1}), \Omega\varsigma(x_{2m}, x_{2m+1})\} = 0 \leq \Omega\varsigma(x_{2m}, x_{2m+1}) = 0 \Rightarrow$ from (3.1) and (3.5), we have,

(3.6) $$\Omega\varsigma(Sx_{2m}, Tx_{2m+1}) \leq \beta\max\{\Omega\varsigma(x_{2m}, x_{2m+1}), \Omega\varsigma(x_{2m+1}, Tx_{2m+1}), \Omega\varsigma(x_{2m}, Sx_{2m})\} + L\Omega\varsigma(x_{2m}, Tx_{2m+1})\Omega\varsigma(x_{2m+1}, Sx_{2m}).$$

Therefore $\Omega\varsigma(x_{2m+1}, x_{2m+2}) \leq \beta\, \Omega\varsigma(x_{2m+1}, x_{2m+2}) \Rightarrow Sx_{2m} = Tx_{2m+1}$.

Hence $x_{2m} = Sx_{2m} = x_{2m+1} = Tx_{2m+1}$.

Thus, k= x_{n_0} is a common fixed point of S and T.

Similarly, when n_0=2m+1, then following on the same steps of above, we get k=x_{n_0} is a fixed point of S and T.

Hence without loss of generality suppose that $x_n \neq x_{n+1}$ for all n $\in\mathbb{N}$.

We now show that $\{x_n\}$ is a Cauchy sequence in N.

Case (a): Assume that n is even, then

½ min$\{\Omega\varsigma(x_{2n}, Sx_{2n}), \Omega\varsigma(x_{2n+1}, Tx_{2n+1})\} \leq \Omega\varsigma(x_{2n}, x_{2n+1}) \Rightarrow$

from (3.1) and (3.4), we have

$$\Omega\varsigma(Sx_{2n}, Tx_{2n+1}) \leq \beta\max\{\Omega\varsigma(x_{2n}, Sx_{2n}), \Omega\varsigma(x_{2n+1}, Tx_{2n+1}), \Omega\varsigma(x_{2n}, x_{2n+1})\} + L\Omega\varsigma(x_{2n}, Tx_{2n+1})\Omega\varsigma(x_{2n+1}, Sx_{2n})$$

$$= \beta\max\{\Omega\varsigma(x_{2n}, x_{2n+1}), \Omega\varsigma(x_{2n+1}, Tx_{2n+1})\}$$

thus

(3.7) $\Omega\varsigma(x_{2n+1}, x_{2n+2}) \leq \beta\max\{\Omega\varsigma(x_{2n}, x_{2n+1}), \Omega\varsigma(x_{2n+1}, x_{2n+2})\}$.

If $\Omega\varsigma(x_{2n+1}, x_{2n+2}) > \Omega\varsigma(x_{2n+1}, x_{2n})$, then from (3.7), we have

$\Omega\varsigma(x_{2n+1}, x_{2n+2}) \leq \beta\,\Omega\varsigma(x_{2n+1}, x_{2n+2}) < \Omega\varsigma(x_{2n+1}, x_{2n+2})$,

contradiction.

Therefore

(3.8) $\Omega\varsigma(x_{2n+1}, x_{2n+2}) \leq \beta\Omega\varsigma(x_{2n}, x_{2n+1})$.

Case (b): Assume that n is odd, then

½ min$\{\Omega\varsigma(x_{2n}, Sx_{2n}), \Omega\varsigma(x_{2n-1}, Tx_{2n-1})\} \leq \Omega\varsigma(x_{2n}, x_{2n-1}) \Rightarrow$ from (3.1) and (3.4), we have

$\Omega\varsigma(x_{2n+1}, x_{2n}) = \Omega\varsigma(Sx_{2n}, Tx_{2n-1})$

$\leq \beta\max\{\Omega\varsigma(x_{2n}, Sx_{2n}), \Omega\varsigma(x_{2n-1}, Tx_{2n-1}), \Omega\varsigma(x_{2n}, x_{2n-1})\}$

$+L\Omega\varsigma(x_{2n}, Tx_{2n-1})\,\Omega\varsigma(x_{2n-1}, Sx_{2n})$

(3.9) $\Omega\varsigma(x_{2n+1}, x_{2n}) \leq \beta\max\{\Omega\varsigma(x_{2n}, x_{2n-1}), \Omega\varsigma(x_{2n+1}, x_{2n})\}$.

If $\Omega\varsigma(x_{2n+1}, x_{2n}) > \Omega\varsigma(x_{2n}, x_{2n-1})$, then from (3.9), we have

$\Omega\varsigma(x_{2n+1}, x_{2n}) \leq \beta\,\Omega\varsigma(x_{2n+1}, x_{2n}) < \Omega\varsigma(x_{2n+1}, x_{2n})$,

contradiction.

Therefore

(3.10) $\Omega\varsigma(x_{2n+1}, x_{2n}) \leq \beta\Omega\varsigma(x_{2n}, x_{2n-1})$.

Hence from Case (a) and Case (b), we conclude that

$\Omega\varsigma(x_n, x_{n+1}) \leq \beta\Omega\varsigma(x_n, x_{n-1})$ for all n in $\mathbb{N}$.

Thus, in view of lemma 1.8 and condition (3.2) of our assumption, it follows that $\{x_n\}$ is a Cauchy sequence in N . Since (N, $\Omega\varsigma$) is complete there exists u in N such that

(3.11) $\lim_{n\to\infty} x_n = u.$

We now show that u is a common fixed point of S and T.

First, we suppose that S is continuous on N, then we have

(3.12) $Su = \lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} x_{2n+1} = u.$

We now assert that Tu=u.

$\frac{1}{2}\min\{\Omega\varsigma(u, Su), \Omega\varsigma(Tu, u)\} = 0 \leq \Omega\varsigma(u, u) = 0 \Rightarrow$ by condition (3.1) and (3.12), we have

$\Omega\varsigma$ (Su, Tu) $\leq \beta max\{\Omega\varsigma(u, Su), \Omega\varsigma(u, Tu), \Omega\varsigma(u, u)\} + L\Omega\varsigma(u, Tu)\, \Omega\varsigma(u, Su)$

$\Longrightarrow \Omega\varsigma$ (Su, Tu) $\leq \beta\ \Omega\varsigma(u, Tu) \Rightarrow$ u=Tu.

Hence u=Su=Tu.

Therefore u is a common fixed point of S and T.

By choosing S=T in Theorem 3.2, we have the following corollary.

Corollary 3.3. Let (N, Ω_ς) be an extended rectangular B-metric space with function ς : N× N→ [1, ∞). Let T be a selfmap on N satisfying the following conditions:

if for any $\beta \in [0, 1)$ and $L \geq 0$, such that $\forall\ \mu, \eta \in N$,

(3.13) $\frac{1}{2}min\{ \Omega\varsigma(\mu, T\mu), \Omega\varsigma(\eta, T\eta)\} \leq \Omega\varsigma(\mu, \eta) \Rightarrow$

$$\Omega\varsigma\ (T\mu, T\eta) \leq \beta \max\{ \Omega\varsigma(\mu, \eta), \Omega\varsigma(\mu, T\mu), \Omega\varsigma(\eta, T\eta)\} + L\ \Omega\varsigma(\mu, T\eta)\Omega\varsigma(\eta, T\mu).$$

Further suppose that

(3.14) for any sequence $\{x_n\}$ in N, *is such that* $\lim_{m,n\to\infty} \theta(x_n, x_m) < \frac{1}{\beta}$.

(3.15) T is continuous.

Then T has a fixed point in N.

By choosing L=0, in Theorem 3.2, we have the following corollary.

Corollary 3.4. Let (N, Ω_ς) be an extended rectangular B-metric space with function ς : N× N→ [1, ∞). Let T be a selfmap on N satisfying the following conditions:

if for any $\beta \in [0, 1)$ such that $\forall\ \mu, \eta \in N$, (3.14) $\frac{1}{2}min\{\ \Omega\varsigma(\mu,T\mu),\Omega\varsigma(\eta,T\eta)\} \leq \Omega\varsigma(\mu,\eta) \Rightarrow$

$\Omega\varsigma$ (Sμ, Tη) ≤ β max$\{\ \Omega\varsigma(\mu,\eta),\Omega\varsigma(\mu,S\mu),\ \Omega\varsigma(\eta,T\eta)\}$.

Further suppose that:

(3.15) for any sequence $\{x_n\}$ in N, $\lim_{m,n\to\infty} \theta(x_{n\,,}x_m) < \frac{1}{\beta}$.

(3.16) Either S or T is continuous.

Then S and T have a common fixed in N.

Example 3.5. Let R=$[0,\frac{2}{3}]$ and S=$(\frac{2}{3}, 1]$ and N= RUS. We define the function ς: N×N→[1, ∞)

A by ς(μ, η)= μ+η +2 and $\Omega\varsigma$: $N \to N$ by

$$\Omega_\varsigma(\mu,\eta) = \begin{cases} 0 & if\ \mu=\eta \\ \frac{2}{3} & \mu \in R, \\ 1 & \mu \in S \\ (\mu+\eta)^2 & otherwise \end{cases}$$

Clearly, (N, $\Omega\varsigma$) is an extended rectangular B- metric space with the function ς.

We define S and T on N by

$T\mu = \frac{\mu}{3}$ for all $\mu \in N$ and $S\mu = \begin{cases} 0 & if\ \ \mu \in R \\ \frac{\mu}{4} & if\ \ \mu \in S \end{cases}$

When $x_0 \in R$, we have $x_n = 0$, so $\lim_{m,n\to\infty} \varsigma(x_n, x_m) = 0+0+2 < \frac{1}{\beta}$, when $\beta = \frac{2}{5}$ and

when $x_0 \in S$, we have $x_n = 0$, so $\lim_{m,n\to\infty} \varsigma(x_n, x_m) = 0+0+2 < \frac{1}{\beta}$, when $\beta = \frac{2}{5}$.

Here we note that T is continuous.

We now verify the inequality (3.1) with for any L>0 and $\beta = \frac{2}{5}$.

Case(a): Let μ, η ∈R

Without loss of generality suppose that μ≠η.

$\frac{1}{2}\ min\{\ \Omega\varsigma(\mu,S\mu),\Omega\varsigma(\eta,T\eta)\} = \frac{1}{2}min\{\Omega\varsigma(\mu,0)\ \ \Omega\varsigma(\eta,\frac{\eta}{3})\} = \frac{1}{3} \leq \frac{2}{3} \Rightarrow$

From (3.1), we have

$\Omega\varsigma\left(0,\frac{\eta}{3}\right) = \frac{2}{3} \leq \beta\ max\left\{\Omega\varsigma(\mu,0),\Omega\varsigma\left(\eta,\frac{\eta}{3}\right),\Omega\varsigma(\mu,\eta)\right\}$+L$\Omega\varsigma\left(\mu,\frac{\eta}{3}\right)\Omega\varsigma(\eta,0)$

$$\leq \beta\frac{2}{3}+L\frac{4}{9} \ = \ \frac{4}{15}+\frac{4}{9}L\,,$$

so inequality (3.1) holds for any L$\geq\frac{9}{10}$.

Case(b): Let μ, η ∈S

Without loss of generality suppose that $\mu\neq$η.

$\frac{1}{2}\ min\{\ \Omega\varsigma(\mu,S\mu),\Omega\varsigma(\eta,T\eta)\}=\frac{1}{2}min\{\left(\mu+\ \frac{\mu}{4}\right)^2,\ (\eta+\frac{\eta}{3})^2\ \}=\frac{1}{2}\ \left(\mu+\ \frac{\mu}{4}\right)^2\leq 1\Rightarrow$from (3.1), we have

$\Omega\varsigma\left(\frac{\mu}{4},\frac{\eta}{3}\right)=\ \frac{2}{3}\leq\beta max\{\Omega\varsigma\left(\mu,\frac{\mu}{4}\right),\Omega\varsigma\left(\eta,\frac{\eta}{3}\right),\Omega\varsigma(\mu,\eta)\}$+L$\Omega\varsigma\left(\mu,\frac{\eta}{3}\right)\Omega\varsigma\left(\eta,\frac{\mu}{4}\right)$

the minimum value of RHS attains at $\eta,\ \mu\rightarrow\frac{2}{3}$,

therefore $\Omega\varsigma\left(\frac{\mu}{4},\frac{\eta}{3}\right)=\frac{2}{3}\leq\beta$+L$\frac{64}{81}\ \frac{25}{36}=\frac{2}{5}+L\,\frac{400}{729}$

so inequality (3.1) holds for any L$\geq\frac{1}{2}$.

Case (c): When μ ∈R, $\eta\in S$, we have

$\frac{1}{2}\ min\{\ \Omega\varsigma(\mu,S\mu),\Omega\varsigma(\eta,T\eta)\}=\frac{1}{2}min\{\ \Omega\varsigma(\mu,0),\Omega\varsigma\left(\eta,\frac{\eta}{3}\right)\quad=\frac{1}{2}min\{(\eta+\frac{\eta}{3})^2,\frac{2}{3}\}=\ \frac{1}{2}\ \ \frac{2}{3}=$ $\frac{1}{3}\leq(\eta+\mu)^2\ \Rightarrow$from (3.1), we have

$\Omega\varsigma\left(\ 0,\frac{\eta}{3}\right)=\frac{2}{3}\leq\beta max\{\Omega\varsigma(\mu,0\ ),\Omega\varsigma\left(\eta,\frac{\eta}{3}\right),\Omega\varsigma(\mu,\eta)\}$+L$\Omega\varsigma\left(\mu,\frac{\eta}{3}\right)\Omega\varsigma(\eta,0)$

the minimum value of RHS attains at $\mu\rightarrow 0,\eta\rightarrow\frac{2}{3}$, in this case

$\Omega\varsigma\left(\ 0,\frac{\eta}{3}\right)=\frac{2}{3}\ \leq\frac{2}{5}\ \frac{64}{81}$+L$\frac{16}{729}$.

so inequality (3.1) holds for any L$\geq\frac{639}{40}$.

Case (d): When μ ∈S, $\eta\in R$, we have

$\frac{1}{2}\ min\{\ \Omega\varsigma(\mu,S\mu),\Omega\varsigma(\eta,T\eta)\}=\frac{1}{2}min\{\ \Omega\varsigma\left(\mu,\frac{\mu}{4}\right),\Omega\varsigma\left(\eta,\frac{\eta}{3}\right)\}=\quad\frac{1}{2}min\{\frac{25\mu^2}{16},\frac{2}{3}\}\leq(\eta+\mu)^2\ \Rightarrow$ from (3.1), we have

$\Omega\varsigma\left(\frac{\mu}{4},\frac{\eta}{3}\right)=\frac{2}{3}\leq\beta max\{\Omega\varsigma\left(\mu,\frac{\mu}{4}\ \right),\Omega\varsigma\left(\eta,\frac{\eta}{3}\right),\Omega\varsigma(\mu,\eta)\}$+L$\Omega\varsigma\left(\mu,\frac{\eta}{3}\right)\Omega\varsigma\left(\eta,\frac{\mu}{4}\right)$

$$=\ \beta\ \ max\{\Omega\varsigma\left(\mu,\frac{\mu}{4}\ \right),\Omega\varsigma\left(\eta,\frac{\eta}{3}\right),\Omega\varsigma(\mu,\eta)\}+L\left(\mu+\ \frac{\eta}{3}\right)^2\ (\eta+\frac{\mu}{4})^2$$

the minimum value of RHS attains at $\mu \to \frac{2}{3}, \eta \to 0$, in this case

$\Omega\varsigma\left(\frac{\mu}{4}, \frac{\eta}{3}\right) = \frac{2}{3} \leq \beta\frac{25}{36}$+L$\frac{1}{81}$

so inequality (3.1) holds for any L≥32.

Thus all the conditions of Theorem 3.2 are satisfied and '0' is a common fixed point of S and T.

Here we note that the inequality (3.14) fails to hold when for any $\beta \in [0, 1)$ *and* L=0 at $\mu = \frac{2}{3}$, $\eta = \frac{1}{3}$, since $\Omega\varsigma\left(0, \frac{1}{9}\right) = \frac{2}{3} \nleq \beta max\{\Omega\varsigma\left(\frac{2}{3}, 0\right), \Omega\varsigma\left(\frac{2}{3}, \frac{1}{3}\right), \Omega\varsigma\left(\frac{1}{3}, \frac{1}{9}\right)\} = \beta\frac{2}{3} < \frac{2}{3}$.

Thus, Theorem 3.2 is a generalization of Corollary 3.4.

Conflict of Interests

The authors declare that there is no conflict of interests.

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