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MEASURE OF MODIFIED ROTATABILITY FOR SECOND ORDER RESPONSE SURFACE DESIGNS

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Abstract: A measure enables us to assess the degree of rotatability for a given response surface designs. In this paper, a new measure of modified rotatability for second order response **s**urface designs is suggested. The method is illustrated using central composite designs for $2 \le v \le 17$.

Keywords: response surface designs; central composite designs; measure of rotatability.

2010 AMS Subject Classification: 62K05.

1. INTRODUCTION

Response surface methodology is a collection of mathematical and statistical techniques useful for analyzing problems where several independent variables influence a dependent variable. The independent variables are often called the input or explanatory variables and the dependent variable is often the response variable. An important step in development of response surface designs was the introduction of rotatable designs by Box and Hunter [1]. Das and Narasimham [2] constructed rotatable designs using balanced incomplete block designs (BIBD). A design is said

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to be rotatable if the variance of the response estimate is a function only of the distance of the point from the design centre. Das et al. [3] introduced modified second order response surface designs. Park et al. [5] introduced measure of rotatability for second order response surface designs. Victorbabu et al. [14] suggested modified second order response surface designs using central composite designs. Victorbabu and Surekha [8] developed measure of rotatability for second order response surface designs using central composite designs. Victorbabu and Vasundharadevi [11] suggested modified second order response surface designs using BIBD. Victorbabu and Surekha [9] studied measure of rotatability for second order response surface designs using BIBD. Victorbabu et al. [13] suggested modified second order response surface designs, rotatable designs using pairwise balanced design. Victorbabu and Vasundharadevi [12] studied modified second order response surface designs, rotatable designs using symmetrical unequal block arrangements (SUBA) with two unequal block sizes. Victorbabu and Surekha [10] studied measure of rotatability for second order response surface designs using incomplete block designs. These measures are useful to enable us to assess the degree of modified rotatability for a given second order response surface designs.

2. CONDITIONS FOR SECOND ORDER ROTATABLE DESIGNS

Suppose we want to use the second order response surface design D= $((x_{iu}))$ to fit the surface,

$$Y_u = b_0 + \sum_{i=1}^{v} b_i x_{iu} + \sum_{i=1}^{v} b_{ii} x_{iu}^2 + \sum_{i<j}\sum b_{ij} x_{iu} x_{ju} + e_u \qquad (2.1)$$

where x_{iu} denotes the level of the i^{th} factor (i =1,2,…,v) in the u^{th} run (u=1,2,…,N) of the experiment, e_u's are uncorrelated random errors with mean zero and variance σ^2. D is said to be second order rotatable design (SORD), if the variance of the estimated response of $\hat{Y}_u$ from the fitted surface is only a function of the distance ($d^2 = \sum_{i=1}^{v} x_i^2$) of the point $(x_1, x_2, ..., x_v)$ from the origin (centre) of the design. Such a spherical variance function for estimation of second order response surface is achieved if the design points satisfy the following conditions [cf. Box and

Hunter [1], Das and Narasimham [2]].

1. $\sum x_{iu}=0$, $\sum x_{iu}x_{ju}=0$, $\sum x_{iu}x_{ju}^2=0$, $\sum x_{iu}x_{ju}x_{ku}=0$, $\sum x_{iu}^3=0$, $\sum x_{iu}x_{ju}^3=0$,

   $\sum x_{iu}x_{ju}x_{ku}^2=0$, $\sum x_{iu}x_{ju}x_{ku}x_{lu}=0$; for $i \neq j \neq k \neq l$; (2.2)

2. (i) $\sum x_{iu}^2 =$ constant $= N\lambda_2$;

   (ii) $\sum x_{iu}^4 =$ constant $= cN\lambda_4$; for all i (2.3)

3. $\sum x_{iu}^2 x_{ju}^2 =$ constant $= N\lambda_4$; for $i \neq j$ (2.4)

4. $\sum x_{iu}^4 = c\sum x_{iu}^2 x_{ju}^2$ (2.5)

5. $\frac{\lambda_4}{\lambda_2^2} > \frac{v}{(c+v-1)}$ (2.6)

where c, λ_2 and λ_4 are constants and the summation is over the design points.

If the above mentioned conditions are satisfied, the variances and covariances of the estimated parameters become,

$$V(\hat{b}_0) = \frac{\lambda_4(c+v-1)\sigma^2}{N\left[\lambda_4(c+v-1)-v\lambda_2^2\right]} ,$$

$$V(\hat{b}_i) = \frac{\sigma^2}{N\lambda_2},$$

$$V(\hat{b}_{ij}) = \frac{\sigma^2}{N\lambda_4},$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2}{(c-1)N\lambda_4}\left[\frac{\lambda_4(c+v-2)-(v-1)\lambda_2^2}{\lambda_4(c+v-1)-v\lambda_2^2}\right],$$

$$Cov(\hat{b}_0,\hat{b}_{ii}) = \frac{-\lambda_2\sigma^2}{N[\lambda_4(c+v-1)-v\lambda_2^2]},$$

$$Cov(\hat{b}_{ii},\hat{b}_{jj}) = \frac{(\lambda_2^2-\lambda_4)\sigma^2}{(c-1)N\lambda_4[\lambda_4(c+v-1)-v\lambda_2^2]} \qquad (2.7)$$

and other covariances are zero.

3. CONDITIONS FOR MODIFIED SECOND ORDER ROTATABLE DESIGNS

The most widely used design for fitting a second order model is the central composite design. Central composite designs are constructed by adding suitable factorial combinations to those obtained from $\frac{1}{2^p}\times 2^v$ fractional factorial design (here $2^{t(v)}=\frac{1}{2^p}\times 2^v$ denotes a suitable fractional replicate of 2^v, in which no interaction with less than five factors is confounded). In coded form the points of $2^v(2^{t(v)})$ factorial have coordinates $(\pm a,\pm a,...,\pm a)$ and 2v axial points have coordinates of the form $((\pm b, 0,...0), (0, \pm b,...0), . . . ,(0, 0, ..., \pm b))$ etc., and n_0 central points. The usual method of construction of SORD is to take combinations with unknown constants, associate a 2^v factorial combinations or a suitable fraction of it with factors each at ± 1 levels to make the level codes equidistant. All such combinations form a design. Generally, SORD need at least five levels (suitably coded) at $0, \pm a, \pm b$ for all factors $((0,0,...0))$- chosen centre of the design, unknown level $\pm a$ and $\pm b$ are to be chosen suitably to satisfy the conditions of the rotatability). Generation of design points this way ensures satisfaction of all the conditions even though the design points contain unknown levels.

Alternatively, by putting some restrictions indicating some relation among $\sum x_{iu}^2, \sum x_{iu}^4$ and $\sum x_{iu}^2 x_{ju}^2$ some equations involving the unknowns are obtained and their solution gives the unknown levels. In SORD the restriction used is $\sum x_{iu}^4 = 3\sum x_{iu}^2 x_{ju}^2$, i.e., $c=3$. Other restrictions are also possible through, it seems, not exploited well. We shall investigate the restriction $(\sum x_{iu}^2)^2 = N\sum x_{iu}^2 x_{ju}^2$ i.e. $\lambda_2^2=\lambda_4$ (cf. Das et al. [3]) to get another series of symmetrical second order response surface designs, which provide more precise estimates of response at specific points of interest than what is available from the corresponding existing designs. Further, the variances and covariances of the estimated parameters are,

$$V(\hat{b}_0) = \frac{(c+v-1)\sigma^2}{N(c-1)}$$

$$V(\hat{b}_i) = \frac{\sigma^2}{N\sqrt{\lambda_4}}$$

$$V(\hat{b}_{ij}) = \frac{\sigma^2}{N\lambda_4}$$

$$V(\hat{b}_{ii}) = \frac{\sigma^2}{(c-1)N\lambda_4}$$

$$Cov(\hat{b}_0, \hat{b}_{ii}) = \frac{-\sigma^2}{N\sqrt{\lambda_4}(c-1)} \qquad (3.1)$$

and other covariances are zero. These modifications of the variances and covariances affect the variance of the estimated response at specific points considerably. Using these variances and covariances, variance of estimated response at any point can be obtained. Let $\hat{Y}_u$ denote the estimated response at the point $(x_{1u}, x_{2u}, \ldots x_{vu})$. Then,

$$V(\hat{Y}_u) = V(\hat{b}_0) + d^2[V(\hat{b}_i) + 2cov(\hat{b}_0, \hat{b}_{ii})] + d^4 V(\hat{b}_{ii}) + (\sum x_{iu}^2 x_{ju}^2)[(c-3)\sigma^2 / (c-1)N\lambda_4]$$

Construction of modified response surface designs is the same as for SORD except that instead of taking c=3 the restriction $(\sum x_{iu}^2)^2 = N\sum x_{iu}^2 x_{ju}^2$ is to be used and this condition will provide different values of the unknowns involved.

4. CONDITIONS FOR MEASURE OF ROTATABILITY FOR SECOND ORDER RESPONSE SURFACE DESIGNS

Following Box and Hunter [1], Das and Narasimham [2], Park et al. [5], conditions (2.2) to (2.6) and (2.7) give the necessary and sufficient conditions for a measure of rotatability for any general second order response surface designs. Further we have,

$V(b_i)$ are equal for i,

$V(b_{ii})$ are equal for i,

$V(b_{ij})$ are equal for i, j, where $i \neq j$,

$$\mathrm{Cov}(b_i,b_{ii})=\mathrm{Cov}(b_i,b_{ij})=\mathrm{Cov}(b_{ii},b_{ij})=\mathrm{Cov}(b_{ij},b_{il})=0 \quad \text{for all} \quad i \neq j, j \neq l, l \neq i. \tag{4.1}$$

Park et al. [5] suggested that if the conditions in (2.2) to (2.6) together with (2.7) and (4.1) are met, then the following measure $(P_v(D))$ given below can be used to assess the degree of rotatability for any general second order response surface design (cf. Park et al. [5], page 661).

$$P_v(D)=\frac{1}{1+R_v(D)}, \tag{4.2}$$

where

$$R_v(D)=\left[\frac{N}{\sigma^2}\right]^2 \frac{6v\left[V(\hat{b}_{ij})+2\,\mathrm{cov}(\hat{b}_{ii},\hat{b}_{jj})-2V(\hat{b}_{ii})\right]^2(v\text{-}1)}{(v+2)^2(v+4)(v+6)(v+8)g^8} \tag{4.3}$$

and g is the scaling factor.

On simplification, numerator of (4.3), $[V(\hat{b}_{ij})+2\,\mathrm{cov}(\hat{b}_{ii},\hat{b}_{jj})-2V(\hat{b}_{ii})]$ using (2.7) becomes $(c\text{-}3)\sigma^2/(c\text{-}1)N\lambda_4$. Thus $R_v(D)$ becomes

$$R_v(D)=\left[\frac{N}{\sigma^2}\right]^2\left(\frac{6v[(c\text{-}3)\sigma^2]^2(v\text{-}1)}{[(c\text{-}1)N\lambda_4]^2(v+2)^2(v+4)(v+6)(v+8)g^8}\right) \tag{4.4}$$

Note. For SORD, we have $c=3$. Substituting the value of $'c'$ in (4.4) and on simplification we get $R_v(D)$ is zero. Hence from (4.2), we get $(P_v(D)$ is one if and only if a design is rotatable and less than one for a nearly rotatable design.

5. Measure of Rotatability for Second Order Response Surface Designs Using CCD

Following Park et al. [5] and Victorbabu and Surekha [8], the method of measure of rotatability for second order response surface design using CCD is given below.

The central composite design (CCD) consists of 2^v or a fraction of 2^v factorial points $(\pm a, \pm a, ... \pm a)$ repeated y_1 times, 2v axial points of the form $(\pm b, 0, ...0)$ repeated y_2 times etc., and a centre point $(0, 0, ..., 0)$. The central point may be replicated n_0 times. Thus the total number of experimental points (N) can be written as $N=2^{t(v)}y_1+2vy_2+n_0$ with level 'a' and 'b' prefixed then $c=\frac{2^{t(v)}y_1a^4+2y_2b^4}{2^{t(v)}y_1}$.

The measure of rotatability for second order response surface designs using CCD is

$$R_v(D)=\left[\frac{(c\text{-}3)}{(c\text{-}1)}\right]^2\frac{6v(v\text{-}1)}{\lambda_4^2(v+2)^2(v+4)(v+6)(v+8)g^8}$$

where

$$g=\begin{cases}\frac{1}{b}, & \text{if } b\ge\sqrt{v}\\ \frac{1}{\sqrt{v}}, & \text{if } b<\sqrt{v}\end{cases}$$

If $P_v(D)$ is 1 if and only if the design is rotatable, and it is smaller than one for a nearly rotatable designs.

6. Measure of Modified Rotatability for Second Order Response Surface Designs Using Central Composite Designs

The proposed measure of modified rotatability for second order response surface designs using CCD is suggested below.

Consider the following set of points:

(i) $2^{t(v)}$ (where $2^{t(v)}$ is resolution V fraction of 2^v) points on cube viz., coordinates $(\pm a,\pm a,\ldots,\pm a)$ repeated y_1 times,

(ii) $2v$ axial points, viz., $(\pm b,0,\ldots,0),(0,\pm b,\ldots,0),\ldots,(0,0,\ldots,\pm b)$ repeated y_2 times,

(iii) n_0 central points, where y_1 and y_2 are chosen to satisfy the criterion of modified rotatability.

Theorem 6.1: The design points,

$$y_1(\pm a,\pm a,\ldots,\pm a)2^{t(v)}\cup y_2(\pm b,0,0,\ldots,0)2^1\cup n_0,$$

will give a v-dimensional measure of modified rotatability for second order response surface designs using central composite designs in

$$N = \frac{(2^{t(v)}a^2y_1+2b^2y_2)^2}{2^{t(v)}a^4y_1}$$ design points, if

$$c = \frac{2^{t(v)}y_1a^4+2y_2b^4}{2^{t(v)}y_1a^4}, \tag{6.2}$$

$$\left(\frac{b}{a}\right)^4 = \left(\frac{2^{t(v)}y_1}{y_2}\right), \tag{6.3}$$

$$n_0 = \left\{\frac{(2^{t(v)}a^2y_1+2b^2y_2)^2}{2^{t(v)}a^4y_1}\right\} - y_1 2^{t(v)} - 2vy_2. \tag{6.4}$$

and n_0 turns out to be an integer.

Proof: For the design points generated from central composite design the conditions in equations (2.2) to (2.6) are satisfied. The conditions in equation (2.3) and (2.4) are true as follows:

$$\sum x_{iu}^2 = 2^{t(v)}y_1a^2+2y_2b^2 = N\lambda_2 \tag{6.5}$$

$$\sum x_{iu}^4 = 2^{t(v)}y_1a^4+2y_2b^4 = cN\lambda_4 \tag{6.6}$$

$$\sum x_{iu}^2x_{ju}^2 = 2^{t(v)}y_1a^4 = N\lambda_4 \tag{6.7}$$

From equations (6.6) and (6.7), we have $2^{t(v)}y_1a^4+2y_2b^4 = c\,2^{t(v)}y_1a^4$, which on simplification lead to equation (6.2). Using the modified condition $\lambda_2^2 = \lambda_4$, from equations (6.5) and (6.7) we get, $N = \frac{(2^{t(v)}a^2y_1+2b^2y_2)^2}{2^{t(v)}a^4y_1}$. Alternatively N may be obtained directly as

$$N = 2^{t(v)}y_1+2vy_2+n_0,$$

where $n_0 = \left\{(2^{t(v)}a^2y_1+2b^2y_2)^2 / 2^{t(v)}a^4y_1\right\} - y_1 2^{t(v)} - 2vy_2$.

From equations (6.5), (6.6) and (6.7) (taking a=1) and on simplification we get

$$b^4 = \left(2^{t(v)}y_1/y_2\right),$$

$$\lambda_2 = \frac{2^{t(v)}y_1+2y_2b^2}{N},$$

$\lambda_4 = \frac{2^{t(v)} y_1}{y_2}$.

To obtain measure of modified rotatability for second order response surface designs using central composite designs

$$P_v(D) = \frac{1}{1+R_v(D)}$$

$$R_v(D) = \left[\frac{(c-3)}{(c-1)}\right]^2 \frac{6v(v-1)}{\lambda_4^2 (v+2)^2 (v+4)(v+6)(v+8) g^8}$$, where g is a scaling factor

$$g = \begin{cases} \frac{1}{b}, & \text{if } b \geq \sqrt{v} \\ \frac{1}{\sqrt{v}} & \text{otherwise} \end{cases}$$

Example: We illustrate the measure of modified rotatability for second order response surface design for $v = 5$ factors in $N = 36$ design points (taking y_1=1, a=1). From (6.5), (6.6) and (6.7), we have

$$\sum x_{iu}^2 = 16+2y_2 b^2 = N\lambda_2 \quad (6.8)$$

$$\sum x_{iu}^4 = 16+2y_2 b^4 = cN\lambda_4 \quad (6.9)$$

$$\sum x_{iu}^2 x_{ju}^2 = 16 = N\lambda_4 \quad (6.10)$$

For $v=5$ factors, to make the design modified rotatable, we take $c=3$. From equations (6.9) and (6.10) we get $b^4=16 \Rightarrow b^2=4 \Rightarrow b=2$ we take $y_2=1$. From the modified condition $\lambda_2^2 = \lambda_4$, we get $N=36$. From equation (6.4) we get $n_0 = 10$.

The proposed measure of modified rotatability for second order response surface designs using central composite design at the modified rotatability value b=2, $y_2 = 1$ and $N=36$, we get $c=3$, scaling factor $g=0.4472$, $R_v(D)=0$ and $P_v(D)=1$. Then the design is modified rotatable.

Instead of taking $b=2$ if we take $b=2.5$ for $v=5$ factors from equations (6.9) and (6.10) we get $c=5.8828$. The scaling factor $g=0.4$, $R_v(D) = 5.1237$ and $P_v(D) = 0.1633$. Here $P_v(D)$

becomes smaller it deviates from modified rotatability.

The measure of modified rotatability for second order response surface designs using central composite designs are presented in table for $2 \le v \le 17$. It can be verified that $P_v(D)$ is 1 if and only if the design is modified rotatable design and it is smaller than one for a nearly modified rotatable design.

Table. Measure of modified rotatability for second order response surface designs using central composite designs. (taking a=1)

$v = 2, N = 16, y_1 = 1, y_2 = 1, n_0 = 8, b = 1.414214$				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.5	0.7071	3.6	0.2174
1.3	2.4280	0.7071	0.0642	0.9397
*1.414214	3	0.7071	0	1
1.6	4.2768	0.625	0.1630	0.8598
1.9	7.5161	0.5263	2.0395	0.3290
2.2	12.7128	0.4545	9.4338	0.9584
2.5	20.5313	0.4	30.7345	0.0315
2.8	31.7328	0.3571	82.5574	0.012
3.1	47.1761	0.3226	195.1523	5.0981×10^{-3}
3.4	67.8168	0.2941	420.1218	2.3746×10^{-3}
3.7	94.7081	0.2703	841.0366	1.1876×10^{-3}
4	129	0.25	1587.6	6.2949×10^{-4}
4.3	171.94	0.2326	2854.074	3.5025×10^{-4}
4.6	224.8728	0.2174	4922.754	2.0310×10^{-4}
4.9	289.2401	0.2041	8193.337	1.2204×10^{-4}
5.2	366.5808	0.1923	13219.1	7.5642×10^{-4}

$v= 3, N= 32, y_1 =1, y_2 = 2, n_0 = 12, b =1.414214$				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.5	0.5774	24.2369	0.0396
1.3	2.4281	0.5774	0.4320	0.6983
*1.414214	3	0.5774	0	1
1.6	4.2768	0.5774	0.4089	0.7098
1.9	7.5161	0.5263	2.7122	0.2694
2.2	12.7128	0.4545	12.5458	0.0738
2.5	20.5313	0.4	40.8729	0.0239
2.8	31.7328	0.3571	109.7906	9.0260×10^{-3}
3.1	47.1761	0.3226	259.5273	3.8384×10^{-3}
3.4	67.8168	0.2941	558.7074	1.7866×10^{-3}
3.7	94.7081	0.2703	1118.47	8.9328×10^{-4}
4	129	0.25	2111.302	4.7417×10^{-4}
4.3	171.94	0.2326	3795.548	2.634×10^{-4}
4.6	224.8728	0.2174	6546.623	1.5273×10^{-4}
4.9	289.2401	0.2041	10896.07	9.1768×10^{-5}
5.2	366.5808	0.1923	17579.69	5.6881×10^{-5}

$v= 4, N= 36, y_1 =1, y_2 =1, n_0 = 12, b = 2$				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.125	0.5	607.5	1.6434×10^{-3}
1.3	1.3570	0.5	57.1828	1.7187×10^{-2}
1.6	1.8192	0.5	5.6097	0.1513
1.9	2.6290	0.5	0.1400	0.8772
*2	3	0.5	0	1
2.2	3.9282	0.4545	0.5815	0.6323
2.5	5.8828	0.4	5.6097	0.1513
2.8	8.6832	0.3571	21.8017	0.0439
3.1	12.5440	0.3226	61.4845	0.0160
3.4	17.7042	0.2941	145.9441	6.8053×10^{-3}
3.7	24.4270	0.2703	309.9039	3.2164×10^{-3}
4	33	0.25	607.5	1.6434×10^{-3}
4.3	43.73501	0.2326	1120.055	8.9202×10^{-4}
4.6	56.9682	0.2174	1965.982	5.0839×10^{-4}
4.9	73.06001	0.2041	3313.174	3.0173×10^{-4}
5.2	92.3952	0.1923	5394.264	1.8535×10^{-4}

$v=5$, $N=36$, $y_1=1$, $y_2=1$, $n_0=10$, $b=2$				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.125	0.4472	1354.672	7.3764×10^{-4}
1.3	1.357013	0.4472	127.5127	7.7813×10^{-3}
1.6	1.8192	0.4472	12.5091	7.4024×10^{-2}
1.9	2.629012	0.4472	0.31224	0.7620
*2	3	0.4472	0	1
2.2	3.9282	0.4472	0.605	0.6231
2.5	5.8828	0.4	5.1237	0.1633
2.8	8.6832	0.3571	19.9130	0.0478
3.1	12.54401	0.3226	56.1583	0.0175
3.4	17.7042	0.2941	133.3013	7.446×10^{-3}
3.7	24.42701	0.2703	283.0576	3.5204×10^{-3}
4	33	0.25	554.8737	1.799×10^{-3}
4.3	43.73501	0.2326	1023.028	9.7654×10^{-4}
4.6	56.9682	0.2174	1795.674	5.5658×10^{-4}
4.9	73.06001	0.2041	3026.161	3.3034×10^{-4}
5.2	92.3952	0.1923	4926.971	2.0292×10^{-4}

$v= 6, N= 72, y_1 =1, y_2 = 2, n_0 = 16, b = 2$				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.125	0.4082	2471.359	4.0447×10^{-4}
1.3	1.3570	0.4082	232.6242	4.2804×10^{-3}
1.6	1.8192	0.4082	22.8206	4.1981×10^{-2}
1.9	2.6290	0.4082	0.5697	0.6371
*2	3	0.4082	0	1
2.2	3.9282	0.4082	1.1037	0.4754
2.5	5.8828	0.4	4.5078	0.1816
2.8	8.6832	0.3571	17.5192	0.054
3.1	12.5440	0.3226	49.4072	0.0198
3.4	17.7042	0.2941	117.2765	8.4548×10^{-3}
3.7	24.4270	0.2703	249.0299	3.9995×10^{-3}
4	33	0.25	488.1696	2.0443×10^{-3}
4.3	43.73501	0.2326	900.0445	1.1098×10^{-3}
4.6	56.9682	0.2174	1579.807	6.3259×10^{-4}
4.9	73.06001	0.2041	2662.372	3.7546×10^{-4}
5.2	92.3952	0.1923	4334.676	2.3064×10^{-4}

v= 7, N= 100, y_1 =1, y_2 =1, n_0 = 22, b = 2.828427				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.03125	0.3779	33744.39	2.9634×10^{-5}
1.3	1.0893	0.3779	3896.547	2.5657×10^{-4}
1.6	1.2048	0.3779	653.2602	1.5284×10^{-3}
1.9	1.4073	0.3779	130.0424	7.6311×10^{-3}
2.2	1.7321	0.3779	25.5061	3.7727×10^{-2}
2.5	2.2207	0.3779	3.4650	0.2240
2.8	2.9208	0.3571	0.0227	0.9778
*2.828427	3	0.3536	0	1
3.1	3.8860	0.3226	2.8464	0.26
3.4	5.1761	0.2941	17.1698	5.5036×10^{-2}
3.7	6.8568	0.2703	53.9352	1.8203×10^{-2}
4	9	0.25	130.5361	7.6025×10^{-3}
4.3	11.6838	0.2326	273.4281	3.6439×10^{-3}
4.6	14.9921	0.2174	521.4529	1.914×10^{-3}
4.9	19.015	0.2041	929.9986	1.0741×10^{-3}
5.2	2..8488	0.1923	1576.124	6.3407×10^{-4}

$v=8$, $N=100$, $y_1=1$, $y_2=1$, $n_0=20$, $b=2.828427$				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.0313	0.3536	49612.5	2.0156×10^{-5}
1.3	1.0893	0.3536	5728.877	1.7452×10^{-4}
1.6	1.2048	0.3536	960.4523	1.0401×10^{-3}
1.9	1.4073	0.3536	191.1942	5.2031×10^{-3}
2.2	1.7321	0.3536	37.5001	2.5974×10^{-2}
2.5	2.2207	0.3536	5.0944	0.1641
2.8	2.9208	0.3536	0.0213	0.9792
*2.828427	3	0.3536	0	1
3.1	3.8860	0.3226	2.4531	0.2896
3.4	5.1761	0.2941	14.7975	6.3301×10^{-2}
3.7	6.8568	0.2703	46.4829	2.1060×10^{-2}
4	9	0.25	112.5	8.8106×10^{-3}
4.3	11.6838	0.2326	235.6487	4.2226×10^{-3}
4.6	14.1121	0.2174	449.4039	2.2567×10^{-3}
4.9	19.015	0.2040816	801.501	1.2461×10^{-3}
5.2	23.8488	0.19230	1358.351	7.3564×10^{-4}

$v= 9$, $N=200$, $y_1 =1$, $y_2 = 2$, $n_0 =36$, $b= 2.828427$				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.03125	0.3333	68470.9	1.4605×10^{-5}
1.3	1.0893	0.3333	7906.502	1.2646×10^{-4}
1.6	1.2048	0.3333	1325.534	7.5385×10^{-4}
1.9	1.4073	0.3333	263.8698	3.7754×10^{-3}
2.2	1.7321	0.3333	51.75445	1.8956×10^{-2}
2.5	2.2207	0.3333	7.030896	0.1245
2.8	2.9208	0.3333	0.0293	0.9715
*2.828427	3	0.3333	0	1
3.1	3.8860	0.3226	2.1136	0.3212
3.4	5.17605	0.2941	12.74948	7.2734×10^{-2}
3.7	6.8568	0.2703	40.0496	2.4361×10^{-2}
4	9	0.25	96.9298	1.0211×10^{-2}
4.3	11.68375	0.2326	203.0345	4.9011×10^{-3}
4.6	14.1121	0.2174	387.2057	2.5759×10^{-3}
4.9	19.015	0.2041	690.5719	1.4459×10^{-3}
5.2	23.8488	0.1923	1170.353	8.5371×10^{-4}

$v=10$, $N = 200$, $y_1 = 1$, $y_2 = 2$, $n_0 = 32$, $b = 2.828427$				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.0313	0.3162	90122.22	1.1096×10^{-5}
1.3	1.0893	0.3162	10406.63	9.6083×10^{-5}
1.6	1.2048	0.3162	1744.683	5.7284×10^{-4}
1.9	1.4073	0.3162	347.3085	2.8710×10^{-3}
2.2	1.7321	0.3162	68.1198	1.4468×10^{-2}
2.5	2.2207	0.3162	9.2542	9.7521×10^{-2}
2.8	2.9208	0.3162	0.0386	0.9628
*2.828427	3	0.3162	0	1
3.1	3.8860	0.3162	2.1401	0.3185
3.4	5.1761	0.2941	11.0100	8.3263×10^{-2}
3.7	6.8568	0.2703	34.5855	0.2810
4	9	0.25	83.7054	1.1806×10^{-2}
4.3	11.6838	0.2326	175.3338	5.6711×10^{-3}
4.6	14.9921	0.2174	334.3779	2.9817×10^{-3}
4.9	19.015	0.2041	596.3549	1.6740×10^{-3}
5.2	23.8488	0.1923	1010.678	9.8846×10^{-4}

v=11, N=200, y_1 =1, y_2 = 2, n_0 = 28, b*= 2.828427				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.0313	0.3015	114355	8.7442×10^{-6}
1.3	1.0893	0.3015	13204.86	7.5724×10^{-5}
1.6	1.2048	0.3015	2213.808	4.5151×10^{-4}
1.9	1.4072	0.3015	440.6958	2.2640×10^{-3}
2.2	1.7321	0.3015	86.4365	1.1437×10^{-2}
2.5	2.2207	0.3015	11.7425	7.8478×10^{-2}
2.8	2.9208	0.3015	0.0490	0.9533
*2.828427	3	0.3015	0	1
3.1	3.8860	0.3015	2.7155	0.2691
3.4	5.1761	0.2941	9.5420	9.4898×10^{-2}
3.7	6.8568	0.2703	29.9742	3.2285×10^{-2}
4	9	0.25	72.5338	1.3597×10^{-2}
4.3	11.68375	0.2326	151.9562	6.5378×10^{-3}
4.6	14.9921	0.2174	289.7946	3.4389×10^{-3}
4.9	19.015	0.2041	516.8417	1.9319×10^{-3}
5.2	23.8488	0.1923	875.9223	1.1404×10^{-3}

$v=12$, $N=324$, $y_1=1$, $y_2=1$, $n_0=44$, $b=4$				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.0078	0.2887	1515172	6.5999×10^{-7}
1.3	1.0223	0.2887	183050	5.463×10^{-6}
1.6	1.0512	0.2887	33757.97	2.9622×10^{-5}
1.9	1.1018	0.2887	8099.36	1.2345×10^{-4}
2.2	1.1830	0.2887	2296.805	4.352×10^{-4}
2.5	1.3052	0.2887	718.6723	1.3895×10^{-3}
2.8	1.4802	0.2887	233.4051	4.2661×10^{-3}
3.1	1.7215	0.2887	73.166	1.3483×10^{-2}
3.4	2.0440	0.2887	19.5378	4.8691×10^{-2}
3.7	2.4642	0.2703	5.2857	0.1591
*4	3	0.25	0	1
4.3	3.6709	0.2326	8.2878	0.1077
4.6	4.4980	0.2174	41.3149	2.3632×10^{-2}
4.9	5.5038	0.2041	115.4141	8.59×10^{-3}
5.2	6.7122	0.1923077	253.7111	3.926×10^{-3}

$v=13$, $N=324$, $y_1=1$, $y_2=1$, $n_0=42$, $b=4$				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.0078	0.2774	1824464	5.4811×10^{-7}
1.3	1.0223	0.2774	220416	4.5369×10^{-6}
1.6	1.0512	0.2774	40648.98	2.4600×10^{-5}
1.9	1.1018	0.2774	9752.681	1.0253×10^{-4}
2.2	1.1830	0.2774	2765.651	3.6145×10^{-4}
2.5	1.3052	0.2774	865.3747	1.1542×10^{-3}
2.8	1.4802	0.2774	281.05	3.5455×10^{-3}
3.1	1.7215	0.2774	88.1013	1.1223×10^{-2}
3.4	2.0440	0.2774	23.526	4.0773×10^{-2}
3.7	2.4642	0.2703	4.6209	0.1779
*4	3	0.25	0	1
4.3	3.6709	0.2325	7.2455	0.1213
4.6	4.4980	0.2174	36.1187	2.6941×10^{-2}
4.9	5.5037	0.2041	100.8983	9.8137×10^{-3}
5.2	6.7122	0.1923	221.8014	4.4883×10^{-3}

$v=14$, $N=324$, $y_1=1$, $y_2=1$, $n_0=40$, $b=4$				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.0078	0.2673	2155064	4.6402×10^{-7}
1.3	1.0223	0.2673	260356.1	3.8409×10^{-6}
1.6	1.0512	0.2673	48014.71	2.0827×10^{-5}
1.9	1.1018	0.2673	11519.9	8.6799×10^{-5}
2.2	1.1830	0.2673	3266.797	3.0602×10^{-4}
2.5	1.3052	0.2673	1022.184	9.7734×10^{-3}
2.8	1.4802	0.2673	331.9772	3.0032×10^{-3}
3.1	1.7215	0.2673	104.0656	9.5179×10^{-3}
3.4	2.0440	0.2673	27.78899	3.4736×10^{-2}
3.7	2.4642	0.2673	4.4382	0.1839
*4	3	0.25	0	1
4.3	3.6709	0.2326	6.3629	0.1358
4.6	4.4980	0.2174	31.71892	0.03056
4.9	5.5038	0.2041	88.6073	1.1159×10^{-2}
5.2	6.7122	0.1923	194.7826	5.1077×10^{-3}

v=15, N=324, y_1 =1, y_2 =1, n_0 =38, b=4				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.0078	0.2582	2505114	3.9918×10^{-7}
1.3	1.0223	0.2582	302646.2	3.3042×10^{-6}
1.6	1.0512	0.2582	55813.82	1.7916×10^{-5}
1.9	1.1018	0.2582	13391.09	7.4671×10^{-5}
2.2	1.1830	0.2582	3797.427	2.6327×10^{-4}
2.5	1.3052	0.2582	1188.218	8.4089×10^{-4}
2.8	1.4802	0.2582	385.9008	2.5846×10^{-3}
3.1	1.7215	0.2582	120.9691	8.1988×10^{-3}
3.4	2.0440	0.2582	32.3028	3.0028×10^{-2}
3.7	2.4642	0.2582	5.1591	0.1624
*4	3	0.25	0	1
4.3	3.6709	0.2326	5.6126	0.1512
4.6	4.4980	0.2174	27.9790	3.45×10^{-2}
4.9	5.5038	0.2041	78.1599	1.263×10^{-2}
5.2	6.7122	0.1923	171.8164	5.786×10^{-3}

$v=16, N=324, y_1=1, y_2=1, n_0=36, b=4$				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.0078	0.25	2872923	3480775×10^{-7}
1.3	1.0223	0.25	347081.7	2.8812×10^{-6}
1.6	1.0512	0.25	64008.58	1.5623×10^{-5}
1.9	1.1018	0.25	57357.22	6.5112×10^{-5}
2.2	1.1830	0.25	4345.978	2.2957×10^{-4}
2.5	1.3052	0.25	1362.677	7.3331×10^{-4}
2.8	1.4802	0.25	442.56	2.2545×10^{-3}
3.1	1.72150	0.25	138.7302	7.1567×10^{-3}
3.4	2.0440	0.25	37.0456	2.6284×10^{-2}
3.7	2.4642	0.25	5.9166	0.1446
*4	3	0.25	0	1
4.3	3.6709	0.2325	4.9722	0.1674
4.6	4.4980	0.2174	24.7865	3.88×10^{-2}
4.9	5.5038	0.2041	69.2414	1.42×10^{-2}
5.2	6.7122	0.1923	152.2111	6.527×10^{-3}

$v=17,\ N=324,\ y_1=1,\ y_2=1,\ n_0=34,\ b=4$				
b	c	g	$R_v(D)$	$P_v(D)$
1	1.0078	0.2425	3256956	3.0704×10^{-7}
1.3	1.0223	0.2425	3934744.83	2.5414×10^{-6}
1.6	1.0512	0.2425	72564.83	1.3781×10^{-5}
1.9	1.1018	0.2425	17410.07	5.7435×10^{-5}
2.2	1.1830	0.2425	4937.123	2.0251×10^{-4}
2.5	1.3052	0.2425	1544.83	6.4690×10^{-4}
2.8	1.4802	0.2425	501.7185	1.9892×10^{-3}
3.1	1.7215	0.2425	157.2747	6.3181×10^{-3}
3.4	2.0440	0.2425	41.9976	2.3257×10^{-2}
3.7	2.4642	0.2425	6.7075	0.1297
*4	3	0.25	0	1
4.3	3.6709	0.2326	4.4230	0.1844
4.6	4.4980	0.2174	22.0489	4.3386×10^{-2}
4.9	5.5038	0.2041	61.5939	1.59×10^{-2}
5.2	6.7122	0.1923	135.4	7.3×10^{-2}

*indicates modified rotatability for second order response surface designs using CCD.

(cf. Victorbabu et al. [14]).

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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