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COMMON FIXED POINT THEOREMS FOR WEAKLY COMPATIBLE MAPPINGS USING GENERALIZED ψ -WEAK CONTRACTION

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Abstract. In this paper, first we introduce generalized ψ – weak contraction condition that involves cubic and quadratic terms of distance function $d(x, y)$. Secondly; we discuss common fixed point theorems for weakly compatible and weakly compatible mappings along with property (E.A.) and common limit range property. At the end, we provide an example and an application of our main theorem satisfying integral type contraction condition.

Keywords: ψ – weak contraction; weakly compatible mappings; property (E.A.); common limit range property.

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1. INTRODUCTION

Fixed point theory for contraction mappings and related mappings has played a fundamental role in many aspects of nonlinear functional analysis for many years. The theory has generally involved an interconnecting of geometrical and topological arguments in a Banach space setting. Fixed point theory results indicate that under certain conditions self-mapping on a

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set admits a fixed point. Among all the results Banach contraction principle is the most celebrated due to its simplicity and ease of application in major areas of mathematics.

Banach fixed point theorem is the basic tool to study fixed point theory and shows the existence and uniqueness of a fixed point under appropriate conditions. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering. Most of the problems of applied mathematics reduce to inequality which in turn their solutions give rise to the fixed points of certain mappings. It was the new era of the fixed point theory literature when the notion of commutativity mappings was used by Jungck [5] to obtain a generalization of Banach's fixed point theorem for a pair of mappings. The first ever attempt to relax the commutativity to weak commutativity was initiated by Sessa [13]. Further, in 1986 Jungck [6] introduced more generalized commutativity, so called compatibility. One can notice that the notion of weak commutativity is a point property, while the notion of compatibility is an iterate of sequence. In 1996, Jungck [8] introduced the notion of weakly compatible mappings and showed that compatible maps are weakly compatible, but not converse may not be true.

In 2002, Aamri and El-Moutawakil [1] introduced the concept of property (E.A.) for the self-mappings, which also includes the notion of the class of non-compatible mappings. Sintunavarat and Kumam [14] further generalized the notion of property (E.A.) by introducing the notion of common limit in the range property (CLR property). The significance of the CLR property and property (E.A.) have the following properties:(i)both the properties relaxes the continuity hypothesis of all the involved mappings and also relaxes the containment condition of the range subspace of the mapping into the range subspaces of the other mappings, which is generally required for constructing the sequences of joint iterates in fixed point results.(ii) the property (E.A.) replaces the completeness requirement of the space (or the range subspaces of the mappings involved) by the condition of the range subspace of the mapping to be closed, whereas (CLR) property ensures that the requirement of the completeness of the space (or range subspaces of any of the mappings involved) can be relaxed entirely and need not to be replaced by any other condition.

2. PRELIMINARIES

Let (X, d) be a metric space. If $T : X \rightarrow X$ satisfies $d(T(x), T(y)) \leq k(d(x, y))$ for all $x, y \in X, k \geq 0$. The smallest k for which the above inequality holds is the Lipschitz constant of T .

If $k \leq 1$ then T is said to be non-expansive, if $0 < k < 1$ T is said to be a contraction. Banach fixed point theorem states that every contraction mapping on a complete metric space has a unique fixed point.

Let (X, d) be a complete metric space. If $T : X \rightarrow X$ satisfies $d(T(x), T(y)) \leq k(d(x, y))$ for all $x, y \in X, 0 \leq k < 1$, then it has a unique fixed point.

In 1969, Boyd and Wong [3] replaced the constant k in Banach contraction principle by a control function ψ as follows:

Let (X, d) be a complete metric space and $\psi : [0, \infty) \rightarrow [0, \infty)$ be upper semi continuous from the right such that $0 \leq \psi(t) < t$ for all $t > 0$.

If $T : X \rightarrow X$ satisfies $d(T(x), T(y)) \leq \psi(d(x, y))$ for all $x, y \in X$, then it has a unique fixed point. Now we discuss weak contractions. The maps which are contractive without being contractions.

Let X be a metric space with a metric d . A map $T : X \rightarrow X$ is a weak contraction if $d(Tx_1, Tx_2) < d(x_1, x_2)$, for all $x_1 \neq x_2$. Being a weak contraction is not in general a sufficient condition for T in order to have a fixed point

In 1997, Alber and Gueree-Delabriere [2] introduced the concept of weak contraction as follows:

A map $T : X \rightarrow X$ is said to be weak contraction if for each $x, y \in X$, there exists a function $\emptyset : [0, \infty) \rightarrow [0, \infty)$, $\emptyset(t) > 0$ for all $t > 0$ and $\emptyset(0) = 0$ such that $d(Tx, Ty) \leq d(x, y) - \emptyset(d(x, y))$.

In connection with control function $\psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ different authors have considered some of the following properties:

- (i) ψ is non decreasing
- (ii) $\psi(t) < 0$ for all $t > 0$
- (iii) $\psi(0) = 0$
- (iv) ψ is continuous
- (v) $\lim_{n \rightarrow \infty} \psi^n(t) = 0$ for all $t \geq 0$
- (vi) $\sum_{n=0}^{\infty} \psi^n(t)$ converges for all $t > 0$, ψ^n is the n th iterate
- (vii) $\psi(t) = 0$ iff $t = 0$
- (viii) $\psi(t) > 0$ for all $t \in \mathbb{R}^+ \setminus \{0\}$
- (ix) $\lim_{r \rightarrow t^+} \psi(r) < 0$ for all $t > 0$
- (x) $\lim_{t \rightarrow \infty} \psi(t) = \infty$
- (xi) ψ is lower semi continuous

Here we note that

- (i) and (ii) implies (iii) ;
- (ii) and (iv) implies (iii)
- (i) and (v) implies (ii)

A function ψ satisfying (i) and (v) that is ψ is non decreasing and $\lim_{n \rightarrow \infty} \psi^n(t) = 0$ for all $t \geq 0$ is called as a comparison function.

Several fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit function.

In 1986 Jungck [6] introduced more generalized commutativity, so called compatibility. The notion of compatibility is an iterate of sequence.

Definition 2.1[6] Two self-mappings f and g on a metric space (X, d) are called compatible if $\lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_n fx_n = \lim_n gx_n = t, \text{ for some } t \text{ in } X.$$

In 1996, Jungck [8] introduced the notion of weakly compatible mappings and showed that compatible maps are weakly compatible, but converse may not be true.

Definition 2.2[8] Two self-mappings f and g on a metric space (X, d) are called weakly compatible if they commute at their coincidence point i.e.,

if $fu = gu$ for some $u \in X$ then $fgu = gfu$.

In 2002, Aamri and El Moutawakil [1] introduced the notion of (E.A.) property follows:

Definition 2.3[1] Let f and g be two self-mappings of a metric space (X, d) . We say that f and g satisfy (E.A) property if there exists a sequence $\{x_n\}$ in X such that

$$\lim_n f x_n = \lim_n g x_n = t, \text{ for some } t \text{ in } X.$$

Pathak et.al [11] has shown that weak compatibility and (E.A.) property are independent to each other.

In 2011, Sintunavarat and Kumam [14] coined the idea of common limit range property (called CLR) which relaxes the requirement of completeness to compute fixed point.

Definition 2.4[14] Two self mappings f and g on a metric space (X, d) are said to satisfy the common limit in the range of g property denoted as CLR $_g$ property if

$$\lim_n f x_n = \lim_n g x_n = gt, \text{ for some } t \text{ in } X.$$

Now we introduce the generalized ψ –weak contraction for a pair of mappings in the following way:

Let A, B, S and T are self mappings on a metric space (X, d) satisfying the following conditions:

$$(C1) \quad S(X) \subset B(X), T(X) \subset A(X);$$

(C2)

$$d^3(Sx, Ty) \leq \psi \left\{ \begin{array}{l} d^2(Ax, Sx)d(By, Ty), d(Ax, Sx)d^2(By, Ty), \\ d(Ax, Sx)d(Ax, Ty)d(By, Sx), \\ d(Ax, Ty)d(By, Sx)d(By, Ty) \end{array} \right\}$$

for all $x, y \in X$, where $\psi: [0, \infty) \rightarrow [0, \infty)$ is a continuous and non-decreasing function with $\psi(t) < t$ for each $t > 0$.

3. MAIN RESULTS

In this section, first we prove a result for weakly compatible mappings that satisfy generalized ψ – weak contraction involving cubic and quadratic terms of distance function.

Theorem 3.1 Let A, B, S and T are four mappings of a complete metric space (X, d) into itself satisfying (C1), (C2) and the following conditions:

(C3) one of subspace AX or BX or SX or TX is complete subspace of X ; then

- (i) A and S have a point of coincidence,
- (ii) B and T have a point of coincidence.

Moreover assume that the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof: Let $x_0 \in X$ be an arbitrary point. From (C1) we can find x_1 such that $S(x_0) = B(x_1) = y_0$, for this x_1 one can find $x_2 \in X$ such that $T(x_1) = A(x_2) = y_1$. Continuing in this way one can construct a sequence $x_n \in X$ such that

$$y_{2n} = S(x_{2n}) = B(x_{2n+1}), y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2}) \text{ for each } n \geq 0. \quad (3.1)$$

For brevity, one can denote $\alpha_{2n} = d(y_{2n}, y_{2n+1})$.

First we prove that $\{\alpha_{2n}\}$ is non increasing sequence and converges to zero.

Case I. If n is even, taking $x = x_{2n}$ and $y = x_{2n+1}$ in (C2), we get

$$d^3(Sx_{2n}, Tx_{2n+1}) \leq \psi \left\{ \begin{array}{l} d^2(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \\ d(Ax_{2n}, Sx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1})' \\ d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n}), \\ d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \end{array} \right\}$$

$$d^3(y_{2n}, y_{2n+1}) \leq \psi \left\{ \begin{array}{l} d^2(y_{2n-1}, y_{2n})d(y_{2n}, y_{2n+1}) \\ d(y_{2n-1}, y_{2n})d^2(y_{2n}, y_{2n+1})' \\ d(y_{2n-1}, y_{2n})d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n}), \\ d(y_{2n-1}, y_{2n+1})d(y_{2n}, y_{2n})d(y_{2n}, y_{2n+1}) \end{array} \right\} \quad (3.2)$$

On using $\alpha_{2n} = d(y_{2n}, y_{2n+1})$ in (3.2), we have

$$\alpha_{2n}^3 \leq \psi \{ \alpha_{2n-1}^2 \alpha_{2n}, \alpha_{2n-1} \alpha_{2n}^2, 0, 0 \} \quad (3.3)$$

If $\alpha_{2n-1} < \alpha_{2n}$ and using property of ψ , then (3.3) reduces to

$\alpha_{2n}^3 < \alpha_{2n}^3$, a contradiction, therefore, $\alpha_{2n} \leq \alpha_{2n-1}$.

Case II. If n is odd, then in a similar way, one can obtain $\alpha_{2n+1} \leq \alpha_{2n}$.

It follows that the sequence $\{\alpha_{2n}\}$ is decreasing.

Let $\lim_{n \rightarrow \infty} \alpha_{2n} = r$, for some $r \geq 0$.

Suppose $r > 0$, then from inequality (C2), we have

$$d^3(Sx_{2n}, Tx_{2n+1}) \leq \psi \left\{ \begin{array}{l} \{d^2(Ax_{2n}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}), \\ d(Ax_{2n}, Sx_{2n})d^2(Bx_{2n+1}, Tx_{2n+1}) \\ d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n}), \\ d(Ax_{2n}, Tx_{2n+1})d(Bx_{2n+1}, Sx_{2n})d(Bx_{2n+1}, Tx_{2n+1}) \} \end{array} \right\}$$

Now by using (3.3), triangular inequality, property of ψ and proceed limits $n \rightarrow \infty$, we get

$r^3 \leq \psi(r^3) < r^3$, a contradiction, therefore we get $r = 0$. Therefore

$$\lim_{n \rightarrow \infty} \alpha_{2n} = \lim_{n \rightarrow \infty} d(y_{2n}, y_{2n+1}) = r = 0. \quad (3.4)$$

Now we show that $\{y_n\}$ is a Cauchy sequence. Suppose we assume that $\{y_n\}$ is not a Cauchy sequence. For given $\epsilon > 0$ we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers k , $n(k) > m(k) > k$.

$$d(y_{m(k)}, y_{n(k)}) \geq \epsilon, d(y_{m(k)}, y_{n(k)-1}) < \epsilon \quad (3.5)$$

Now $\epsilon \leq d(y_{m(k)}, y_{n(k)}) \leq d(y_{m(k)}, y_{n(k)-1}) + d(y_{n(k)-1}, y_{n(k)})$.

Letting $k \rightarrow \infty$, we get $\lim_{k \rightarrow \infty} d(y_{m(k)}, y_{n(k)}) = \epsilon$

Now from the triangular inequality we have,

$$|d(y_{n(k)}, y_{m(k)+1}) - d(y_{m(k)}, y_{n(k)})| \leq d(y_{m(k)}, y_{m(k)+1}).$$

Taking limits as $k \rightarrow \infty$ and using (3.4) and (3.5), we have

$$\lim_{k \rightarrow \infty} d(y_{n(k)}, y_{m(k)+1}) = \epsilon.$$

Again from the triangular inequality, we have

$$|d(y_{m(k)}, y_{n(k)+1}) - d(y_{m(k)}, y_{n(k)})| \leq d(y_{n(k)}, y_{n(k)+1}).$$

Taking limits as $k \rightarrow \infty$ and using (3.4) and (3.5), we have

$$\lim_{k \rightarrow \infty} d(y_{m(k)}, y_{n(k)+1}) = \epsilon.$$

Similarly on using triangular inequality, we have

$$|d(y_{m(k)+1}, y_{n(k)+1}) - d(y_{m(k)}, y_{n(k)})| \leq d(y_{m(k)}, y_{m(k)+1}) + d(y_{n(k)}, y_{n(k)+1}).$$

Taking limit as $k \rightarrow \infty$ in the above inequality and using (3.4) and (3.5), we have

$$\lim_{k \rightarrow \infty} d(y_{n(k)+1}, y_{m(k)+1}) = \epsilon.$$

On putting $x = x_{m(k)}$ and $y = x_{n(k)}$ in (C2), we get

$$d^3(Sx_{m(k)}, Tx_{n(k)}) \leq \psi \left\{ \begin{array}{l} d^2(Ax_{m(k)}, Sx_{m(k)})d(Bx_{n(k)}, Tx_{n(k)}), \\ d(Ax_{m(k)}, Sx_{m(k)})d^2(Bx_{n(k)}, Tx_{n(k)}) \\ d(Ax_{m(k)}, Sx_{m(k)})d(Ax_{m(k)}, Tx_{n(k)})d(Bx_{n(k)}, Sx_{m(k)}), \\ d(Ax_{m(k)}, Tx_{n(k)})d(Bx_{n(k)}, Sx_{m(k)})d(Bx_{n(k)}, Tx_{n(k)}) \end{array} \right\}$$

Using (3.1), we obtain

$$d^3(y_{m(k)}, y_{n(k)}) \leq \psi \left\{ \begin{array}{l} d^2(y_{m(k)-1}, y_{m(k)})d(y_{n(k)-1}, y_{n(k)}), \\ d(y_{m(k)-1}, y_{m(k)})d^2(y_{n(k)-1}, y_{n(k)}) \\ d(y_{m(k)-1}, y_{m(k)})d(y_{m(k)-1}, y_{n(k)})d(y_{n(k)-1}, y_{m(k)}), \\ d(y_{m(k)-1}, y_{n(k)})d(y_{n(k)-1}, y_{m(k)})d(y_{n(k)-1}, y_{n(k)}) \end{array} \right\}$$

Letting $k \rightarrow \infty$, and using property of ψ , we have

$$\epsilon^3 \leq 0,$$

which is a contradiction.

Thus $\{y_n\}$ is a Cauchy sequence in X .

Now suppose that AX is complete subspace of X , then there exist $z \in X$ such that

$$y_{2n+1} = T(x_{2n+1}) = A(x_{2n+2}) \rightarrow z \text{ as } n \rightarrow \infty.$$

Consequently, we can find $w \in X$ such that $Aw = z$. Further a Cauchy sequence $\{y_n\}$ has a convergent subsequence $\{y_{2n+1}\}$, therefore the sequence $\{y_n\}$ converges and hence a subsequence $\{y_{2n}\}$ also converges. Thus we have $y_{2n} = S(x_{2n}) = B(x_{2n+1}) \rightarrow z$ as $n \rightarrow \infty$.

On setting $x = w$ and $y = x_{2n+1}$ in (C2) and proceeding limit, we have

$$d^3(Sw, z) \leq \psi \left\{ \begin{array}{l} d^2(Aw, Sw)d(Bz, Tz), d(Aw, Sw)d^2(Bz, Tz), \\ d(Aw, Sw)d(Aw, Tz)d(Bz, Sw), \\ d(Aw, Tz)d(Bz, Sw)d(Bz, Tz) \end{array} \right\}$$

$$\text{Therefore, } d^3(Sw, z) \leq \psi \left\{ \begin{array}{l} d^2(z, Sw)d(z, z), d(z, Sw)d^2(z, z), \\ d(z, Sw)d(z, z)d(z, Sw), \\ d(z, z)d(z, Sw)d(z, z) \end{array} \right\}$$

This implies that $Sw = z$ and hence $Sw = Aw = z$. Therefore, w is a coincidence point of A and S . Since $z = Sw \in SX \subset BX$ there exist $v \in X$ such that $z = Bv$.

Next we claim that $Tv = z$. Now putting $x = x_{2n}$ and $y = v$ in (C2), we have

$$d^3(Sx_{2n}, Tv) \leq \psi \left\{ \begin{array}{l} d^2(Ax_{2n}, Sx_{2n})d(Bv, Tv), d(Ax_{2n}, Sx_{2n})d^2(Bv, Tv), \\ d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tv)d(Bv, Sx_{2n}), \\ d(Ax_{2n}, Tv)d(Bv, Sx_{2n})d(Bv, Tv) \end{array} \right\}$$

Therefore,

$d^3(z, Tv) \leq \psi\{0,0,0,0\}$, this gives $z = Tv$ and hence $z = Tv = Bv$. Therefore, v is a coincidence point of B and T . Since the pairs A, S and B, T are weakly compatible, we have

$$Sz = S(Aw) = A(Sw) = Az, \quad Tz = T(Bv) = B(Tv) = Bz.$$

Next we show that $Sz = z$. For this put $x = z$ and $y = x_{2n+1}$ in (C2)

For this put $x = z$ and $y = x_{2n+1}$ in (C2) and proceeding limit as $n \rightarrow \infty$,

$$d^3(Sz, Tx_{2n+1}) \leq \psi \left\{ \begin{array}{l} d^2(Az, Sz)d(z, z), d(Az, Sz)d^2(z, z), \\ d(Az, Sz)d(Az, z)d(z, Sz), \\ d(Az, z)d(z, Sz)d(z, z) \end{array} \right\}$$

Therefore, we get

$$d^3(Sz, z) \leq \psi \{0,0,0,0\} \text{ Using property of } \psi, \text{ we have}$$

Thus we get $d^2(Sz, z) = 0$. This implies that $Sz = z$ and hence $Sz = Az = z$.

Next we claim that $Tz = z$. Now put $x = x_{2n}$ and $y = z$ in (C2)

$$d^3(Sx_{2n}, Tz) \leq \psi \left\{ \begin{array}{l} d^2(Ax_{2n}, Sx_{2n})d(Bz, Tz), d(Ax_{2n}, Sx_{2n})d^2(Bz, Tz) \\ d(Ax_{2n}, Sx_{2n})d(Ax_{2n}, Tz)d(Bz, Sx_{2n}), \\ d(Ax_{2n}, Tz)d(Bz, Sx_{2n})d(Bz, Tz) \end{array} \right\}$$

Hence we get

$$d^3(z, Tz) \leq \psi \{0,0,0,0\}$$

This gives $z = Tz$ and hence $z = Tz = Bz$. Therefore z is a common fixed point of A, B, S and T .

Similarly we can complete the proofs for the cases in which BX or SX or TX is complete.

Uniqueness: Suppose $z \neq w$ be two common fixed point of S, T, A and B .

Put $x = z$ and $y = w$ in (C2), we have

$$d^3(Sz, Tw) \leq \psi \left\{ \begin{array}{l} d^2(Az, Sz)d(Bw, Tw), d(Az, Sz)d^2(Bw, Tw), \\ d(Az, Sz)d(Az, Tw)d(Bw, Sz), \\ d(Ax, Ty)d(By, Sx)d(By, Ty) \end{array} \right\}$$

$$i. e., \quad d^3(Sz, Tw) \leq \psi \{0,0,0,0\}$$

i. e., $d^2(z, w) = 0$, this implies $z = w$.

This completes the proof.

If we put $S = T$ in theorem 3.1, then we obtain the following results

Corollary 3.1 Let S, A and B are three self-mappings of a complete metric space (X, d) satisfying (C3) and the following conditions:

$$(C4) \quad S(X) \subset B(X), S(X) \subset A(X);$$

$$(C5) \quad d^3(Sx, Sy) \leq \psi \left\{ \begin{array}{l} d^2(Ax, Sx)d(By, Sy), d(Ax, Sx)d^2(By, Sy), \\ d(Ax, Sx)d(Ax, Sy)d(By, Sx), \\ d(Ax, Sy)d(By, Sx)d(By, Sy) \end{array} \right\}$$

$\psi: [0, \infty) \rightarrow [0, \infty)$ is a continuous and non-decreasing function with $\psi(t) < t$ for each $t > 0$ and.

Assume that the pairs (A, S) and (B, S) are weakly compatible, then S, A and B have a unique common fixed point.

In Theorem 3.1, if we put $A = B = I$, we obtain the following result.

Corollary 3.2 Let S and T be mappings of a complete metric space (X, d) into itself satisfying the following conditions:

$$d^3(Sx, Ty) \leq \psi \left\{ \begin{array}{l} d^2(x, Sx)d(y, Ty), d(x, Sx)d^2(y, Ty), \\ d(x, Sx)d(x, Ty)d(y, Sx), \\ d(x, Ty)d(y, Sx)d(y, Ty) \end{array} \right\}, \text{ for all } x, y \in X \text{ and}$$

$\psi: [0, \infty) \rightarrow [0, \infty)$ is a continuous and non decreasing function with $\psi(t) < t$ for each $t > 0$. If one of subspace SX or TX is complete then S and T have a unique common fixed point.

Now we prove the above theorem for weakly compatible mappings in a metric space by dropping the condition of completeness of subspaces as follows:

Theorem 3.2 Let S, T, A and B are four mappings of a complete metric space (X, d) into itself satisfying (C1), (C2) and the following condition:

(C6) one of subspace AX or BX or SX or TX is closed subset of X .

Assume that the pairs (A, S) and (B, T) are weakly compatible. Then S, T, A and B have a unique common fixed point.

Proof. As we know that the subspace of a complete metric space is complete if and only if it is closed. By Theorem 3.1, this conclusion holds.

Now we prove a result for weakly compatible mappings along with property (E.A.)

Theorem 3.3 Let S, T, A and B are four mappings of a complete metric space (X, d) into itself satisfying (C1), (C2), (C6) and assume the following:

(C7) The pairs (A, S) and (B, T) are weakly compatible;

(C8) The pairs (A, S) and (B, T) are satisfies property (E.A.).

Then S, T, A and B have a unique common fixed point.

Proof. Suppose that the pair (A, S) satisfies E.A. property then there exists a sequence $\{x_n\}$ in X such that $\lim_n Ax_n = \lim_n Sx_n = z$ for some z in X . Since $S(X) \subset B(X)$, there exists a sequence $\{y_n\}$ in X such that $By_n = Sx_n$. Hence $\lim_n By_n = z$. Also $T(X) \subset A(X)$ so there exists a sequence $\{w_n\}$ in X such that $Tw_n = Ax_n$. Hence $\lim_n Tw_n = z$.

Now suppose that BX is closed subset of X , then there exists $u \in X$ such that $z = Bu$.

Subsequently, we have

$$\lim_n Ax_n = \lim_n Sx_n = \lim_n Tw_n = \lim_n By_n = z = Bu, \text{ for some } u \in X.$$

First we claim that $Tu = z$.

Now putting $x = x_n$ and $y = u$ in (C2)

$$d^3(Sx_n, Tu) \leq \psi \left\{ \begin{array}{l} d^2(Ax_n, Sx_n)d(Bu, Tu), \\ d(Ax_n, Sx_n)d^2(Bu, Tu), \\ d(Ax_n, Sx_n)d(Ax_n, Tu)d(Bu, Sx_n), \\ d(Ax_n, Tu)d(Bu, Sx_n)d(Bu, Tu) \end{array} \right\}$$

Therefore, we get

$$d^3(z, Tu) \leq \psi \{0,0,0,0\} \quad .$$

Using property of ψ , we have, $z = Tu$ and hence $z = Tu = Bu$. Since $T(X) \subset A(X)$ therefore there exists $v \in X$ such that $Tu = z = Av$.

Next we claim that $Sv = z$. On setting $x = v$ and $y = u$ in (C2), we get

$$d^3(Sv, Tu) \leq \psi \left\{ \begin{array}{l} d^2(Av, Sv)d(Bu, Tu), \\ d(Av, Sv)d^2(Bu, Tu) \\ d(Av, Sv)d(Av, Tu)d(Bu, Sv), \\ d(Av, Tu)d(Bu, Sv)d(Bu, Tu) \end{array} \right\}$$

Therefore, we get

$$d^3(Sv, z) \leq \psi \left\{ \begin{array}{l} d^2(z, Sv)d(z, z), d(z, Sv)d^2(z, z), \\ d(z, Sv)d(z, z)d(z, Sv), \\ d(z, z)d(z, Sv)d(z, z) \end{array} \right\}$$

Using property of ψ , we have $Sv = z$ and hence $Sv = Av = z$ so $Av = Sv = Tu = Bu = z$. Since the pairs (A, S) and (B, T) are weakly compatible and v and u are their coincidence point respectively, so we have

$$Az = A(Sv) = S(Av) = Sz, \quad Bz = B(Tu) = T(Bu) = Tz.$$

Now we prove that z is a common fixed point of A, B, S and T . For this we prove that $Sv = Tz$. On setting $x = v$ and $y = z$ in (C2), we get

$$d^3(Sv, Tz) \leq \psi \left\{ \begin{array}{l} d^2(Av, Sv)d(Bz, Tz), d(Av, Sv)d^2(Bz, Tz) \\ d(Av, Sv)d(Av, Tz)d(Bz, Sv), \\ d(Av, Tz)d(Bz, Sv)d(Bz, Tz) \end{array} \right\}$$

Therefore, we get

$$d^3(Sv, Tz) \leq \psi \left\{ \begin{array}{l} d^2(z, z)d(Bz, Tz), d(z, z)d^2(Bz, Tz), \\ d(z, z)d(Sv, Tz)d(Tz, Sv), \\ d(Sv, Tz)d(Tz, Sv)d(Bz, Tz) \end{array} \right\}$$

Using property of ψ , we have $Sv = Tz$ and hence $z = Sv = Tz$ and $z = Tz = Bz$, So z is a common fixed point of B and T . Also we can prove that $Sv = z$ is also a common fixed point of A and S . Similarly we can complete the proof for cases in which AX or SX or TX is closed subset of X . The uniqueness follows easily. This completes the proof.

We shall continue our discussion to find fixed point for the mapping satisfying weakly compatible mappings along with common limit range property.

Theorem 3.4 Let S, T, A and B are four mappings of a complete metric space (X, d) into itself satisfying (C1), (C2), (C6), (C7) and the following:

(C9) The pairs (A, S) satisfies CLR_A property or the pair (B, T) satisfies CLR_B property.

Then S, T, A and B have a unique common fixed point.

Proof. If the pair (B, T) satisfies CLR_B property so there exists a sequence $\{x_n\}$ in X such that $\lim_n Bx_n = \lim_n Tx_n = z \in BX$. Since $T(X) \subset A(X)$ so for each $\{x_n\}$ in X there corresponds a sequence $\{y_n\}$ in X such that $Tx_n = Ay_n$. Therefore, $\lim_n Ay_n = \lim_n Tx_n = z \in BX$. Thus we have $\lim_n Ay_n = \lim_n Bx_n = \lim_n Tx_n = z$.

Now suppose that BX is a closed subset of X , there exists a point $u \in X$ such that $Bu = z$.

Now we show that $\lim_n Sy_n = z$. Putting $x = y_n$ and $y = x_n$ in (C2), we have

$$d^3(Sy_n, Tx_n) \leq \psi \left\{ \begin{array}{l} d^2(Ay_n, Sy_n)d(Bx_n, Tx_n), \\ d(Ay_n, Sy_n)d^2(Bx_n, Tx_n), \\ d(Ay_n, Sy_n)d(Ay_n, Tx_n)d(Bx_n, Sy_n), \\ d(Ay_n, Tx_n)d(Bx_n, Sy_n)d(Bx_n, Tx_n) \end{array} \right\},$$

$$d^3(Sy_n, z) \leq \psi \left\{ \begin{array}{l} [d^2(z, Sy_n)d(z, z), d(z, Sy_n)d^2(z, z)], \\ d(z, Sy_n)d(z, z)d(z, Sy_n), \\ d(z, z)d(z, Sy_n)d(z, z) \end{array} \right\}$$

Using property of ψ , we have $\lim_n d(Sy_n, z) = 0$. Hence $\lim_n Ay_n = \lim_n Bx_n = \lim_n Tx_n = \lim_n Sy_n = z = Bu$, for some u in X . From the proof of Theorem 3.3, we can easily prove that z is a common fixed point of A, B, S and T . Also one can easily prove the Theorem 3.4 if the pair (A, S) satisfies CLR_A property. Similarly we can complete the proof for cases in which AX or TX or SX is a closed subset of X . This completes the proof.

Example 3.1 Let $X = [2,20]$ with the metric d defined by $d(x, y) = |x - y|$. Define the self mappings A, B, S and T on X by

$$Ax = \begin{cases} 12 & \text{if } 2 < x \leq 5 \\ x - 3 & \text{if } x > 5 \\ 2 & \text{if } x = 2. \end{cases}, \quad Bx = \begin{cases} 2 & \text{if } x = 2 \\ 6 & \text{if } x > 2 \end{cases}$$

$$Sx = \begin{cases} 6 & \text{if } 2 < x \leq 5 \\ x & \text{if } x = 2 \\ 2 & \text{if } x > 5. \end{cases} \quad \text{and} \quad Tx = \begin{cases} x & \text{if } x = 2 \\ 3 & \text{if } x > 2. \end{cases}$$

Define $\psi: [0, \infty) \rightarrow [0, \infty)$ is a continuous and non-decreasing function with $\psi(t) < t$ for each $t > 0$. Taking $\langle x_n \rangle = \langle 5 + \frac{1}{n} \rangle$, it is clear that the pair (S, A) and (B, T) are weakly compatible mappings. Therefore, all the condition of Theorem 3.1 are satisfied, then we can obtain $S2 = T2 = A2 = B2 = 2$, so 2 is a common fixed point of S, T, A and B . In fact, 2 is the unique common fixed point of S, T, A and B .

4. APPLICATION

In 2002 Branciari [1] obtained a fixed point theorem for a single mapping satisfying an analogue of a Banach contraction principle for integral type inequality.

Theorem 4.1. Let (X, d) be a complete metric space. $f : X \rightarrow X$ a mapping such that, for each $x, y \in X$,

$$\int_0^{d(x,y)} \varphi(t) dt \leq c \int_0^{d(x,y)} \varphi(t) dt$$

$c \in [0, 1)$, where $\varphi : R^+ \rightarrow R^+$ is a ‘‘Lebesgue-integrable function’’ which is summable, nonnegative, and such that, for each $\epsilon > 0$, $\int_0^\epsilon \varphi(t) dt > 0$. Then f has a unique fixed point $z \in X$ such that, for each $x \in X$, $\lim_{n \rightarrow \infty} f^n = z$.

Now we prove the following theorem as an application of theorem 3.1.

Theorem 4.2 Let A, B, S and T be self mappings on a metric space (X, d) satisfying the following conditions:

(C1), (C3) and the following:

$$\int_0^{d^3(Sx, Ty)} \varphi(t) dt \leq \int_0^{M(x, y)} \varphi(t) dt$$

$$M(x, y) = \psi \left\{ \begin{array}{l} d^2(Ax, Sx)d(By, Ty), d(Ax, Sx)d^2(By, Ty), \\ d(Ax, Sx)d(Ax, Ty)d(By, Sx), \\ d(Ax, Ty)d(By, Sx)d(By, Ty) \end{array} \right\}$$

for all $x, y \in X$, where $\psi : [0, \infty) \rightarrow [0, \infty)$ is a continuous and non-decreasing function with $\psi(t) < t$ for each $t > 0$. Further, where $\varphi : R^+ \rightarrow R^+$ is a “Lebesgue-integrable over R^+ function” which is summable on each compact subset of R^+ , non-negative, and such that for each $\epsilon > 0$,

$$\int_0^\epsilon \varphi(t) dt > 0.$$

Moreover, assume that the pairs (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point.

Proof. The proof of the theorem follows on the same lines of the proof of the theorem 3.1. On setting $\varphi(t) = 1$, we get theorem 3.1.

Remark 4.1. Every contractive condition of integral type automatically includes a corresponding contractive condition not involving integrals, by setting $\varphi(t) = 1$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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