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FEKETE SZEGÖ PROPERTIES FOR THE CLASS OF MOCANU FUNCTIONS ASSOCIATED WITH $q-$RUSCHEWEYH OPERATOR

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Abstract. The objective of this paper is to introduce and investigate new subclass of analytic functions involving $q-$derivative Ruscheweyh operator. For functions belonging to this class, we obtain coefficient estimates on Taylor - Maclaurin series and the results on the famous Fekete Szegö inequality.

Keywords: univalent function; subordination; $q-$derivative; Fekete-Szegö problem.

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1. Introduction

The q-difference calculus or quantum calculus was initiated at the beginning of 19th century, that was initially developed by Jackson [8, 9]. The $q-$calculus is one of the tool which is used to introduce and investigate many number of subclasses of analytic functions. Basic definitions and properties of q-difference calculus can be found in the book mentioned in [10]. The origin of fractional q-difference calculus has been found in the works by Al.Salam [3] and Agarwal

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[2]. Due to the application of $q-$calculus in various branches of science, recently, the area of q-calculus has attracted the serious attention of researchers. Later, geometrical interpretation of $q-$analysis has been recognised through studies on quantum groups. Mohammed and Darus [14] studied approximation and geometric properties of these q-operators for some subclasses of analytic functions in compact disk.

Let $\mathscr{A}$ denote the class of all functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}) \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : \quad |z| < 1\}$.

Let $\mathscr{S}$ be the subclass of $\mathscr{A}$ consisting of all univalent functions in $\mathbb{U}$.

If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, then we say that the function $f(z)$ is *subordinate* to $g(z)$, if there exists a Schwarz function $w(z)$, analytic in $\mathbb{U}$ with

$$w(0) = 0 \quad \textit{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

We denote this subordination by

$$f \prec g \quad \textit{or} \quad f(z) \prec g(z) \quad (z \in \mathbb{U}).$$

In particular, if the function $g(z)$ is univalent in $\mathbb{U}$, the above subordination is equivalence to

$$f(0) = g(0) \quad \textit{and} \quad f(\mathbb{U}) \subseteq g(\mathbb{U}).$$

A q-analog of the class of starlike functions was first introduced in 1990 [7] by means of the q-difference operator $D_q f(z)$ acting on functions $f \in \mathscr{A}$ given by (1.1) and $0 < q < 1$, the q-derivative of a function $f(z)$ is defined by (see [8, 9])

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z} \quad (z \neq 0). \tag{1.2}$$

$D_qf(0)=f'(0)$ *and* $D_q^2f(z)=D_q(D_qf(z))$. From (1.2), we deduce that,

$$D_qf(z)=1+\sum_{k=2}^{\infty}[k]_qa_kz^{k-1} \tag{1.3}$$

where

$$[k]_q=\frac{1-q^k}{1-q}. \tag{1.4}$$

As $q\to 1^-$, $[k]_q\to k$. For a function $h(z)=z^k$, we observe that,

$$D_q(h(z))=D_q(z^k)=\frac{1-q^k}{1-q}z^{k-1}=[k]_qz^{k-1},$$

$$\lim_{q\to 1^-}(D_q(h(z)))=\lim_{q\to 1^-}([k]_qz^{k-1})=kz^{k-1}=h'(z),$$

where h' is the ordinary derivative.

As a right inverse, Jackson [9] introduced the q-integral

$$\int_0^z f(t)d_qt=z(1-q)\sum_{k=0}^{\infty}q^kf(zq^k),$$

provided that the series converges. For a function $h(z)=z^k$, we have

$$\int_0^z h(t)d_qt=\int_0^z t^kd_qt=\frac{z^{k+1}}{[k+1]_q}\quad (k\neq -1)$$

$$\lim_{q\to 1^-}\int_0^z h(t)d_qt=\lim_{q\to 1^-}\frac{z^{k+1}}{[k+1]_q}=\frac{z^{k+1}}{k+1}=\int_0^z h(t)dt,$$

where $\int_0^z h(t)dt$ is the ordinary integral. Note that the q-difference operator plays an important role in the theory of hypergeometric series and quantum physics (see for instance [4, 5, 6, 12, 16]). Kanas and Răducanu in [11] used the Ruscheweyh q-differential operator to introduce and study some properties of (q,k) uniformly starlike functions of order α. One can clearly see that $D_qf(z)\to f'(z)$ as $q\to 1^-$. This difference operator helps us to generalize the class of starlike functions S^* analytically.

Ma and Minda [13] unified various subclasses of starlike and convex functions for which either of quantity $\frac{zf'(z)}{f(z)}$ (or) $1+\frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. For this purpose, they considered an analytic function $\phi(z)$ with positive real part in the unit disc $\mathbb{U}$, with $\phi(0)=1$, $\phi'(0)>0$ and ϕ maps $\mathbb{U}$ onto a region starlike, with respect to the real axis.

The classes of Ma-Minda starlike functions consists of functions $f(z) \in \mathscr{A}$ satisfying the subordination $\frac{zf'(z)}{f(z)} \prec \phi(z)$. Similarly, the class of Ma-Minda convex functions $f \in \mathscr{A}$ satisfying the subordination $1+\frac{zf''(z)}{f'(z)} \prec \phi(z)$.

2. Preliminaries

In this session, we present some of the known concepts and new definitions defined in the open unit disc $\mathbb{U}$.

Definition 2.1. For $0 \leq \alpha \leq 1$; a function $f \in \mathscr{A}$ is in $\mathscr{M}_\alpha(\phi)$ if

$$(1-\alpha)\frac{zf^{'}(z)}{f(z)}+\alpha\left(1+\frac{zf^{''}(z)}{f^{'}(z)}\right) \prec \phi(z).$$

Quite recently, Abdullah and Darus in [1] introduced the new differential operator $\mathscr{D}^{m,v}_{q,\mu,\delta,k,\lambda}$ by

$$\mathscr{D}^{m,v}_{q,\mu,\delta,k,\lambda} f(z) = z+\sum_{i=2}^{\infty} \Omega^{m,v}_{k,\lambda,\delta,\mu}\left([i]_q\right) a_i z^i, \tag{2.1}$$

where

$$(\delta,k,\lambda,\mu \geq 0), k>\lambda, \delta>\mu, m \in \mathbb{N}_0.$$

$$\Omega^{m,v}_{k,\lambda,\delta,\mu} = (k-\lambda)(\delta-\mu)\left[\left([i]_q-1\right)+1\right]^m \frac{[v-1+i]_q!}{\left[v_q\right]![i-1]!}.$$

Remark 2.2. For different values of v,k,λ,δ,β and μ, we get various differential operators explained as Remark in [1].

By inspiring the works of Abdullah and Darus [1], we now define the new subclass $\mathscr{M}^{m,v}_q(\phi)$ of $\mathscr{A}$ associated with the differential operator(2.1).

Definition 2.3. Let $f \in \mathscr{A}$ and $0 \leq \alpha \leq 1$, then f is said to be in the class $\mathscr{M}_q^{m,v}(\phi)$ if it satisfies the following subordination condition:

$$(1-\alpha)\left(\frac{z\partial_q \mathscr{D}_q^{m,v} f(z)}{\mathscr{D}_q^{m,v} f(z)}\right) + \alpha\left(\frac{\partial_q\left(z\partial_q \mathscr{D}_q^{m,v} f(z)\right)}{\partial_q \mathscr{D}_q^{m,v} f(z)}\right) \prec \phi(z). \tag{2.2}$$

In order to prove the main result, we need the following lemma.

Lemma 2.4. [15] *If $p(z) = 1 + c_1 z + c_2 z^2 + \dots$ is an analytic function with positive real part in $\mathbb{U}$, then*

$$|c_2 - v c_1^2| \leq 2max\{1, |2v-1|\}.$$

the result is sharp for the function $p(z) = \dfrac{1+z^2}{1-z^2}$ *and* $p(z) = \dfrac{1+z}{1-z}$.

we also need the following results for our investigation.

Lemma 2.5. [13] *If $p(z) = 1 + c_1 z + c_2 z^2 + \dots$.is an analytic function with positive real part in $\mathbb{U}$, then*

$$\left|c_2 - v c_1^2\right| \leq \begin{cases} -4v+2 & \text{if} \quad v \leq 0 \\ 2 & \text{if} \quad 0 \leq v \leq 1 \\ 4v-2 & \text{if} \quad v \geq 1 \end{cases} \tag{2.3}$$

3. Main Results

The main purpose of this paper is to obtain the Fekete-Szegö inequality for certain class of analytic functions defined by the differential operator involving $q-$Ruscheweyh operator.

Theorem 3.1. *Let $0 \leq \alpha \leq 1$. Also let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$, where the coefficients B_n are real with $B_1 > 0$. If $f(z)$ given by* (1.1) *belongs to $\mathscr{M}_q^{m,v}(\phi)$, then*

$$|a_3 - \nu a_2^2| \leq \frac{B_1}{2(1-\alpha+[3]_q\alpha)\left([3]_q-1\right)\Omega^{m,v}_{k,\lambda,\delta,\mu}([3]_q)} \times$$

$$max\left\{1; \left|\frac{B_2}{B_1} + \frac{B_1(1-\alpha+([2]_q)^2\alpha)}{\left([2]_q-1\right)\left(1-\alpha+[2]_q\alpha\right)^2}\right.\right. \tag{3.1}$$

$$\left.\left.\left(1 - \frac{(1-\alpha+[3]_q\alpha)\left([3]_q-1\right)\Omega^{m,v}_{k,\lambda,\delta,\mu}([3]_q)}{(1-\alpha+([2]_q)^2\alpha)\left([2]_q-1\right)\left(\Omega^{m,v}_{k,\lambda,\delta,\mu}([2]_q)\right)^2}\nu\right)\right|\right\}.$$

Proof. Observe that the condition in (2.2) can be written as follows:

$$(1-\alpha)\left(\frac{z\partial_q \mathscr{D}_q^{m,v} f(z)}{\mathscr{D}_q^{m,v} f(z)}\right) + \alpha\left(\frac{\partial_q\left(z\partial_q \mathscr{D}_q^{m,v} f(z)\right)}{\partial_q \mathscr{D}_q^{m,v} f(z)}\right) = \phi(\omega(z)). \tag{3.2}$$

Here, the function $\omega(z)$ is analytic in $\mathbb{U}$ with the condition $\omega(0) = 0$ and $|\omega(z)| < 1$ in $\mathbb{U}$.

Let $h(z)$ be an analytic function defined in $\mathbb{U}$ with $\Re\{h(z)\} > 0$ and $h(0) = 1$ be given by $h(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n$ for $z \in \mathbb{U}$, then

$$h(z) = \frac{1+\omega(z)}{1-\omega(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots. \tag{3.3}$$

Since $\omega(z)$ is a Schwarz function, we have

$$\phi(\omega(z)) = \phi\left(\frac{\omega(z)-1}{\omega(z)+1}\right) = 1 + \frac{1}{2}B_1 c_1 z + \left(\frac{1}{2}B_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}B_2 c_1^2\right) z^2 + \ldots. \tag{3.4}$$

Upon computation we get,

(3.5)

$$(1-\alpha)\left(\frac{z\partial_q \mathscr{D}_q^{m,v} f(z)}{\mathscr{D}_q^{m,v} f(z)}\right) + \alpha\left(\frac{\partial_q\left(z\partial_q \mathscr{D}_q^{m,v} f(z)\right)}{\partial_q \mathscr{D}_q^{m,v} f(z)}\right) =$$

$$1 + \left[\left([2]_q-1\right)\Omega^{m,v}_{k,\lambda,\delta,\mu}([2]_q)\left(1-\alpha+[2]_q\alpha\right)\right] a_2 z$$

$$+ \left\{\left[\left([3]_q-1\right)\Omega^{m,v}_{k,\lambda,\delta,\mu}([3]_q)\left(1-\alpha+[3]_q\alpha\right)\right] a_3\right.$$

$$\left.+ \left[\left(-[2]_q+1\right)\left(\Omega^{m,v}_{k,\lambda,\delta,\mu}([2]_q)\right)^2\left(1-\alpha+([2]_q)^2\alpha\right)\right] a_2^2\right\} z^2 + \ldots.$$

From (3.4) and (3.5), we obtain

$$a_2 = \frac{B_1 c_1}{2\left(1-\alpha+[2]_q\alpha\right)\left([2]_q-1\right)\Omega^{m,v}_{k,\lambda,\delta,\mu}([2]_q)} \tag{3.6}$$

and

$$\begin{aligned} a_3 =& \frac{B_1}{2\left(1-\alpha+[3]_q\alpha\right)\left([3]_q-1\right)\Omega^{m,v}_{k,\lambda,\delta,\mu}([3]_q)} \\ & \left\{c_2 - \frac{1}{2}\left[1-\frac{B_2}{B_1}-\frac{B_1(1-\alpha+([2]_q)^2\alpha)}{([2]_q-1)(1-\alpha+[2]_q\alpha)^2}\right]c_1^2\right\}. \end{aligned} \tag{3.7}$$

Therefore,

$$a_3-\mu a_2^2 = \frac{B_1}{2\left(1-\alpha+[3]_q\alpha\right)\left([3]_q-1\right)\Omega^{m,v}_{k,\lambda,\delta,\mu}([3]_q)}(c_2-\sigma c_1^2). \tag{3.8}$$

Where

$$\begin{aligned} \sigma =& \frac{1}{2}\left\{1-\frac{B_2}{B_1}-\frac{B_1(1-\alpha+([2]_q)^2\alpha)}{([2]_q-1)(1-\alpha+[2]_q\alpha)^2}\right. \\ & \left.\left(1-\frac{\left([3]_q-1\right)(1-\alpha+[3]_q\alpha)\Omega^{m,v}_{k,\lambda,\delta,\mu}([3]_q)}{\left([2]_q-1\right)(1-\alpha+([2]_q)^2\alpha)(\Omega^{m,v}_{k,\lambda,\delta,\mu}([2]_q))^2}v\right)\right\}. \end{aligned} \tag{3.9}$$

We get our desired result by applying Lemma 2.4. This completes the proof of Theorem 3.1. □

Remark 3.2. If we set $\alpha = 0$ and $\alpha = 1$, then we have the results of Theorem 5 and Theorem 6 obtained by Abdullah and Darus [1] respectively.

If we set $m = 0$ and $v = 0$ in Theorem 3.1, we thus obtain the following:

Corollary 3.3. *Let $\phi(z) = 1+B_1z+B_2z^2+B_3z^3+\ldots$, with $B_1 > 0$, and if $f(z)$ given by* (1.1) *belongs to $\mathscr{M}_q(\phi)$, then*

$$\begin{aligned} |a_3-\nu a_2^2| \leq & \frac{B_1}{\left([3]_q-1\right)(1-\alpha+[3]_q\alpha)} max\left\{1;\left|\frac{B_2}{B_1}+\frac{B_1\left(1-\alpha+([2]_q)^2\alpha\right)}{\left([2]_q-1\right)(1-\alpha+[2]_q\alpha)^2}\right.\right. \\ & \left.\left.\left(1-\frac{([3]_q-1)(1-\alpha+[3]_q\alpha)}{([2]_q-1)(1-\alpha+([2]_q)^2\alpha)}\sigma\right)\right|\right\}. \end{aligned}$$

The result is sharp.

Theorem 3.4. *Let* $0 \leq \alpha \leq 1$. *Also let* $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \ldots$, *where the coefficients* B_n *are real with* $B_1 > 0$ *and* $B_2 \geq 0$. *If* $f(z)$ *given by* (1.1) *belongs to* $\mathscr{M}_q^{m,\nu}(\phi)$, *then*

$$|a_3 - \nu a_2^2| \leq \begin{cases} F_1 & if \quad \nu \leq \chi_1, \\ \dfrac{B_2}{([3]_q - 1)\Omega_{k,\lambda,\delta,\mu}^{m,\nu}([3]_q)(1-\alpha+[3]_q\alpha)} & if \quad \chi_1 \leq \nu \leq \chi_2, \\ F_2 & if \quad \nu \leq \chi_2. \end{cases} \tag{3.10}$$

Where,

$$\begin{aligned} F_1 = & \frac{B_2}{(1-\alpha+[3]_q\alpha)([3]_q-1)\Omega_{k,\lambda,\delta,\mu}^{m,\nu}([3]_q)} \\ & + \frac{B_1^2}{([2]_q-1)^2}\left[\frac{([2]_q-1)(1-\alpha+([2]_q)^2\alpha)}{(1-\alpha+[3]_q\alpha)(1-\alpha+[2]_q\alpha)^2([3]_q-1)\Omega_{k,\lambda,\delta,\mu}^{m,\nu}([3]_q)} \right. \\ & \left. - \frac{\nu}{(1-\alpha+[2]_q\alpha)^2(\Omega_{k,\lambda,\delta,\mu}^{m,\nu}([2]_q))^2}\right], \end{aligned}$$

$$\begin{aligned} F_2 = & \frac{B_1^2}{([2]_q-1)^2}\left[\frac{\nu}{(1-\alpha+[2]_q\alpha)^2(\Omega_{k,\lambda,\delta,\mu}^{m,\nu}([2]_q))^2} \right. \\ & \left. - \frac{([2]_q-1)(1-\alpha+([2]_q)^2\alpha)}{(1-\alpha+[3]_q\alpha)(1-\alpha+[2]_q\alpha)^2([3]_q-1)\Omega_{k,\lambda,\delta,\mu}^{m,\nu}([3]_q)}\right] \\ & - \frac{B_2}{(1-\alpha+[3]_q\alpha)([3]_q-1)\Omega_{k,\lambda,\delta,\mu}^{m,\nu}([3]_q)} \end{aligned}$$

and

$$\chi_1 = \frac{([2]_q-1)(\Omega_{k,\lambda,\delta,\mu}^{m,\nu}([2]_q))^2}{([3]_q-1)\Omega_{k,\lambda,\delta,\mu}^{m,\nu}([3]_q)}\left[\frac{([2]_q-1)(1-\alpha+([2]_q)\alpha)^2(B_2+B_1)+(1-\alpha+([2]_q)^2\alpha)B_1^2}{B_1^2(1-\alpha+([3]_q)\alpha)}\right],$$

$$\chi_2 = \frac{([2]_q-1)(\Omega_{k,\lambda,\delta,\mu}^{m,\nu}([2]_q))^2}{([3]_q-1)\Omega_{k,\lambda,\delta,\mu}^{m,\nu}([3]_q)}\left[\frac{([2]_q-1)(1-\alpha+([2]_q)\alpha)^2(B_2-B_1)+(1-\alpha+([2]_q)^2\alpha)B_1^2}{B_1^2(1-\alpha+([3]_q)\alpha)}\right].$$

Proof. The Proof is followed by Lemma 2.5.

Using (3.8) and (3.9), we have the following cases:

Case (i): If $\nu \leq \chi_1$,

$$|a_3-\nu a_2^2| \leq \frac{B_1}{2(1-\alpha+[3]_q\alpha)([3]_q-1)\Omega^{m,\nu}_{k,\lambda,\delta,\mu}([3]_q)}[2-4\sigma]$$

$$\leq \frac{B_1}{(1-\alpha+[3]_q\alpha)([3]_q-1)\Omega^{m,\nu}_{k,\lambda,\delta,\mu}([3]_q)}\left[\frac{B_2}{B_1}+\frac{(1-\alpha+([2]_q)^2\alpha)B_1}{(1-\alpha+([2]_q\alpha)^2)([2]_q-1)}\left(1-\frac{(1-\alpha+[3]_q\alpha)([3]_q-1)\Omega^{m,\nu}_{k,\lambda,\delta,\mu}([3]_q)}{(1-\alpha+([2]_q)^2\alpha)([2]_q-1)(\Omega^{m,\nu}_{k,\lambda,\delta,\mu}([2]_q))^2}\nu\right)\right].$$

Therefore,

$$\begin{aligned}|a_3-\nu a_2^2| \leq{}& \frac{B_2}{(1-\alpha+[3]_q\alpha)([3]_q-1)\Omega^{m,\nu}_{k,\lambda,\delta,\mu}([3]_q)}+\frac{B_1^2}{([2]_q-1)^2}\\ &+\left[\frac{(1-\alpha+([2]_q)^2\alpha)([2]_q-1)}{(1-\alpha+[2]_q\alpha)^2(1-\alpha+[3]_q\alpha)([3]_q-1)\Omega^{m,\nu}_{k,\lambda,\delta,\mu}([3]_q)}\right.\\ &\left.-\frac{\nu}{(1-\alpha+[2]_q\alpha)^2(\Omega^{m,\nu}_{k,\lambda,\delta,\mu}[2]_q)^2}\right].\end{aligned} \tag{3.11}$$

Case (ii): If $\chi_1\nu \leq \chi_2$,

$$|a_3-\nu a_2^2| \leq \frac{B_1}{(1-\alpha+[3]_q\alpha)([3]_q-1)\Omega^{m,\nu}_{k,\lambda,\delta,\mu}([3]_q)}. \tag{3.12}$$

Case (iii): If $\nu \geq \chi_2$,

$$|a_3-\nu a_2^2| \leq \frac{B_1}{2(1-\alpha+[3]_q\alpha)([3]_q-1)\Omega^{m,\nu}_{k,\lambda,\delta,\mu}[3]_q}[4\sigma-2]$$

$$\leq -\frac{B_1}{(1-\alpha+[3]_q\alpha)([3]_q-1)\Omega^{m,\nu}_{k,\lambda,\delta,\mu}([3]_q)}\left[\frac{B_2}{B_1}+\frac{(1-\alpha+([2]_q)^2\alpha)B_1}{(1-\alpha+([2]_q\alpha)^2)([2]_q-1)}\left(1-\frac{(1-\alpha+[3]_q\alpha)([3]_q-1)\Omega^{m,\nu}_{k,\lambda,\delta,\mu}([3]_q)}{(1-\alpha+([2]_q)^2\alpha)([2]_q-1)(\Omega^{m,\nu}_{k,\lambda,\delta,\mu}([2]_q))^2}\right)\right].$$

Therefore,

$$\begin{aligned}|a_3-\nu a_2^2| \leq{}& -\frac{B_2}{(1-\alpha+[3]_q\alpha)([3]_q-1)\Omega^{m,\nu}_{k,\lambda,\delta,\mu}([3]_q)}-\frac{B_1^2}{([2]_q-1)^2}\\ &\left[\frac{(1-\alpha+([2]_q)^2\alpha)([2]_q-1)}{(1-\alpha+[2]_q\alpha)^2(1-\alpha+[3]_q\alpha)([3]_q-1)\Omega^{m,\nu}_{k,\lambda,\delta,\mu}([3]_q)}\right.\\ &\left.-\frac{\nu}{(1-\alpha+[2]_q\alpha)^2(\Omega^{m,\nu}_{k,\lambda,\delta,\mu}[2]_q)^2}\right].\end{aligned} \tag{3.13}$$

This completes the proof of the Theorem 3.4.

Remark 3.5. If we set $\alpha = 0$ and $\alpha = 1$, then we have the results of Theorem 10 and Theorem 11 obtained by Abdullah and Darus [1] respectively.

If we set $m = 0$ and $\nu = 0$ in Theorem 3.4, we thus obtain Fekete Szegö inequality for the subclass $\mathscr{M}_q(\phi)$:

Corollary 3.6. *Let $\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \dots$, with $B_1 > 0$, and if $f(z)$ given by* (1.1) *belongs to $\mathscr{M}_q(\phi)$, then*

$$|a_3 - \nu a_2^2| \leq \begin{cases} G_1 & if \quad \nu \leq \Psi_1, \\ \dfrac{B_2}{([3]_q - 1)(1 - \alpha + [3]_q \alpha)} & if \quad \Psi_1 \leq \nu \leq \Psi_2, \\ G_2 & if \quad \nu \leq \Psi_2. \end{cases} \tag{3.14}$$

Where,

$$G_1 = \frac{B_2}{(1-\alpha+[3]_q\alpha)([3]_q-1)} + \frac{B_1^2}{([2]_q-1)^2}\left[\frac{([2]_q-1)(1-\alpha+([2]_q)^2\alpha)}{(1-\alpha+[3]_q\alpha)(1-\alpha+[2]_q\alpha)^2([3]_q-1)} - \frac{\nu}{(1-\alpha+[2]_q\alpha)^2}\right],$$

$$G_2 = \frac{B_1^2}{([2]_q-1)^2}\left[\frac{\nu}{(1-\alpha+[2]_q\alpha)^2} - \frac{([2]_q-1)(1-\alpha+([2]_q)^2\alpha)}{(1-\alpha+[3]_q\alpha)(1-\alpha+[2]_q\alpha)^2([3]_q-1)}\right] - \frac{B_2}{(1-\alpha+[3]_q\alpha)(([3]_q-1)}.$$

And

$$\Psi_1 = \frac{([2]_q-1)}{([3]_q-1)}\left[\frac{([2]_q-1)(1-\alpha+([2]_q)\alpha)^2(B_2+B_1)+(1-\alpha+([2]_q)^2\alpha)B_1^2}{B_1^2(1-\alpha+([3]_q)\alpha)}\right],$$

$$\Psi_2 = \frac{([2]_q-1)}{([3]_q-1)}\left[\frac{([2]_q-1)(1-\alpha+([2]_q)\alpha)^2(B_2-B_1)+(1-\alpha+([2]_q)^2\alpha)B_1^2}{B_1^2(1-\alpha+([3]_q)\alpha)}\right].$$

The result is sharp.

□

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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