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A NOTE ON GRAPH QUASI-CONTINUOUS AND GRAPH CLIQUISH FUNCTIONS

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Abstract: In this paper we study under which conditions there is exactly one quasi-continuous function whose graph is contained in the closure of the graph of a graph quasi-continuous function. Also, we study the relation between a graph cliquish function and a cliquish function whose graph is contained in the closure of the graph of the graph cliquish function.

Keywords: quasi-continuity; graph quasi-continuity; cliquish functions; graph cliquish functions; graph continuity.
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1. INTRODUCTION AND BASIC NOTATIONS

Z. Grande [1] in 1977 introduced the notion of graph continuity of functions. K. Sakalava [10], [11] studied the relation between graph continuous function f and a continuous function g such that $G(g) \subseteq cl(G(f))$ and showed that this relation is neither one-to-one nor onto i.e., there is a continuous function g whose graph is contained in the closure of the graph of infinitely many graph continuous functions and also there is a graph continuous function f such that the closure of the graph of f contains the graph of several continuous functions. A. Mikuka [5] in 2003

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introduced the notion of graph quasi-continuity and studied the relations between graph continuous functions and other class of functions. Some operations on graph continuous and graph quasi-continuous function was investigated by Mikuka [6]. We introduced the notion of graph cliquish functions and studied the relation between graph cliquish functions and other types of continuous functions in [4].

In this paper, we deal with the result that for a continuous function f there is one and only one quasi-continuous function g with $G(g) \subseteq cl(G(f))$ and also the result on graph cliquish functions.

In what follows X,Y are topological spaces and M is a metric space with metric d. For a subset $A \subseteq X, cl(A)$, $int(A)$ denote the closure and the interior of A respectively. If $G(f)$ denotes the graph of $f: X \to Y(f: X \to M)$ then the symbol $cl(G(f))$ denotes the closure of $G(f)$ in the product topology of $X \times Y(X \times M_d, M_d$ being the topology on M induced by $d)$. By $C(f), Q(f), AE(f)$ we denote the set of all points at which f is continuous, quasi-continuous and almost continuous (in the sense of Husain) respectively.

The letter $\mathbb{R}$ stands for the set of all real numbers, $\emptyset$ denotes the empty set and $S(x, r)$ denotes the open sphere with centre x and radius r.

Let us recall some basic definitions which will be used throughout this paper.

Definition 1.1: A subset A of X is called semi-open if there exists an open set O such that $O \subseteq A \subseteq cl(O)$[3].

Definition 1.2: A function $f: X \to Y$ is said to be:

-quasi-continuous at a point $x_0 \in X$ if for each open neighbourhood U of x_0 and each open neighbourhood V of $f(x_0)$, there exists a non-empty open set $G \subseteq U$ such that $f(G) \subseteq V$ [7].

-almost continuous (in the sense of Husain) at a point $x \in X$ if for any neighbourhood V of $f(x)$, the set $int(cl(f^{-1}(V)))$ is a neighbourhood of x [2].

f is called quasi-continuous (almost continuous) if it is such at each point.

Definition 1.3: A function $f: X \to M$ is said to be cliquish at a point $x \in X$ if for each $\epsilon > 0$ and each open neighbourhood U of x, there exists a non-empty open set $G \subseteq U$ such that $d(f(y), f(z)) < \epsilon$ whenever $y, z \in G$ [12].

f is called cliquish if it is such at every point.

Definition 1.4: A function $f: X \to Y$ is said to be graph continuous [1] (graph quasi-continuous [5], graph cliquish [4]) if there exists a continuous (quasi-continuous, cliquish) function $g: X \to Y$ such that $G(g) \subseteq cl(G(f))$.

The following implications follow from the above definitions:

Continuity	$\Longrightarrow$	quasi-continuity	$\Rightarrow$	cliquish
$\Downarrow$		$\Downarrow$		$\Downarrow$
Graph continuity	$\Rightarrow$	graph quasi-continuity	$\Rightarrow$	graph cliquish

And all of these are not invertible [4, 5].

2. RESULTS ON GRAPH QUASI-CONTINUOUS FUNCTIONS

A fundamental result given in [11] shows that for any continuous function g with $G(g) \subseteq cl(G(f))$ and for any point x of quasi-continuity of f we have $f(x) = g(x)$. The following theorem gives an essential connection between graph quasi-continuous functions and quasi-continuous functions.

Theorem 2.1: Let $f: X \to Y$ be graph quasi-continuous where Y is a Hausdroff space. Then for each quasi-continuous function $g: X \to Y$ such that $G(g) \subseteq cl(G(f))$ and each $x \in C(f)$ we have $f(x) = g(x)$.

Proof: If possible, let $f(x) \neq g(x)$ for some $x \in C(f)$.

Since Y is a Hausdorff space, there exists an open neighbourhood $V$of $f(x)$ and an open neighbourhood $W$of $g(x)$ such that $V \cap W = \emptyset$.

Since $x \in C(f)$, there exists an open neighbourhood U of x such that $f(U) \subseteq V$.

Now $x \in Q(g)$, so there exists a non-empty open set $H (\subseteq U)$ such that $g(H) \subseteq W$.

Let $x_1 \in H$. Then $(x_1, g(x_1)) \in cl(G(f))$.

So, $(H \times W) \cap G(f) \neq \emptyset$.

Choose $x_2 \in H$ such that $f(x_2) \in W$. Then $f(x_2) \in V \cap W$. This contradicts that $V \cap W = \emptyset$.

Remark 2.1: Let $f: X \to Y$ be continuous then f is graph quasi-continuous. Theorem 2.1 states that there is unique quasi-continuous function (viz f itself) whose graph is contained in the closure of the graph of f under the condition Y being a Hausdroff space.

Remark 2.2: The assumption "Y is a Hausdroff space" is essential in the Theorem 2.1. It follows from the following example.

Example 2.1: Consider $\mathbb{R}$ with the usual topology τ_u and $\mathbb{R}$ with the topology $\tau = \{A \subseteq \mathbb{R}: O \in A\} \cup \{\emptyset\}$. $(\mathbb{R}, \tau)$ is not a Hausdroff space. The functions $f, g: (\mathbb{R}, \tau_u) \to (\mathbb{R}, \tau)$ are defined as $f(x) = 0$; $\forall\, x \in \mathbb{R}$ and $g(x) = \begin{cases} 0, & x \leq 0 \\ 1; & x > 0 \end{cases}$. Now, f is continuous and g is quasi-continuous. Also, $cl(G(f)) = \mathbb{R} \times \mathbb{R}$. So, $G(g) \subseteq cl(G(f))$. But $f \neq g$.

Remark 2.3: In the Theorem 2.1, the assumption '$x \in C(f)$' cannot be replaced by '$x \in Q(f)$'. We give the following example.

Example 2.2: Consider the real line $\mathbb{R}$. Let $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 0, & x \leq 0 \\ 1; & x > 0 \end{cases}$ and $g(x) = \begin{cases} 0, & x < 0 \\ 1; & x \geq 0 \end{cases}$. g is quasi-continuous and $G(g) \subseteq cl(G(f))$. Here $O \in Q(f)$ and $O \notin C(f)$. But $f(0) \neq g(0)$.

The next theorem gives a sufficient condition for a function to be graph quasi-continuous.

Theorem 2.2: Let $f: X \to Y$ and A be a dense subset of X. Let $g: X \to Y$ be a quasi-continuous function such that $f(x) = g(x)$ for any $x \in A$. Then f is graph quasi-continuous.

Proof: It is sufficient to show that $G(g) \subseteq cl(G(f))$.

Let $x \in X$, U be an open neighbourhood of x and V be an open neighbourhood of $g(x)$.

Since $x \in Q(g)$, there is a non-empty open set $H \subseteq U$ such that $g(H) \subseteq V$.

Choose $x_1 \in H \cap A$.

Then $x_1 \in H$, $f(x_1) = g(x_1) \in V$.

So, $\left(x_1, f(x_1)\right) \in (U \times V) \cap G(f)$.

Hence $\left(x, g(x)\right) \in cl\left(G(f)\right)$.

Theorem 2.3: For a function $f: X \to Y$ (Y being a Hausdroff space), where $C(f)$ is dense in X, the followings are equivalent:

i) f is graph quasi-continuous

ii) there exists a quasi-continuous function $g: X \to Y$ and a dense subset A of X such that $f(x) = g(x)$ for any $x \in A$.

Proof: It easily follows from the Theorem 2.1 and the Theorem 2.2.

3. RESULTS ON GRAPH CLIQUISH FUNCTIONS

The following lemmas, theorems, results are known.

Lemma 3.1: Let $A \subseteq W \subseteq X$. If A is semi-open in X then A is semi-open in the subspace W [3].

Lemma 3.2: If a set A is dense and semi-open in X and a set B is dense in X then $A \cap B$ is dense in X [5].

Theorem 3.1: Let $f: X \to M$ be graph cliquish. Then for any $\epsilon > 0$ the set $A(f, g, \epsilon) = \left\{x \in X: d\left(f(x), g(x)\right) < \epsilon\right\}$ is dense in X, for any cliquish function $g: X \to M$ with $G(g) \subseteq cl\left(G(f)\right)$ [4].

Theorem 3.2: If $f: X \to M$ and $g: X \to M$ are cliquish functions such that $G(g) \subseteq cl\left(G(f)\right)$ then $A(f, g, \epsilon)$ is semi-open for any $\epsilon > 0$ [4].

Theorem 3.3: If $f: X \to M$ is cliquish then $X \setminus C(f)$ is of first category [8].

Theorem 3.4: In a Baire space the complement of every set of first category is dense [9].

Result 3.1: If $f: X \to Y$ is almost continuous at a point $x \in X$ then there exists an open neighbourhood U of x such that $f^{-1}(V)$ is dense in U for any neighbourhood V of $f(x)$.

It easily follows from the definition of almost continuity.

Theorem 3.5: Let $f: X \to M$ be quasi-continuous and $g: X \to M$ be cliquish such that $G(g) \subseteq cl\big(G(f)\big)$. Then $f(x) = g(x)$ for each $x \in AE(g)$.

Proof: If possible, let $f(x) \neq g(x)$ for some $x \in AE(g)$.

Suppose $r = d\big(f(x), g(x)\big)$. Then $r > 0$.

Since $x \in AE(g)$, there is an open neighbourhood $U$of x such that $g^{-1}\left(S(g(x), \frac{r}{4})\right)$ is dense in U by the Result 3.1.

Using the Theorem 3.1 we can say that $A\left(f, g, \frac{r}{4}\right)$ is dense in X and hence dense in the open subspace $U$of X.

Also, $A\left(f, g, \frac{r}{4}\right)$ is semi-open in U by the Theorem 3.2 and using the Lemma 3.1.

Hence by the Lemma 3.2, $A\left(f, g, \frac{r}{4}\right) \cap g^{-1}\left(S(g(x), \frac{r}{4}\right)$ is dense in U.

Now since $x \in Q(f)$, there exists a non-empty open set $H \subseteq U$ such that $f(H) \subseteq S\left(f(x), \frac{r}{2}\right)$.

Choose $x_1 \in H \cap \ A\left(f, g, \frac{r}{4}\right) \cap g^{-1}\left(S(g(x), \frac{r}{4}\right)$.

Then $x_1 \in H$, $d(f(x_1), g(x_1)) < \frac{r}{4}$, $d(g(x_1), g(x)) < \frac{r}{4}$.

Now, $d(f(x_1), g(x)) \leq \ d\big(f(x_1), g(x_1)\big) + d\big(g(x_1), g(x)\big) < \frac{r}{2}$.

So, $f(x_1) \in S\left(g(x), \frac{r}{2}\right)$

Then $f(x_1) \in S\left(g(x), \frac{r}{2}\right) \cap S\left(f(x), \frac{r}{2}\right)$.

Thus, we arrive at a contradiction as $S\left(g(x), \frac{r}{2}\right) \cap S\left(f(x), \frac{r}{2}\right) = \emptyset$.

Remark 3.1: In the theorem 3.5 the quasi-continuity of f can not be replaced by the cliquishness of f even if g is continuous.

It follows from the following example

Example 3.1: Consider $\mathbb{R}$ with the topology $\tau = \{A \subseteq \mathbb{R}: O \in A\} \cup \{\emptyset\}$ and $\mathbb{R}$ with the usual metric d.

The functions $f, g: (\mathbb{R}, \tau) \to (\mathbb{R}, d)$ are defined as $f(x) = \begin{cases} 0, & x = 0 \\ 1; & x \neq 0 \end{cases}$ and $g(x) = 0\ \forall x \in \mathbb{R}$.

f is cliquish on $(\mathbb{R}, \tau)$ and fails to be quasi-continuous at any $x \neq 0$.

g is continuous on $(\mathbb{R}, d)$.

Here $G(g) \subseteq cl\big(G(f)\big)$. But $f(x) \neq g(x)$ for any $x \neq 0$.

Theorem 3.6: Let X be a Baire space and $f: X \to M$ be graph cliquish. Then for each cliquish function $g: X \to M$ such that $G(g) \subseteq cl\big(G(f)\big)$ and each $x \in C(f) \cap Q(g)$ we have $f(x) = g(x)$.

Proof: If possible, let $f(x) \neq g(x)$ for some $x \in C(f) \cap Q(g)$.

Suppose $r = d\big(f(x), g(x)\big)$. Then $r > 0$.

Since $x \in C(f)$, there is an open neighbourhood $U$of x such that $f(U) \subseteq S(f(x), \frac{r}{2})$.

Since $x \in Q(g)$, there exists a non-empty open set $H \subseteq U$ such that $g(H) \subseteq S(g(x), \frac{r}{6})$.

By the theorems 3.3 and 3.4, $C(g)$ is dense in X and so $Q(g)$ is dense in X.

Choose $x_1 \in H \cap Q(g)$.

Then there exists a non-empty open set $H' \subseteq H$ such that $g(H') \subseteq S(g(x), \frac{r}{6})$.

By the theorem 3.1, $A\left(f, g, \frac{r}{6}\right)$ is dense in X.

Choose $x_2 \in H'$ such that $d\big(f(x_2), g(x_2)\big) < \frac{r}{6}$

Now, $d(f(x_2), g(x)) \leq\ d\big(f(x_2), g(x_2)\big) + d\big(g(x_2), g(x_1)\big) + d(g(x_1), g(x))$

$$< \frac{r}{6} + \frac{r}{6} + \frac{r}{6} = \frac{r}{2}$$

So, $f(x_2) \in S\left(g(x), \frac{r}{2}\right)$

Also, $f(x_2) \in S\left(f(x), \frac{r}{2}\right)$

Thus, we arrive at a contradiction as $S\left(g(x), \frac{r}{2}\right) \cap S\left(f(x), \frac{r}{2}\right) = \emptyset$.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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