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OPTIMIZATION OF A QUEUEING SYSTEM WITH INVERSE-GAUSSIAN SERVICE PATTERN

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Abstract: In this paper parameters involvedin a single server waiting line system with poisson arrivals and Inversegaussian service times are estimated. Also, the same result has been obtained when it is assumed that the service time distribution is a finite range model namely, Mukheerji-Islam model, which is a well-known life testing model.

Keywords: waiting line system; service time distribution; mean; variance; inversegaussian service time distribution.
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1. INTRODUCTION

The Inversegaussian family of distributions are often used in analyzing many of the realistic situations arising at life testing, economical analysis, insurance studies etc. The major advantage of this distribution is the interpretation of the inversegaussian random variable as the first passage time distribution of Brownian motion with positive drift.In textile industries the printing or bleaching processed are distribution approximately as Inversegaussian distribution . Here unit of cloth is to be taken as customer, the printing or bleaching is viewed as service. In spite of wide applicability of the inversegaussian distribution as approximate model of skewed data and having simple exact sampling theory. It has been not much utilized in analyzing waiting line systems.

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2. QUEUEING MODEL WITH INVERSEGAUSSIAN SERVICE TIME DISTRIBUTION

Consider a single server queueing with infinite capacity having FCFS (First Come First Serve) queue discipline. we assume that the arrivals are Poission with arrival rate $\boldsymbol{\lambda}$. The service time distribution of the process is aninversegaussian of the form.

$$f(t;\mu,\theta)=\left(\frac{\theta}{2\pi t^{3}}\right)^{1/2} exp-\left\{\frac{\theta(\mu t-1)^{2}}{2.t}\right\};\ \mu,\theta\geq 0; 0\leq t<\infty \tag{1}$$

with mean = $\frac{1}{\mu}$ and Variance = $\frac{1}{\theta\mu^{3}}$.

2.1Maximum Likelihood Estimates

The μ and θ are parameters involved in the service time distribution given in eq^n.(1).Consider a random sample T_1 , T_2,……T_n from the population with p.d.feq^n.(3)The likelihood function is given as

$$L(t;\mu,\theta)=\prod_{i=1}^{n} f_i(t_i;\mu,\theta)$$

$$=\prod_{i=1}^{n}\left(\frac{\theta}{2\pi {t_i}^{3}}\right)^{1/2} exp-\left\{\frac{\theta(\mu t_i-1)^{2}}{2.t_i}\right\}$$

$$=\left(\frac{\theta}{2\pi}\right)^{n/2}\prod_{i=1}^{n}\left(\frac{1}{t_i}\right)^{3/2} exp-\left\{\frac{\theta}{2}\sum_{i=1}^{n}\frac{(\mu t_i-1)^{2}}{t_i}\right\} \tag{2}$$

Taking logarithm both sides, we have

$$LogL(t;\mu,\theta)=\frac{n}{2}log\theta-\frac{n}{2}\log(2\pi)-\frac{3}{2}\sum_{i=1}^{n} logt_i-\frac{\theta}{2}\sum_{i=1}^{n}\frac{(\mu t_i-1)^{2}}{t_i} \tag{3}$$

Now, to obtain the maximum likelihood estimator of the parameter θ partially differentiating eq^n.(3) with respect to$\theta$and equating the resultant to zero, we get

$$\frac{\partial}{\partial\theta}LogL=0$$

$$\frac{\partial}{\partial\theta}\left[\frac{n}{2}log\theta-\frac{n}{2}\log(2\pi)-\frac{3}{2}\sum_{i=1}^{n} logt_i-\frac{\theta}{2}\sum_{i=1}^{n}\frac{(\mu t_i-1)^{2}}{t_i}\right]=0$$

$$\frac{n}{2.\theta}-\frac{1}{2}\sum_{i=1}^{n}\frac{(\mu t_i-1)^{2}}{t_i}=0 \tag{4}$$

Similarly, to obtain the maximum likelihood estimator of the parameter μ partially differentiating eq^n.(3) with respect toμ and equating the resultant to zero, we get

$$\frac{\partial}{\partial \mu} LogL = 0$$

$$\frac{\partial}{\partial \mu}\left[\frac{n}{2} log\theta - \frac{n}{2}\log(2\pi) - \frac{3}{2}\sum_{i=1}^{n} logt_i - \frac{\theta}{2}\sum_{i=1}^{n}\frac{(\mu t_i - 1)^2}{t_i}\right] = 0$$

$$-\theta\sum_{i=1}^{n}(\mu t_i - 1) = 0$$

$$\mu\sum_{i=1}^{n} t_i - n = 0$$

$$\hat{\mu} = \frac{n}{\sum_{i=1}^{n} t_i} = \frac{1}{\bar{\bar{T}}}$$

(5)

Now, substituting the maximum likelihood estimate of μ in eq^n.(4) , we get

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n}\frac{(\mu t_i - 1)^2}{t_i}}$$

$$= \frac{n\bar{T}^2}{\sum_{i=1}^{n}\frac{(\mu t_i - 1)^2}{t_i}}$$

(6)

2.2Analysis of the Model

To analyze the model we will obtain probability generating function of H_n , the probability that there are n arrivals during the service time of a customer.

Let H_n be the probability that there are n arrivals during the service time of a customer. Let $H(z)$ denotes the probability generating function (p.d.f) of H_n given as

$$H(z) = \sum_{i=1}^{n} H_n z^n \quad ; \quad |z| \leq 1$$

(7)

Following heuristic argument Kendall [12] and Gross and Hariss [9] , the probability H_n that there are n arrivals during the service time is given by

$$H_n = \int_0^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!}\left(\frac{\theta}{2\pi t^3}\right)^{1/2} exp - \left\{\frac{\theta(\mu t - 1)^2}{2.t}\right\} dt$$

(8)

Then the probability generating function of H_n is

$$H(z) = \sum_{n=0}^{\infty} z^n \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!}\left(\frac{\theta}{2\pi t^3}\right)^{1/2} exp - \left\{\frac{\theta(\mu t-1)^2}{2.t}\right\} dt$$

$$= \left(\frac{\theta}{2\pi}\right)^{1/2} \int_0^\infty e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda z t)^n}{n!}\left(\frac{1}{t^{3/2}}\right). exp - \frac{\theta\mu^2}{2}\left\{\frac{t^2+\frac{1}{\mu^2}-\frac{2t}{\mu}}{t}\right\} dt$$

$$= \left(\frac{\theta}{2\pi}\right)^{1/2} \int_0^\infty e^{-\lambda t} e^{\lambda z t}\left(\frac{1}{t^{3/2}}\right). exp - \frac{\theta\mu^2}{2}\left\{\frac{t^2+\frac{1}{\mu^2}-\frac{2t}{\mu}}{t}\right\} dt$$

$$= \left(\frac{\theta}{2\pi}\right)^{1/2} e^{\theta\mu} \int_0^\infty exp - \{(\lambda-\lambda z)t\}\left(\frac{1}{t^{3/2}}\right). exp - \frac{\theta\mu^2}{2}\left\{t+\frac{1}{t\mu^2}\right\} dt$$

$$= \left(\frac{\theta}{2\pi}\right)^{1/2} e^{\theta\mu} \int_0^\infty \left(\frac{1}{t^{3/2}}\right). exp - \frac{\theta\mu^2}{2}\left\{\frac{2(\lambda-\lambda z)t}{\theta\mu^2}+t+\frac{1}{t\mu^2}\right\} dt$$

$$= \left(\frac{\theta}{2\pi}\right)^{1/2} e^{\theta\mu} \int_0^\infty \left(\frac{1}{t^{3/2}}\right). exp - \frac{\theta\mu^2}{2}\left\{\left(1+\frac{2(\lambda-\lambda z)t}{\theta\mu^2}\right)t+\frac{1}{t\mu^2}\right\} dt$$

On further simplification, we get

$$H(z) = exp\left\{\theta\mu\left[1-\left(1+\frac{2(\lambda-\lambda z)t}{\theta\mu^2}\right)^{1/2}\right]\right\} \tag{9}$$

The average number of arrivals during the service time is

$$H'(z)\,\Big|_{z=1} = \frac{\lambda}{\mu} \tag{10}$$

Let P_n be the probability that are n customers in the system that are steady state. Let $P(z)$ be the probability generating function of P_n.Therefore,

$$P(z) = \frac{\left(1-\frac{\lambda}{\mu}\right)(1-z)exp\left\{\theta\mu\left[1-\left(1+\frac{2(\lambda-\lambda z)t}{\theta\mu^2}\right)^{1/2}\right]\right\}}{exp\left\{\theta\mu\left[1-\left(1+\frac{2(\lambda-\lambda z)t}{\theta\mu^2}\right)^{1/2}\right]\right\}-z} \tag{11}$$

By expanding $P(z)$ and collecting the coefficients of z^n , we get P_n be the probability that are n customers in the system.

The probability that the system is empty is

$$P_o = 1 - \frac{\lambda}{\mu} \tag{12}$$

The maximum likelihood estimate of the parameter μ is given in equation (5).

After substituting estimated value of μ in equation (12), we can obtain P_o for various values of λ . we also observe that P_o is independent of θ i.e. P_o is not influenced by the variability of the service time.

The average number of customers in the system can be obtained as

$$L = P'(z)\Big|_{z=1}$$

$$= \frac{\lambda[\lambda+\theta\mu(2\mu-\lambda)]}{2(\mu-\lambda)\theta\mu^2} \tag{13}$$

From the equation (13) it can be observe that the average number of customers in the system influenced by θ . The value of L can be computed by uing estimated values of μand θ from equation (5) and (6) and various values of λ .

The variability of the system size can be obtained by using the formula.

$$V = \left[P''(z) + P'(z) - \left(P'(z)\right)^2\right]\Big|_{z=1}$$

$$= \frac{3A^2 + (\rho-1)(2\rho-3).A - 4.B - 6\rho(2\rho-1)(\rho-1)}{12(\rho-1)^2} \tag{14}$$

Where

$$A = \frac{\lambda^2}{\mu^2}\left(\frac{1}{\theta\mu} + 1\right)$$

$$B = \frac{\lambda^3}{\mu^3}\left(\frac{3}{\theta\mu^2} + \frac{3}{\theta\mu} + 1\right)$$

and $$\rho = \frac{\lambda}{\mu}$$

The coefficient of variation (C.V.) of the system size will be

$$C.V = \frac{\sqrt{V}}{L} \times 100 \tag{15}$$

To various values of λ and estimated values of $\mu$and , the values of variability of the system size and the coefficient can be computed.

It has been seen by Murty (1993) that the variability of the system size decreases as μ increases for fixed value of λ and θ. As λ increases the variability of the system size increases fixed value of θ and μ. Further, it has been observed that the coefficient of variation increases as μ increases for fixed value of λ and θ . As λ increases the coefficient of variation decreases for fixed value of and λ . As θ increases the coefficient of variation decreases for fixed value of λ and θ . For given arrival and service rates, the mean value length of M/M/1 and M/1G/1 model are compared and it has been observed that when $\frac{1}{\mu} \leq \theta$, the mean queue length of M/1G/1 is less than that of the M/M/1 model.

It means that by controlling $'\theta'$ we can control the mean queue length of M/1G/1 model, which has influence on the optimal operating policies of the system. Further we can analyze the model in a better by using estimated values of θ and μ in place of hypothetical value of θ and. In this model we have used only hypothetical value to λ .

1.3 QUEUEING MODEL WITH MUKHEERJI – ISLAM SERVICE TIME DISTRIBUTION

Again consider a single server queueing with infinite capacity having FCFS (First Come First Serve) queue discipline. we assume that the arrivals are poission with arrival rate **λ**. But the service time distribution of the process is a new finite range probability distribution which is introduced by Mukheerji-Islam (1983) as a life testing model .

$$f(t;\mu,\theta) = (p/\theta^p)t^{p-1}; \quad p,\theta \geq 0; t \geq 0 \tag{16}$$

The above model is monotonic decreasing and highly skewed to the right. The graph is J-shaped thereby showing the unimodel feature. The distribution function of above model will be

$$F(t) = [t/\theta]^p \tag{17}$$

with $mean = \frac{p}{p+1}.\theta$

and $Variance = \frac{p}{(p+1)^2(p+2)}.\theta^2$

1.3.1 MAXIMUM LIKELIHOOD ESTIMATES

Under the same consideration as in section 1.2.1 the likelihood function for the model (16) is given by

$$L(t;\theta,p) = p^n\theta^{-np}\prod_{i=1}^{n} t_i{}^{p-1} \tag{18}$$

Taking log onboth the sides, we get

$$logL = n\,logp - nplog\theta + (p-1)\sum logt_i \tag{19}$$

Differentiating the equation (19) partially with respect to p and equating it to zero,

$$\frac{\partial logL(t)}{\partial p} = \frac{n}{p} - nlog\theta + \sum logt_i = 0$$

The M.L.E of p is finally obtained as

$$\hat{p} = \frac{n}{nlog\theta - \sum logt_i} \tag{20}$$

Again, differentiating partially the equation (19) with respect to $\theta$and equating it to zero to obtain the M.L.E of θ

$$\frac{\partial logL(t)}{\partial \theta} = \frac{np}{\theta} = 0$$

In the solution for M.L.E of θ the traditional method is not applicable. The M.L.E is obtained through order statistic technique. Since the upper limit of the model is , it is convincing to take $t_{(n)}$i.e. maximum t_i as the M.L.E for the parameter θ

$$i.e. \quad \hat{\theta} = t_{(n)} = max(t_1, t_2, \ldots\ldots\ldots\ldots.. t_n) \tag{21}$$

1.3.2 ANALYSIS OF THE MODEL

To analyze the model we will obtain probability generating function of H_n , the probability that there are n arrivals during the service time of a customer.

Let H_n be the probability that there are n arrivals during the service time of a customer. Let $H(z)$ denotes the probability generating function (p.d.f) of H_n given as

$$H(z) = \sum_{i=1}^{n} H_n z^n \quad ; \quad |z| \leq 1$$

Following heuristic argument Kendall [12] and Gross and Hariss [9] , the probability H_n that there are n arrivals during the service time is given by

$$H_n = \int_0^\theta \frac{e^{-\lambda t}(\lambda t)^n}{n!}\left(\frac{p}{\theta^p}\right) t^{p-1} dt \tag{22}$$

Then the probability generating function of H_n is

$$H(z) = \sum_{n=0}^{\infty} z^n \int_0^\theta \frac{e^{-\lambda t}(\lambda t)^n}{n!}\left(\frac{p}{\theta^p}\right) t^{p-1} dt$$

$$= \left(\frac{p}{\theta^p}\right) \int_0^\theta \sum_{n=0}^{\infty} z^n \frac{e^{-\lambda t}(\lambda t)^n}{n!} t^{p-1} dt$$

$$= \left(\frac{p}{\theta^p}\right)\int_0^{\theta} e^{-\lambda t}\sum_{n=0}^{\infty}\frac{(\lambda z t)^n}{n!}t^{p-1}dt$$

$$= \left(\frac{p}{\theta^p}\right)\int_0^{\theta} e^{-\lambda t}e^{\lambda z t}\, t^{p-1}dt$$

$$= \left(\frac{p}{\theta^p}\right)\int_0^{\theta} e^{-(\lambda-\lambda z)t}\, t^{p-1}dt$$

$$= \left(\frac{p}{\theta^p}\right)\int_0^{\theta} \sum_{j=0}^{\infty}\frac{(-(\lambda-\lambda z)t)^j}{j!}t^{p-1}dt$$

$$= \left(\frac{p}{\theta^p}\right)\sum_{j=0}^{\infty}\frac{(-(\lambda-\lambda z)t)^j}{j!}\int_0^{\theta} t^{p+j-1}\,dt$$

$$H(z) = p.\sum_{j=0}^{\infty}\frac{(-(\lambda-\lambda z)t)^j}{j!}\frac{\theta^j}{p+j}$$

(23)

The average number of arrivals during the service time is

$$H'(z)\,\Big|_{z=1} = \frac{p}{p+1}.\theta.\boldsymbol{\lambda} \tag{24}$$

Let we denote that $\mu = \frac{p}{p+1}.\theta$ (the reciprocal of the mean)then

$$H'(z)\,\Big|_{z=1} = \frac{\lambda}{\mu} \tag{25}$$

Now, let P_n be the probability that are $'n'$customers in the system at the steady state and $P(z)$ be the probability generating function of P_n.Then by expanding $P(z)$ and collecting the coefficient of z^n, we get P_n.

Furthermore, the analysis can be carried out in the same manner as in the section 1.2 for inversegaussian service time distribution system.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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