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NUMBER OF IRREDUCIBLE FACTORS AND DEGREE IN DIVISOR GRAPH OF $Z_p[x,n]$

PRADEEP MAAN[1,*], AMIT SEHGAL[1], ARCHANA MALIK[2]

[1]Pt. NRS Govt. College, Rohtak, India

[2]Maharshi Dayanand University, Rohtak, India

Abstract. The divisor graph, denoted by $D(Z_p[x,n])$, is the graph whose vertex set is the set of all polynomials of degree at most n whose coefficients are from field Z_p and its any two distinct vertices are adjacent if one is a divisor of the other. In this paper, (i) we determine the degree of each vertex of $D(Z_p[x,3])$ and also discuss its girth, size, degree sequence, irregularity index etc. (ii) We also establish that two polynomials of same degree k in $Z_p[x,n]$ having different number of irreducible factors, the one with fewer number of irreducible factors has smaller degree. (iii) Further, if two polynomials of same degree k in $Z_p[x,n]$ having same number of irreducible factors but different number of distinct irreducible factors, the one with fewer number of distinct irreducible factors has smaller degree.

Keywords: irreducible factors; divisor graph; degree sequence.

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1. Introduction

The study of algebraic structures through graphs and investigation of these graphs has been a growing area of research. Firstly, the idea of divisor graph was given by Singh and Santhosh [2] and further investigated by Chartrand, Muntean, Saenpholphant and Zhang [1]. This concept

*Corresponding author

E-mail address: pradeepmaan89@gmail.com

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also studied several divisor graphs and their properties. A divisor graph $G(V)$ is an ordered pair (V,E) where vertex set V is subset of set of integers and for all $u \neq u \in V, uv \in E$ if and only if $u|v$ or $v|u$. The degree of a vertex associated with a finite group is also used to study the structural properties of graph[6, 12], motivated by this concept we obtain results for degree of vertices in divisor graph.

In section 3, we obtain results for the degree of vertices of $D(Z_p[x,3])$ of the type ax^3+bx^2+cx+d where $a,b,c,d \in Z_p$ which are generalization of results in [7]. We also characterize the minimum degree vertex as the irreducible polynomial and show that for any prime p, the graph $D(Z_p[x,3])$ is neither complete nor cyclic. We also obtain the degree sequence and irregularity index for $D(Z_p[x,3])$.

In section 4, we prove that two polynomial of same degree k in $D(Z_p[x,n])$, but number of irreducible factors of polynomials are different, then degree of polynomial in graph $D(Z_p[x,n])$ has degree least degree which has less number of irreducible factors. Also, we prove that two polynomial of same degree k in $D(Z_p[x,n])$ and same number of number of irreducible factors of polynomials but number of distinct number of irreducible factors of polynomials are different, then degree of polynomial in graph $D(Z_p[x,n])$ has degree least degree which has less number of distinct number of irreducible factors of polynomials.

2. Number of Irreducible Polynomials

Let $\pi(r,k), \pi^*(r,k)$ denotes number of monic polynomials and non-monic polynomials of degree r in $Z_p[x]$ having k associative irreducible factors respectively. By [9], we know that $\pi(r,1) = \frac{1}{r}\sum_{d|r}\mu(d)p^{\frac{r}{d}}$ where μ denotes the Moebius function. Further, the irreducible polynomials of degree r are nothing but associates of monic irreducible polynomials of degree r. So $\pi^*(r,1) = \pi(r,1)$.Assumption $\pi(r,1) = 0$ if r is not a positive integer.

Theorem 1. $\pi^*(r,k) = \begin{cases} \pi^*(r/k,1) & if\ k\ divides\ r \\ 0 & otherwise. \end{cases}$

Proof. Depending upon whether k divides r, we consider following cases:

1. If k divides r, then every polynomial of degree r having k associate irreducible factors can be written as product of irreducible polynomial of degree r/k with their $k-1$ monic

associates. Number of irreducible polynomials of degree r/k are $\pi^*(r/k,1)$ and has unique monic associate. Hence $\pi^*(r,k) = \pi^*(r/k,1)$.

2. If k does not divide r, then no polynomial of degree r having k associate irreducible factors. Hence $\pi^*(r,k) = 0$.

This proves the theorem. □

Theorem 2. *The number $\pi^*(r,(k-1)(1)+1(1))$ is given by*

$$\begin{cases} \frac{\sum_{i=1}^{r-1}(p-1)\pi(i,1)\pi(r-i,1)-(p-1)\pi(\frac{r}{2},1)}{2} & \text{if } k=2 \\ \sum_{i=1}^{\lfloor\frac{r-1}{k-1}\rfloor}(p-1)\pi(i,1)\pi(r-(k-1)i,1)-(p-1)\pi(\frac{r}{k},1) & \text{if } k>2. \end{cases}$$

Proof. Let $f(x) \in Z_p[x]$ be a polynomial of degree r which can be factorised into k irreducible polynomial such that at least $k-1$ factors are associate. So, we can write $f(x)$ as $a_r g(x)^{k-1}h(x)$ where $g(x), h(x)$ are irreducible monic polynomial of at-least one degree and a_r is non-zero coefficient of highest power in $f(x)$.

It is given that $deg(f(x)) = r$, so $h(x)$ is $r-(k-1)deg(g(x))$. Here $f(x) = a_r g(x)^{k-1}h(x)$, so possibilities of degree of $g(x)$ is from 1 to $\lfloor\frac{r-1}{k-1}\rfloor$ because degree $h(x)$ is at-least one.

Possibilities of irreducible monic polynomial of degree i are $\pi(i,1)$ and possibilities of monic irreducible polynomial of degree $r-(k-1)i$ are $\pi(r-(k-1)i,1)$ where $\lfloor\frac{r-1}{k-1}\rfloor$.

Here, we want to count such cases where $g(x)$ and $h(x)$ must be non-associate. If $g(x)$ and $h(x)$ are associate, then degree of $f(x)$ is $k \times deg(h(x))$, so $k|r$ and degree of $g(x)$ is $\frac{r}{k}$.

Case 1: For $k > 2$, $g(x)^{k-1}$ and $h(x)$ are different groups, so possibilities of $f(x)$ when $g(x)$ and $h(x)$ may or may not be associate are $\sum_{i=1}^{\lfloor\frac{r-1}{k-1}\rfloor}(p-1)\pi(i,1)\pi(r-(k-1)i,1)$. Possibilities of $f(x)$ when $g(x)$ and $h(x)$ are associate of each other are $(p-1)\pi(\frac{r}{k},1)$. Hence, we get

$$\pi^*(r,(k-1)(1)+1(1)) = \sum_{i=1}^{\lfloor\frac{r-1}{k-1}\rfloor}(p-1)\pi(i,1)\pi(r-(k-1)i,1)-(p-1)\pi(\frac{r}{k},1).$$

Case 2: For $k=2$, $g(x)$ and $h(x)$ are same groups and no distinction is possible between the groups, then the two groups can be interchanged without giving a new group, so possibilities of $f(x)$ when $g(x)$ and $h(x)$ may or may not be associate are $\frac{\sum_{i=1}^{r-1}(p-1)\pi(i,1)\pi(r-i,1)}{2}$. Possibilities of

$f(x)$ when $g(x)$ and $h(x)$ are associate of each other are $\frac{(p-1)\pi(\frac{r}{2},1)}{2}$. Hence, we get

$$\pi^*(r,(1)(1)+(1)(1)) = \frac{\sum_{i=1}^{r-1}(p-1)\pi(i,1)\pi(r-i,1)-(p-1)\pi(\frac{r}{2},1)}{2}.$$

□

Theorem 3. *The number $\pi^*(r,1+1+1)$ is given by*

$$\frac{\sum_{i=1}^{r-2}\sum_{j=1}^{r-1-i}(p-1)\pi(i,1)\pi(j,1)\pi(r-i-j,1)-3\pi^*(r,2+1)-\pi^*(\frac{r}{3},1)}{6}.$$

Proof. Let $f(x) \in Z_p[x]$ of degree r be such as which can be factorized into three irreducible factors. So, we can write $f(x)$ as $a_r g(x)h(x)s(x)$ where $g(x), h(x), s(x)$ are irreducible monic polynomials of degree at-least one and a_r is non zero coefficient of highest power in f(x). It is given that $deg(f(x)) = r$, so, if we assume that $degg(x) = i$ and $degh(x) = j$, then $degs(x) = r-i-j$. Here $f(x) = a_r g(x)h(x)s(x)$.So, possibilities of degree of g(x) is from 1 to $r-2$ because $h(x)$ and $s(x)$ will have at-least one degree and accordingly possibilities of degree of $h(x)$ is from 1 to $r-1-deg(g(x))$. Possibilities of irreducible monic polynomial of degree i are $\pi(i,1)$.

Here we want to count such cases where $g(x), h(x)$ and $s(x)$ must be mutually non-associate,so we have exclude the cases when two are associate and when all the three are associate.Here possibilities of monic irreducible polynomial $f(x)$ are $\pi(i,1)$.

Possibilities of $a_r g(x)h(x)s(x)$ are $\sum_{i=1}^{r-2}\sum_{j=1}^{r-1-i}(p-1)\pi(i,1)\pi(j,1)\pi(r-i-j,1)$, possibilities of $a_r g(x)h(x)s(x)$ when two out of three $g(x),h(x),s(x)$ being associate are $3C_1(p-1)\mu(r,2+1)$. Possibilities when $a_r g(x)h(x)s(x)$ when all three $g(x),h(x),s(x)$ being associate are $(p-1)\mu(\frac{r}{3},1)$. Possibilities when $a_r g(x)h(x)s(x)$ when all three $g(x),h(x),s(x)$ are monic and non associate are $\sum_{i=1}^{r-1}\sum_{j=1} r-i-1(p-1)\mu(i,1)\mu(j,1)\mu(r-i-j,1)-3(p-1)\mu(r,2+1)-(p-1)\mu(\frac{r}{3},1)$ In this case $g(x),h(x)$, and $s(x)$ are same groups and no distiction is possible between the groups then any two of them can be interchanged without giving a new group, so total possibilities for polynomial $f(x)$ reduces to $\frac{\sum_{i=1}^{r-1}\sum_{j=1}^{r-i-1}(p-1)\mu(i,1)\mu(j,1)\mu(r-i-j,1)-3(p-1)\mu(r,2+1)-(p-1)\mu(\frac{r}{3},1)}{6}$. This completes the proof. □

3. THE DIVISOR GRAPH $D(Z_p[x,3])$

Theorem 4. *Let $V = Z_p[x,3]$ be the vertex set, then Divisor graph $\mathscr{D}(V)$ has following results for the degree of their vertices.*

(i). $\deg(0x^3+0x^2+0x+0) = p^4-1$.

(ii). $\deg(0x^3+0x^2+0x+d) = p^4-1$ *where* $d \neq 0$.

(iii). $\deg(0x^3+0x^2+cx+d) = p^3+p-2$ *where* $c \neq 0$.

(iv). $\deg(bx^2+cx+d) = p^2+p-2$ *where* $b \neq 0$ *and* bx^2+cx+d *is irreducible polynomial.*

(v). $\deg(bx^2+cx+d) = p^2+2p-3$ *where* $b \neq 0$ *and* bx^2+cx+d *is reducible and has identical roots.*

(vi). $\deg(bx^2+cx+d) = p^2+3p-4$ *where* $b \neq 0$ *and* bx^2+cx+d *is reducible and has distinct roots.*

(vii). $\deg(ax^3+bx^2+cx+d) = 2p-2$ *where* $a \neq 0$ *and* ax^3+bx^2+cx+d *is irreducible.*

$(viii)$. $\deg(ax^3+bx^2+cx+d) = 4p-4$ *where* $a \neq 0$ *and* ax^3+bx^2+cx+d *has a linear factor and the other quadratic factor is further irreducible.*

(ix). $\deg(ax^3+bx^2+cx+d) = 4p-4$ *where* $a \neq 0$ *and* (ax^3+bx^2+cx+d) *is reducible to linear factors and has three identical roots.*

(x). $\deg(ax^3+bx^2+cx+d) = 6p-6$ *where* $a \neq 0$ *and* (ax^3+bx^2+cx+d) *is reducible to linear factors and has two identical roots.*

(xi). $\deg(ax^3+bx^2+cx+d) = 8p-8$ *where* $a \neq 0$ *and* (ax^3+bx^2+cx+d) *is reducible to linear factors and has three distinct roots provided* $p > 2$.

Proof. We know that every unit of $Z_p[x]$ is a non-zero polynomial of degree zero of the form $0x^3+0x^2+0x+d$, where $d \neq 0$, so the number of units in $Z_p[x]$ are $p-1$. Then every polynomial of degree one or two have $p-1$ associates. Also the number of polynomials of degree one and degree two exactly are $p(p-1)$ and $p^2(p-1)$ respectively.

(i) Every polynomial except zero polynomial divides zero polynomial, but no polynomial is divisible by zero polynomial.

We get $\deg^+(0x^3+0x^2+0x+0) = p^4-1$, $\deg^-(0x^3+0x^2+0x+0) = 0$, $\deg^{+-}(0x^3+0x^2+0x+0) = 0$.

Hence $\deg(0x^3+0x^2+0x+0) = \deg^+(0x^3+0x^2+0x+0)+\deg^-(0x^3+0x^2+0x+0)-\deg^{+-}(0x^3+0x^2+0x+0) = p^4-1+0-0 = p^4-1$

(ii) Every unit of $Z_p[x]$ divides zero polynomial, every non-zero polynomial of degree zero, every polynomial of degree one, two and three. Furthermore it is easy to see that units are divisible only by units.

We get $\deg^+(0x^3+0x^2+0x+d) = (p-1)-1 = p-2$, $\deg^-(0x^3+0x^2+0x+d) = 1+(p-1)+p(p-1)+p^2(p-1)+p^3(p-1)-1 = p^4-1$, $\deg^{+-}(0x^3+0x^2+0x+d) = (p-1)-1 = p-2$.

Hence $\deg(0x^3+0x^2+0x+d) = \deg^+(0x^3+0x^2+0x+d)+\deg^-(0x^3+0x^2+0x+d)-\deg^{+-}(0x^3+0x^2+0x+d) = p-2+p^4-1-(p-2) = p^4-1$.

(iii) One can easily show that the polynomial $cx+d$, where $c \neq 0$ and $c,d \in Z_p$ has exactly one root in Z_p(say α), so $cx+d$ can be rewritten as $b(x-\alpha)$. Hence every polynomial of degree one having root α is associate with polynomial $bx+c$. No polynomial other than units and associates of polynomial $cx+d$ divides $cx+d$. We get $\deg^+(0x^3+0x^2+cx+d) = (p-1)+(p-1)-1 = 2p-3$.

The zero polynomial and every associate of polynomial $cx+d$ is divisible by $cx+d$. Furthermore reducible polynomials of degree two and three with at-least one of the roots as α are divisible by $cx+d$. Reducible polynomials of degree two with at-least one root as α are of the form $\alpha_1(x-\beta)(x-\alpha)$, where $\alpha_1 \neq 0$,and number of such polynomials are exactly $p(p-1)$ and that of degree three are of the form $\alpha_1(x^2+\beta_1 x+\gamma_1)(x-\alpha)$ and number of such polynomials are $p^2(p-1)$ respectively.

We get $\deg^{+}(0x^3+0x^2+cx+d)=(p-1)+(p-2)=2p-3$, $\deg^{-}(0x^3+0x^2+cx+d)=1+(p-2)+p(p-1)+p^2(p-1)=p^3-1$, $\deg^{+-}(0x^3+0x^2+cx+d)=(p-1)-1=p-2$. Hence $\deg(0x^3+0x^2+cx+d)=\deg^{+}(0x^3+0x^2+cx+d)+\deg^{-}(0x^3+0x^2+cx+d)-\deg^{+-}(0x^3+0x^2+cx+d)=2p-3+p^3-1-(p-2)=p^3+p-2$.

(iv) Let bx^2+cx+d be an irreducible polynomial with $b\neq 0$ and $b,c,d\in Z_p$. Then bx^2+cx+d divides zero polynomial, associates of bx^2+cx+d and polynomials of degree three having one of the quadratic factors as (bx^2+cx+d) of the form $\alpha_1(x-\beta)(bx^2+cx+d)$, number of such polynomils over Z_p are $p(p-1)$ but (bx^2+cx+d) is divisible only by units of $Z_p[x]$ and associates of polynomial bx^2+cx+d.

We get $\deg^{+}(bx^2+cx+d)=(p-1)+(p-2)=2p-3$, $\deg^{-}(ax^2+bx+c)=1+(p-2)+p(p-1)=p^2-1$, $\deg^{+-}(ax^2+bx+c)=(p-1)-1=p-2$. Hence $\deg(bx^2+cx+d)=\deg^{+}(bx^2+cx+d)+\deg^{-}(bx^2+cx+d)-\deg^{+-}(ax^2+bx+c)=(2p-3)+(p^2-1)-(p-2)=p^2+p-2$.

(v) Let bx^2+cx+d be a reducible polynomial with $b\neq 0$ and $b,c,d\in Z_p$ and both roots are identical (say α,α). So bx^2+cx+d can be rewritten as $b(x-\alpha)^2$. Thus bx^2+cx+d divides zero polynomial, associates of bx^2+cx+d and polynomials of degree three of the form $\alpha_1(x-\beta)(bx^2+cx+d)$ having one of the quadratic factors as bx^2+cx+d. But is divisible by units of $Z_p[x]$, associates of polynomial $(x-\alpha)$ and associates of polynomial bx^2+cx+d.

We get $\deg^{+}(bx^2+cx+d)=(p-1)+(p-1)+(p-1)-1=3p-4$, $\deg^{-}(bx^2+cx+d)=1+(p-1)-1+p(p-1)=p^2-1$, $\deg^{+-}(bx^2+cx+d)=(p-1)-1=p-2$. Hence $\deg(bx^2+cx+d)=\deg^{+}(ax^2+bx+c)+\deg^{-}(ax^2+bx+c)-\deg^{+-}(bx^2+cx+d)=(3p-4)+(p^2-1)-(p-2)=p^2+2p-3$.

(vi) Let bx^2+cx+d be a reducible polynomial with $b\neq 0$ and $b,c,d\in Z_p$ and both roots are distinct (say α,β). Then bx^2+cx+d divides zero polynomial, associates of ax^2+bx+c, and polynomials of degree three of the form $\alpha_1(x-\beta)(bx^2+cx+d)$ having one of the quadratic factors as bx^2+cx+d. But is divisible by units of $Z_p[x]$, associates of polynomial $(x-\alpha)$, associates of polynomial $(x-\beta)$ and associates of polynomial bx^2+cx+d.

(vii) Let ax^3+bx^2+cx+d be an irreducible polynomial with $a \neq 0$ and $a,b,c,d \in Z_p$. Then ax^3+bx^2+cx+d divides zero polynomial and associates of ax^3+bx^2+cx+d, but is divisible by units of $Z_p[x]$ and associates of polynomial ax^3+bx^2+cx+d.

We get $\deg^+(bx^2+cx+d) = (p-1)+(p-1)-1 = 2p-3$, $\deg^-(ax^2+bx+c) = 1+(p-1)-1 = p-1$, $\deg^{+-}(ax^2+bx+c) = (p-1)-1 = p-2$. Hence $\deg(bx^2+cx+d) = \deg^+(bx^2+cx+d)+\deg^-(bx^2+cx+d)-\deg^{+-}(ax^2+bx+c) = (2p-3)+(p-1)-(p-2) = 2p-2$.

(viii) Let ax^3+bx^2+cx+d be a reducible polynomial with $a \neq 0$ and $a,b,c,d \in Z_p$ which has only one root (say α) in Z_p and the other quadratic factor is further irreducible over Z_p. That is of the form $a(x-\alpha)(x^2+mx+n)$ where $a,\alpha,m,n \in Z_p$. Then ax^3+bx^2+cx+d divides zero polynomial and associates of ax^3+bx^2+cx+d but is divisible by units of $Z_p[x]$, associates of polynomial $(x-\alpha)$, associates of polynomial (x^2+mx+n) and associates of polynomial ax^3+bx^2+cx+d.

We get $\deg^+(ax^3+bx^2+cx+d) = (p-1)+(p-1)+(p-1)+(p-1)-1 = 4p-5$, $\deg^-(ax^3+bx^2+cx+d) = 1+(p-1)-1 = p-1$, $\deg^{+-}(ax^3+bx^2+cx+d) = (p-1)-1 = p-2$. Hence $\deg(ax^3+bx^2+cx+d) = \deg^+(ax^3+bx^2+cx+d)+\deg^-(ax^3+bx^2+cx+d)-\deg^{+-}(ax^3+bx^2+cx+d) = (4p-5)+(p-1)-(p-2) = 4p-4$.

(ix) Let ax^3+bx^2+cx+d be a reducible polynomial with $a \neq 0$ and $a,b,c,d \in Z_p$ with $p>2$ which has three identical roots (say α,α,α) in Z_p and is of the form $a(x-\alpha)^3$. Then ax^3+bx^2+cx+d divides zero polynomial and associates of ax^3+bx^2+cx+d but is divisible by units of $Z_p[x]$, associates of polynomial $(x-\alpha)$, $(x-\alpha)^2$ and $(x-\alpha)^3$.

We get $\deg^+(ax^3+bx^2+cx+d) = (p-1)+(p-1)+(p-1)+(p-1)-1 = 4p-5$, $\deg^-(ax^3+bx^2+cx+d) = 1+(p-1)-1 = p-1$, $\deg^{+-}(ax^3+bx^2+cx+d) = (p-1)-1 = p-2$. Hence $\deg(ax^3+bx^2+cx+d) = \deg^+(ax^3+bx^2+cx+d)+\deg^-(ax^3+bx^2+cx+d)-$

$\deg^{+-}(ax^3+bx^2+cx+d) = (4p-5)+(p-1)-(p-2) = 4p-4.$

(x) Let ax^3+bx^2+cx+d be a reducible polynomial with $a \neq 0$ and $a,b,c,d \in Z_p$ which has two identical roots (say α,α,β) in Z_p and is of the form $a(x-\alpha)^2(x-\beta)$. Then ax^3+bx^2+cx+d divides zero polynomial and associates of ax^3+bx^2+cx+d but is divisible by units of $Z_p[x]$, associates of polynomial $(x-\alpha)$, associates of polynomial $(x-\alpha)^2$, associates of polynomial $(x-\beta)$ and associates of polynomial $(x-\alpha)(x-\beta)$.

We get $\deg^{+}(ax^3+bx^2+cx+d) = (p-1)+(p-1)+(p-1)+(p-1)+(p-1)+(p-1)-1 = 6p-7$, $\deg^{-}(ax^3+bx^2+cx+d) = 1+(p-1)-1 = p-1$, $\deg^{+-}(ax^3+bx^2+cx+d) = (p-1)-1 = p-2$. Hence $\deg(ax^3+bx^2+cx+d) = \deg^{+}(ax^3+bx^2+cx+d)+\deg^{-}(ax^3+bx^2+cx+d)-\deg^{+-}(ax^3+bx^2+cx+d) = (6p-7)+(p-1)-(p-2) = 6p-6.$

(xi) Let ax^3+bx^2+cx+d be a reducible polynomial with $a \neq 0$ and $a,b,c,d \in Z_p$ which has three distinct roots (say $\alpha,\beta,\gamma,$) in Z_p and is of the form $a(x-\alpha)(x-\beta)(x-\gamma)$. Then ax^3+bx^2+cx+d divides zero polynomial and associates of ax^3+bx^2+cx+d but is divisible by units of $Z_p[x]$, associates of polynomial $(x-\alpha)$, associates of polynomial $(x-\beta)$, associates of polynomial $(x-\gamma)$, associates of polynomials $(x-\alpha)(x-\beta),(x-\beta)(x-\gamma),(x-\gamma)(x-\alpha)$ and associate of polynomial ax^3+bx^2+cx+d.

We get $\deg^{+}(ax^3+bx^2+cx+d) = (p-1)+3(p-1)+3(p-1)+(p-1)-1 = 8p-9$, $\deg^{-}(ax^3+bx^2+cx+d) = 1+(p-1)-1 = p-1$, $\deg^{+-}(ax^3+bx^2+cx+d) = (p-1)-1 = p-2$. Hence $\deg(ax^3+bx^2+cx+d) = \deg^{+}(ax^3+bx^2+cx+d)+\deg^{-}(ax^3+bx^2+cx+d)-\deg^{+-}(ax^3+bx^2+cx+d) = (8p-9)+(p-1)-(p-2) = 8p-8.$

□

Corollary 1. *For any prime p, the divisor graph $D(Z_p[x,3])$ never becomes a complete graph.*

Proof. For any prime p, there exist at least two non-associate as well as irreducible polynomial x and $x+1$, so neither x divides $x+1$ nor $x+1$ divides x. Hence, these vertices are not connected.Therefore, $D(Z_p[x,3])$ is not complete.This completes the proof. □

Corollary 2. *For any prime p, the divisor graph $D(Z_p[x,3])$ never becomes a cyclic graph.*

Proof. For any prime p, polynomial $x^3+(p-1)x^2$ has three roots $0,0,1$, so by part (x) of Theorem 4 degree of this vertex is 6p-6 which never becomes two. Hence we get a vertex whose degree is not two, so divisor graph of set of all polynomials of degree at most three from $Z_p[x]$ never becomes a cyclic graph. □

Corollary 3. *The girth of divisor graph $D(Z_p[x,3]$ is three for every prime p.*

Proof. If p is any prime, then $(0x^2+0x+1)|(x^2+0x+0)$, $(0x^2+x+0)|(x^2+0x+0)$ and $(0x^2+0x+1)|(0x^2+x+1)$. So, we get a cycle formed with vertices $0x^2+0x+1, x^2+0x+0, 0x^2+x+0$ of length three in simple graph $D(Z_p[x,3])$, hence girth of $D(Z_p[x,3])$ is three. □

Corollary 4. *The divisor graph $D(Z_p[x,3])$ is Eulerian if and only if p is an odd prime.*

Proof. Suppose $D(Z_p[x,3])$ is Eulerian and if possible, let p be an even prime. Clearly, $p=2$. Then by Theorem 4 the degree of verities $0x^2+0x+0, 0x^2+0x+1, x^2+0x+1$ in divisor graph must be odd. Hence we get more than 2 vertices with odd degree, then graph cannot have Euler circuit, that is a contradiction. Hence p must be an odd prime.

Conversely, if p is an odd prime then $p^4-1, p^3+p-2, p^2+p-2, p^2+2p-3, 2p-2, 4p-4, 6p-6, 8p-8$ are all even. Thus degree of each vertex of divisor graph is even, hence divisor graph is Eulerian. □

Corollary 5. *A vertex corresponding to any irreducible polynomial of degree three has minimum degree among all vertices of divisor graph $D(Z_p[x,3])$.*

Proof. For every prime p, we have $2p-2<4p-4<6p-6<8p-8$, $2p-2<p^2+p-2<p^2+2p-3, 2p-2<p^3+p-2$ and $2p-2<p^4-1$because $p>1$. By Theorem 4, every irreducible polynomial of degree three has degree $2p-2$ and remaining vertices have degree p^4-1 or p^3+p-2 or p^2+p-2 or p^2+2p-3 or $4p-4$ or $6p-6$ or $8p-8$. Hence we get the desired result. □

Corollary 6. *For any prime p,$D(Z_p[x,3])$ is a non-planer graph.*

Proof. To show this, we find two partition each with three vertices such that each vertices of first partition is adjacent to all the vertices of second partition in divisor graph $D(Z_p[x,3])$.We take vertices $0,1,x$ in first partition and vertices x^2, x^3, x^2+x in second partition, so that each vertices in first is adjacent to second partition , so divisor graph $D(Z_p[x,3])$ has sub-graph which is isomorphic to $K_{3,3}$. Hence, divisor graph $D(Z_p[x,3])$ is a non-planer graph for every prime p. □

Corollary 7. *The size of divisor graph* $D(Z_p[x,3])$ *is* $\frac{p(p-1)(7p^3-p^2-p-1)}{2}$.

Proof. From parts (i) and (ii) of Theorem 4, every polynomial of degree zero and zero polynomial has degree p^4-1 and the exact number of such polynomials are p. By theorem 1,possibilities for polynomials of the type $\pi^*(1,1)$ are $p(p-1)$, by part (iii) of Theorem 4, degree of vertices corresponding to such polynomials is p^3+p-2.By theorem 1,possibilities for polynomials of the type $\pi^*(2,1)$ are $\frac{p(p-1)^2}{2}$, by part (iv) of Theorem 4, degree of vertices corresponding to such polynomials is p^2+p-2.By theorem 1,possibilities for polynomials of the type $\pi^*(2,2)$ are $p(p-1)$, by part (v) of Theorem 4, degree of vertices corresponding to such polynomials is p^2+2p-3.By theorem 2,possibilities for polynomials of the type $\pi^*(2,1+1)$ are $\frac{p(p-1)^2}{2}$, by part (vi) of Theorem 4, degree of vertices corresponding to such polynomials is p^2+3p-4.By theorem 1,possibilities for polynomials of the type $\pi^*(3,1)$ are $\frac{p(p-1)^2(p+1)}{3}$, by part (vii) of Theorem 4, degree of vertices corresponding to such polynomials is $2p-2$.By theorem 2,possibilities for polynomials of the type $\pi^*(3,1+1)$ are $\frac{p^2(p-1)^2}{2}$, by part (viii) of Theorem 4, degree of vertices corresponding to such polynomials is $4p-4$.By theorem 1,possibilities for polynomials of the type $\pi^*(3,3)$ are $p(p-1)$, by part (ix) of theorem 4, degree of vertices corresponding to such polynomials is $4p-4$.By theorem 2,possibilities for polynomials of the type $\pi^*(3,2+1)$ are $p(p-1)^2$, by part (x) of Theorem 4, degree of vertices corresponding to such polynomials is $6p-6$.By theorem 3,possibilities for polynomials of the type $\pi^*(3,1+1+1)$ are $\frac{p(p-1)^2(p-2)}{3!}$, by part (xi) of Theorem 4, degree of vertices corresponding to such polynomials is $8p-8$.

Therefore the sum of degrees of all the vertices in $D_(Z_p[x,3])$ is $p(p-1)(7p^3-p^2-p-1)$. By Fundamental theorem of graph theory [8], sum of degrees of all the vertices in the graph $D(Z_p[x,3])$ is twice the sum of edges in it. We know that size of graph is equal to number of edges in it. Hence the size of graph $D(Z_p[x,3])$ is $\frac{p(p-1)(7p^3-p^2-p-1)}{2}$. □

Corollary 8. *The divisor graph $D(Z_p[x,3])$ has a not Hamiltonian circuit for any prime p.*

Proof. If p is a prime, we have to show that divisor graph $D(Z_p[x,3])$ does not possess Hamiltonian circuit.

If possible, let C be a Hamiltonian circuit of divisor graph $D(Z_p[x,3])$. We know that there exists $\frac{p(p-1)^2(p+1)}{3}$ irreducible polynomials of degree three and the number of units and zero polynomial are p in $Z_p[x,3]$. Since every irreducible polynomial of degree three in $Z_p[x,3]$ is either adjacent with zero element, unit element or its associate polynomial only. Pick any unit or zero polynomial from Hamiltonian circuit C which moves to irreducible polynomial of degree three (say) x_1, one can move to at most $p-2$ associates of x_1 continuously before moving to remaining unit or zero element in C and C contains at most $p-1$ such moves because C has at-least one move from unit or zero polynomial to irreducible polynomial of degree three so that these vertices also get covered. So C forms Hamiltonian circuit only if

$\frac{\#\ irreducible\ polynomials\ of\ degree\ three\ from\ Z_p[x,3]}{p-1} \leq p-1$

$\implies \frac{p(p-1)^2(p+1)}{2(p-1)} \leq (p-1)$

$\implies p(p+1) \leq 3$, but no prime satisfies this.Hence we get contradiction with the fact that C is Hamiltonian circuit. ☐

Corollary 9. *The degree sequence and irregularity index of $D(Z_p[x,3])$ are given by*

$$DS(D(Z_p[x,3])) = \underset{p\ times}{p^4-1}, \underset{p(p-1)\ times}{p^3+p-2}, \underset{\frac{p(p-1)^2}{2}\ times}{p^2+p-2}, \underset{p(p-1)\ times}{p^2+2p-3}, \underset{\frac{p(p-1)^2}{2}\ times}{p^2+3p-4},$$
$$\underset{\frac{p(p-1)^2(p+1)}{3}\ times}{8p-8}, \underset{\frac{p^2(p-1)^2}{2}\ times}{6p-6}, \underset{p^2(p-1)\ times}{4p-4}, \underset{\frac{p(p-1)^2(p-2)}{3!}\ times}{2p-2} \quad and$$

$$t(D(Z_p[x,3])) = \begin{cases} 6, & if\ p\ is\ an\ even\ prime, \\ 8, & if\ p\ is\ 3\ or\ 5. \\ 9, & if\ p\ is\ a\ odd\ prime. \end{cases}$$

Proof. By Theorem 4, the degree of vertices are $p^4-1, p^3+p-2, p^2+3p-4, p^2+2p-3, p^2+p-2, 8p-8, 6p-6, 4p-4, 2p-2$ with multiplicity
$p, p(p-1), \frac{p(p-1)^2}{2}, p(p-1), \frac{p(p-1)^2}{2}, \frac{p(p-1)^2(p+1)}{3}, \frac{p^2(p-1)^2}{2}, p(p-1)+p(p-1)^2, \frac{p(p-1)^2(p-2)}{3}$, respectively .This completes the proof of first part. For irregularity index, we have $p^4-1, p^3+p-2, p^2+3p-4, p^2+2p-3, p^2+p-2, 8p-8, 6p-6, 4p-4, 2p-2$ all are pairwise unequal for every prime greater than 5.If $p=2$, we get degree's of various vertices as $15, 8, 6, 5, 4, 2$.If

$p = 3$, we get degree's of various vertices as $80, 28, 16, 14, 12, 10, 8, 4$.If $p = 5$, we get degree's of various vertices as $624, 128, 36, 32, 28, 24, 16, 8$.Hence $t(D(Z_p[x,3]))$ is $6, 8, 8, 8, 9$ for p being even prime, 3 ,5 or odd prime greater than 5 respectively. □

4. Results on the Basis of Number of Irreducible Factors

Theorem 5. *The degree of irreducible polynomial of degree $1 \leq k \leq n$ as vertex of divisor graph $D(Z_p[x,n])$ is $p^{n-k+1} + p - 2$*

Proof. Let $f(x)$ be an irreducible polynomial of degree $1 \leq k \leq n$. Then $f(x)$ divides polynomial of form $f(x)g(x)$ where $g(x)$ is a polynomial of degree less than or equal to $n-k$, but is divisible by units of $Z_p[x]$ and associates of polynomial $f(x)$. Possibilities for all polynomial of degree less than or equal to $n-k$ are p^{n-k+1}

We get $\deg^+(f(x)) = (p-1) + (p-1) - 1 = 2p - 3$, $\deg^-(f(x)) = p^{n-k+1} - 1$, $\deg^{+-}(f(x)) = (p-1) - 1 = p - 2$.

Hence $\deg(f(x)) = \deg^+(f(x)) + \deg^-(f(x)) - \deg^{+-}(f(x)) = (2p-3) + (p^{n-k+1} - 1) - (p - 2) = p^{n-k+1} + p - 2$. □

Theorem 6. *The degree of vertex which is polynomial of degree $2 \leq k \leq n$ and decomposed as product of two non-associate irreducible polynomials in divisor graph $D(Z_p[x,n])$ is $p^{n-k+1} + 3p - 4$*

Proof. Let $f(x)$ be a reducible polynomial of degree $2 \leq k \leq n$ and which can expressed as product of exactly two irreducible non associative non constant factors (say $f_1(x)$ and $f_2(x)$), then $f(x)$ divides polynomial of form $f(x)g(x)$ where $g(x)$ is a polynomial of degree less than or equal to $n-k$, but is divisible by units of $Z_p[x]$, associates of polynomial $f_1(x)$,associates of polynomial $f_2(x)$ and associates of polynomial $f(x)$.

Possibilities for all polynomial of degree less than or equal to $n-k$ are p^{n-k+1}

We get $\deg^+(f(x)) = (p-1) + (p-1) + (p-1) + (p-1) - 1 = 4p - 5$, $\deg^-(f(x)) = p^{n-k+1} - 1$, $\deg^{+-}(f(x)) = (p-1) - 1 = p - 2$.

Hence $\deg(f(x)) = \deg^+(f(x)) + \deg^-(f(x)) - \deg^{+-}(f(x)) = (4p-5) + (p^{n-k+1} - 1) - (p - 2) = p^{n-k+1} + 3p - 4$. □

Theorem 7. *The degree of vertex which is polynomial of degree* $3 \leq k \leq n$ *and decomposed as product of three mutually non-associate irreducible polynomials in divisor graph* $D(Z_p[x,n])$ *is* $p^{n-k+1} + 7p - 8$

Proof. Let $f(x)$ be a reducible polynomial of degree $3 \leq k \leq n$ and which can expressed as product of exactly three mutually non-associate irreducible polynomials and non constant factors (say $f_1(x)$,$f_2(x)$ and $f_3(x)$), then $f(x)$ divides polynomial of form $f(x)g(x)$ where $g(x)$ is a polynomial of degree less than or equal to $n-k$, but is divisible by units of $Z_p[x]$, associates of polynomial $f_1(x)$,$f_2(x)$, $f_3(x)$, $f_1(x)f_2(x)$,$f_1(x)f_3(x)$, $f_2(x)f_3(x)$ and $f(x)$.
Possibilities for all polynomial of degree less than or equal to $n-k$ are p^{n-k+1}

We get $\deg^+(f(x)) = 7(p-1)-1 = 7p-8$, $\deg^-(f(x)) = p^{n-k+1}-1$, $\deg^{+-}(f(x)) = (p-1)-1 = p-2$.
Hence $\deg(f(x)) = \deg^+(f(x)) + \deg^-(f(x)) - \deg^{+-}(f(x)) = (7p-8)+(p^{n-k+1}-1)-(p-2) = p^{n-k+1}+7p-8$. □

The following theorem generalizes last three results.

Theorem 8. *The degree of vertex which is polynomial of degree* $1 \leq s \leq k \leq n$ *and decomposed as product of s mutually non-associate irreducible polynomials in divisor graph* $D(Z_p[x,n])$ *is* $p^{n-k+1} + (2^s-1)p - 2^s$

Corollary 10. *Two polynomials of same degree k in* $Z_p[x,n]$ *having mutually non-associate irreducible, the one with fewer number of irreducible factors has smaller degree.*

Proof. Let $f(x)$ and $g(x)$ be two polynomials of same degree k having s_1 and s_2 mutually non-associate irreducible factors with $s_1 \leq s_2$, then $deg(f(x)) = p^{n-k+1} + (2^{s_1}-1)p - 2^{s_1}$ and $deg(g(x)) = p^{n-k+1} + (2^{s_2}-1)p - 2^{s_2}$

By use of $s_1 \leq s_2$, we have $deg(g(x)) - deg(f(x)) = (2^{s_2}-2^{s_1})p - (2^{s_2}-2^{s_1}) = (2^{s_2}-2^{s_1})(p-1) > 0$.Hence $deg(f(x)) \leq deg(g(x))$ □

Theorem 9. *The degree of vertex which is polynomial of degree* $2 \leq k \leq n$ *and decomposed as product of two irreducible polynomials but both irreducible polynomials are associates in divisor graph* $D(Z_p[x,n])$ *is* $p^{n-k+1} + 2p - 3$.

Proof. Let $f(x)$ be a reducible polynomial of degree $2 \leq k \leq n$ and which can expressed as product of exactly two irreducible associates non constant factors i.e., $f(x) = a(f_1(x))^2$ where $f_1(x)$ is irreducible polynomial and $af_1(x)$ where a is units in $z_p[x]$, then $f(x)$ divides polynomial of form $f(x)g(x)$ where $g(x)$ is a polynomial of degree less than or equal to $n-k$, but is divisible by units of $Z_p[x]$, associates of polynomial $f_1(x)$ and associates of polynomial $f(x)$.
Possibilities for all polynomial of degree less than or equal to $n-k$ are p^{n-k+1}

We get $\deg^+(f(x)) = (p-1)+(p-1)+(p-1)-1 = 3p-4$, $\deg^-(f(x)) = p^{n-k+1}-1$, $\deg^{+-}(f(x)) = (p-1)-1 = p-2$.
Hence $\deg(f(x)) = \deg^+(f(x)) + \deg^-(f(x)) - \deg^{+-}(f(x)) = (3p-4)+(p^{n-k+1}-1)-(p-2) = p^{n-k+1}+2p-3$. □

Now generalization of above result, we have following result

Theorem 10. *The degree of vertex which is polynomial of degree $2 \leq s \leq k \leq n$ and decomposed as product of s irreducible polynomials but exactly two irreducible polynomials are associates in divisor graph $D(Z_p[x,n])$ is $p^{n-k+1}+(2^s-2)p-(2^s-1)$*

Theorem 11. *The degree of vertex which is polynomial of degree $3 \leq k \leq n$ and decomposed as product of three irreducible polynomials which are mutually associates in divisor graph $D(Z_p[x,n])$ is $p^{n-k+1}+3p-4$.*

Proof. Let $f(x)$ be a reducible polynomial of degree $3 \leq k \leq n$ and which can expressed as product of exactly two irreducible associates non constant factors i.e., $f(x) = a(f_1(x))^3$ where $f_1(x)$ is irreducible polynomial and $af_1(x)$ where a is units in $z_p[x]$, then $f(x)$ divides polynomial of form $f(x)g(x)$ where $g(x)$ is a polynomial of degree less than or equal to $n-k$, but is divisible by units of $Z_p[x]$, associates of polynomial $f_1(x)$, associates of polynomial $(f_1(x))^2$and associates of polynomial $f(x)$.

Possibilities for all polynomial of degree less than or equal to $n-k$ are p^{n-k+1}.

We get $\deg^+(f(x)) = (p-1)+(p-1)+(p-1)+(p-1)-1 = 4p-5$, $\deg^-(f(x)) = p^{n-k+1}-1$, $\deg^{+-}(f(x)) = (p-1)-1 = p-2$.
Hence $\deg(f(x)) = \deg^+(f(x)) + \deg^-(f(x)) - \deg^{+-}(f(x)) = (4p-5)+(p^{n-k+1}-1)-(p-2) = p^{n-k+1}+3p-4$. □

Now we generalize the above result.

Theorem 12. *The degree of vertex which is polynomial of degree* $1 \leq s \leq k \leq n$ *and decomposed as product of* s *irreducible polynomials which are mutually associates in divisor graph* $D(Z_p[x,n])$ *is* $p^{n-k+1} + sp - s - 1$.

5. Conclusion

In this paper, we obtained relation of number of monic and non monic polynomials of a particular degree with its degree as a vertex in its corresponding divisor graph. The polynomial with fewer number of distinct irreducible factors will have the smaller degree.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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