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TENSOR PRODUCT TECHNIQUE AND ATOMIC SOLUTION OF FRACTIONAL BATE MAN BURGERS EQUATION

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Abstract. Sometimes, it is not possible to find a general solution for some differential equations using some classical methods, like separation of variables. In such a case, one can try to use theory of tensor product of Banach spaces to find certain solutions, called atomic solutions. The goal of this paper is to find atomic solution for Bate Man Burgers equation.

Keywords: Burger equation; fractional derivative; atomic solution.

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1. Introduction

In [8], a new definition called $\alpha-$conformable fractional derivative was introduced:

Let $\alpha \in (0,1)$, and $f : E \subseteq (0,\infty) \to R$. For $x \in E$ let:

$$D^{\alpha} f(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon x^{1-\alpha}) - f(x)}{\varepsilon}.$$

If the limit exists then it is called the $\alpha-$ conformable fractional derivative of f at x.

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For $x=0$, if f is $\alpha-$differentiable on $(0,r)$ for some $r>0$, and $\lim_{x\to 0} D^{\alpha} f(0)$ exists then we define $D^{\alpha} f(0)=\lim_{x\to 0} D^{\alpha} f(0)$.

The new definition satisfies:

(1) $T_{\alpha}(af+bg)=aT_{\alpha}(f)+bT_{\alpha}(g)$, for all $a,b\in\mathbb{R}$.

(2) $T_{\alpha}(\lambda)=0$, for all constant functions $f(t)=\lambda$.

Further, for $\alpha\in(0,1]$ and f,g are $\alpha-$differentiable at a point t, with $g(t)\neq 0$. Then

(3) $T_{\alpha}(fg)=fT_{\alpha}(g)+gT_{\alpha}(f)$.

(4) $T_{\alpha}(\frac{f}{g})=\frac{gT_{\alpha}(f)-fT_{\alpha}(g)}{g^2}$, $g(t)\neq 0$.

We list here the fractional derivatives of certain functions,

(1) $T_{\alpha}(t^p)=p\,t^{p-\alpha}$.

(2) $T_{\alpha}(\sin\frac{1}{\alpha}t^{\alpha})=\cos\frac{1}{\alpha}t^{\alpha}$.

(3) $T_{\alpha}(\cos\frac{1}{\alpha}t^{\alpha})=-\sin\frac{1}{\alpha}t^{\alpha}$.

(4) $T_{\alpha}(e^{\frac{1}{\alpha}t^{\alpha}})=e^{\frac{1}{\alpha}t^{\alpha}}$.

On letting $\alpha=1$ in these derivatives, we get the corresponding classical rules for ordinary derivatives.

One should notice that a function could be $\alpha-$conformable differentiable at a point but not differentiable, for example, take $f(t)=2\sqrt{t}$. Then $T_{\frac{1}{2}}(f)(0)=1$.

This is not the case for the known classical fractional derivatives, since $T_1(f)(0)$ does not exist.

For more on fractional calculus and its applications we refer to [1]-[13].

2. Atomic Solution

Let X and Y be two Banach spaces and X^* be the dual of X. Assume $x\in X$ and $y\in Y$. The operator $T:X^*\to Y$, defined by

$$T(x^*)=x^*(x)y$$

is a bounded one rank linear operator. We write $x\otimes y$ for T. Such operators are called atoms. Atoms are among the main ingredient in the theory of tensor products.

Atoms are used in theory of best approximation in Banach spaces, [13].

One of the known result, see [16], that we need in our paper is: if the sum of two atoms is an atom, then either the first components are dependent or the second are dependent.

For more On tensor products of Banach spaces we refer to [14] and [16].

Our main object in this paper is to find an atomic solution of the equation.

$$D_t^{\beta} U + U D_x^{\alpha} U = D_x^{\alpha} D_x^{\alpha} U + U^2 \quad \text{..............(1)}$$

This is called the fractional Bate Man Burgers equation.

3. PROCEDURE

Step (i).

$$u(x,t) = P(x) \otimes Q(t) \text{ ....}(*)$$

Substitute $(*)$ in equation (1) to get:

$$P(x) \otimes Q^{\beta}(t) + P(x)P^{\alpha}(x) \otimes Q^2(t) = P^{2\alpha}(x) \otimes Q(t) + P^2(x) \otimes Q^2(t) \quad \text{.......... (2)}$$

Here $P(x)$ and $Q(t)$ are the unknowns.

We will assume that

$$P(0) = 1 \text{ and } Q(0) = 1 \text{..........}(**)$$

Step (ii). Collecting terms in (2) to get:

$$P(x) \otimes Q^{\beta}(t) + (P(x)P^{\alpha}(x) - P^2(x)) \otimes Q^2(t) = P^{2\alpha}(x) \otimes Q(t) \quad \text{............ (3)}$$

In equation (3), we have the sum of two atoms is an atom. Hence using [15], we have two cases to consider:

Case (i):

$$P(x) = P(x)P^{\alpha}(x) - P^2(x)$$

we divide both sides by $P(x)$ where $P(x) \neq 0$ we get:

$$P^{\alpha}(x) - P(x) - 1 = 0..........(4)$$

This is a fractional linear differential equations. Hence we can use the result in [16], or use the fact that $P^{\alpha}(x) = x^{(1-\alpha)}P'(x)$ to get:

$$x^{(1-\alpha)}P'(x) = P(x) + 1$$

So,

$$\frac{P'(x)}{P(x)+1} = x^{\alpha-1}$$

Thus :

$$\ln(P(x)+1) = \frac{x^{\alpha}}{\alpha} + k$$

where K is an constant .

Hence:

$$P(x) + 1 = ce^{\frac{x^{\alpha}}{\alpha}}$$

where $c = e^k$.

consequently:

$$P(x) = ce^{\frac{x^{\alpha}}{\alpha}} - 1$$

conditions $(*)$ implies that $c = 2$.

Hence, the solution of equation (4) is

$$P(x) = 2e^{\frac{x^{\alpha}}{\alpha}} - 1 \text{ ....}(4).$$

Now we go back to equation (3) to get:

$$(2e^{\frac{x^{\alpha}}{\alpha}} - 1)Q^{\beta}(t) + (2e^{\frac{x^{\alpha}}{\alpha}} - 1)(2e^{\frac{x^{\alpha}}{\alpha}})Q^2(t) = (2e^{\frac{x^{\alpha}}{\alpha}})Q(t) + (2e^{\frac{x^{\alpha}}{\alpha}} - 1)^2 Q^2(t) \quad (5)$$

$$i.e, \quad (2e^{\frac{x^{\alpha}}{\alpha}} - 1)Q^{\beta}(t) + [4e^{\frac{2x^{\alpha}}{\alpha}} - 2e^{\frac{x^{\alpha}}{\alpha}} - 4e^{\frac{2x^{\alpha}}{\alpha}} + 4e^{\frac{x^{\alpha}}{\alpha}} - 1]Q^2(t) - 2e^{\frac{x^{\alpha}}{\alpha}}Q(t) = 0$$

Then

$$(2e^{\frac{x^\alpha}{\alpha}}-1)Q^{\beta}(t)+(2e^{\frac{x^\alpha}{\alpha}}-1)Q^2(t)-2e^{\frac{x^\alpha}{\alpha}}Q(t)=0$$

Hence we obtain this equation:

$$Q^{\beta}(t)+Q^2(t)-\frac{2e^{\frac{x^\alpha}{\alpha}}}{(2e^{\frac{x^\alpha}{\alpha}}-1)}Q(t)=0 \qquad (6)$$

Now (6) is well defined for all x. In particular, it is true for $x=0$ **:**

So, equation (6) becomes:

$$Q^{\beta}(t)+Q^2(t)-2Q(t)=0$$

This is a fractional nonlinear differential equation. hence using properties of conformable fractional derivative we get:

$$t^{1-\beta}Q^{'}(t)+Q^2(t)-2Q(t)=0$$

Hence,

$$\frac{dQ}{Q^2-2Q}+t^{\beta-1}dt=0$$

From which we get:

$$\frac{1}{2}\ln\left|\frac{(Q-2)}{Q}\right|+\frac{t^{\beta}}{\beta}=c$$

Using the conditions in $(**)$, we get $c=0$. Consequently,

$$Q(t)=\frac{2}{1-e^{\frac{-2t^{\beta}}{\beta}}}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

So equations (4) and (7) give us the atomic solution $P(x)\otimes Q(t)$ of (1) for the first case.

Case (ii):

$$Q^{\beta}(t)=Q^2(t) \qquad (ii)$$

Hence,

$$t^{1-\beta}Q'(t)=Q^2(t)$$

So :

$$\frac{dQ}{Q^2(t)}=t^{\beta-1}dt$$

This implies:

$$\frac{-1}{Q(t)}=\frac{t^{\beta}}{\beta}+c$$

Hence, using conditions in $(**)$ we get $Q(0) = 1$. Thus $c = -1$. Consequently,

$$Q(t) = -\frac{-1}{\frac{t^\beta}{\beta} - 1} = \frac{\beta}{\beta - t^\beta} \text{..................}(8)$$

Now we back to equation (3) we get:

$$P(x)(\frac{\beta}{\beta - t^\beta})^2 + P(x)P^\alpha(x)(\frac{\beta}{\beta - t^\beta})^2 = P^{2\alpha}(x)(\frac{\beta}{\beta - t^\beta}) + P^2(x)(\frac{\beta}{\beta - t^\beta})^2$$

This is well defined for every t. In particular, for $t = 0$, we get:

$$P^{2\alpha}(x) - P(x)P^\alpha(x) - P(x) + P^2(x) = 0 \qquad (9)$$

This is a differential equation in which x is missing. So put $P^\alpha(x) = \theta$.

Then,

$$P^{2\alpha}(x) = D^\alpha P^\alpha(x) = D^\alpha \theta = x^{1-\alpha}\frac{d\theta}{dx} = x^{1-\alpha}\frac{d\theta}{dP}\frac{dP}{dx} = \theta\theta'$$

where ,

$$\theta' = \frac{d\theta}{dP}$$

. Thus equation (9) becomes

$$\theta\theta' - P\theta - P + P^2 = 0 \text{................}(10)$$

Clearly, $\theta = P$ is a solution for (9). But $P^\alpha(x) = \theta$. Hence $P^\alpha(x) = P$. This has a solution

$$P(x) = ce^{\frac{t^\alpha}{\alpha}}$$

Since $P(0) = 1$, we get $c = 1$. Thus

$$P(x) = e^{\frac{t^\alpha}{\alpha}} \text{................}(11)$$

From (7) and (10) we get the atomic solution for the second case.

Remark: Results in this paper are part of the Ph.D thesis of A. Bushnaque under the supervision of Professors: Al-Horani and Khalil.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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