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CHARACTERIZATION OF A HEART-ORIENTED PARALETRIX

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Abstract. This paper presents more results in the theory of paraletrix. These results are simply a characterization of a heart-oriented paraletrix ring, which include paraletrix integral domain, paraletrix polynomial, paraletrix ring functions, differentiation and integration.

Keywords: rhotrix; matrix; paraletrix; ring.

Mathematics Subject Classification: 20M10.

1. INTRODUCTION

In [4], Atanassov and Shanaon discussed arrays of numbers that are in some way, between two-dimensional vectors and (2×2)-dimensional matrices in their paper titled matrix-tertions and noitrets. As an extension, Ajibade [1] in 2003 introduced objects which are in some ways, between (2×2)-dimensional and (3×3)-dimensional matrices. This field of science now known as rhotrix theory was defined in [1] for dimension three as:

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$$R = \left\{ \left\langle \begin{matrix} & a & \\ b & c & d \\ & e & \end{matrix} \right\rangle \, : \, a, b, c, d, e \in \mathbb{R} \right\},$$

where $c = h(R)$ is called the heart of any rhotrix R and $\mathbb{R}$ is the set of real numbers.

It is worthy to note that these heart-oriented rhotrices are always of odd dimension. Thereafter, Mohammed [10] in his PhD thesis extended the idea to rhotrix set of size n.

It is known in [1] that addition and multiplication of two heart-oriented rhotrices are as follows:

$$R + Q = \left\langle \begin{matrix} & a & \\ b & h(R) & d \\ & e & \end{matrix} \right\rangle + \left\langle \begin{matrix} & f & \\ g & h(Q) & j \\ & k & \end{matrix} \right\rangle = \left\langle \begin{matrix} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{matrix} \right\rangle$$

and $$R \circ Q = \left\langle \begin{matrix} & ah(Q) + fh(R) & \\ bh(Q) + gh(R) & h(R)h(Q) & dh(Q) + jh(R) \\ & eh(Q) + kh(R) & \end{matrix} \right\rangle$$

respectively. Furthermore, Mohammed [10] and Ezegwu et al [5] gave a generalization of this heart-oriented rhotrices.

A row-column multiplication of heart-oriented rhotrices was given by Sani [15] as:

$$R \circ Q = \left\langle \begin{matrix} & af + dg & \\ bf + eg & h(R)h(Q) & aj + dk \\ & bj + ek & \end{matrix} \right\rangle.$$

Sani [16] also gave a generalization of this row-column multiplication of heart-oriented rhotrices as:

$$R_n \circ Q_n = \langle a_{i_1 j_1}, c_{l_i k_1} \rangle \circ \langle b_{i_2 j_2}, d_{l_2 k_2} \rangle = \langle \sum_{i_2 j_{1=1}}^{t} \left(a_{i_1 j_1} b_{i_2 j_2} \right), \sum_{l_2 k_1 = 1}^{t-1} \left(c_{l_i k_1} d_{l_2 k_2} \right) \rangle \ , t = \frac{n+1}{2},$$

where R_n and Q_n denote n-dimensional rhotrices (with n rows and n columns).

Mohammed [9] classified the heart-oriented rhotrices as abstract structures of rings, fields, integral domains and unique factorization domain. The rhotrix quadratic polynomial presented as part of a note on rhotrix exponent rule and its applications in [10] was extended in [18]. Rhotrix polynomial and its extension to construction of rhotrix polynomial ring was studied in [19]. Also in [11], some construction of rhotrix semigroup was given and then extended by in [13] to rhotrix type A semigroup. The study of non-commutative full rhotrix ring and its subring was carried out by Mohammed in [12]. Patil [14] gave a characterization of ideals of rhotricesover a ring and its application.

Consequently, Isere [6, 7] introduced rhotrices without a heart, and these rhotrices were found to be even-dimensional rhotrices.

Tudunkaya and Makanjuola [17] studied rhotrices and the construction of finite fields.

The concept of paraletrix was introduced by Aminu and Michael [2] as an extension of rhotrix [1] when the number of rows is not equal to the number of columns. It is worthy to note that not all paraletrix has a heart as seen in [2].

Suppose P and P' are two 3×7 paraletrices such that the heart of the paraletrix exist;

$$P = \left\langle \begin{array}{ccccc} & a_1 & & & \\ a_2 & a_3 & a_4 & & \\ & a_5 & a_6 & a_7 & \\ & & a_8 & a_9 & a_{10} \\ & & & a_{11} & \end{array} \right\rangle, P' = \left\langle \begin{array}{ccccc} & b_1 & & & \\ b_2 & b_3 & b_4 & & \\ & b_5 & b_6 & b_7 & \\ & & b_8 & b_9 & b_{10} \\ & & & b_{11} & \end{array} \right\rangle,$$

then the multiplication and addition of two 3×7 heart-oriented paraletrices P and P' using Ajibade's multiplication of rhotrix [1] are as follows:

$$P \circ P' = \left\langle \begin{array}{ccccc} & c_1 & & & \\ c_2 & c_3 & c_4 & & \\ & c_5 & c_6 & c_7 & \\ & & c_8 & c_9 & c_{10} \\ & & & c_{11} & \end{array} \right\rangle$$

where each $c_i = a_i h(P') + b_i h(P) \ni i = 1,2,3, \dots ,11 \ni i \neq 6$ and $c_6 = h(P')h(P)$ for the heart of the paraletrix $P \circ P'$ and $h(P') = b_6$, $h(P) = a_6$ are hearts of paraletrices P' and P respectively.

$$P + P' = \left\langle \begin{array}{ccccc} & d_1 & & & \\ d_2 & d_3 & d_4 & & \\ & d_5 & d_6 & d_7 & \\ & & d_8 & d_9 & d_{10} \\ & & & d_{11} & \end{array} \right\rangle$$

where $d_i = a_i + b_i \ni i = 1,2,3, \dots ,11.$

It is important to note that the multiplication of paraletrix P and P' using Sani [15] is only possible whenever the number of columns of P is equal to the number of rows of P' for any arbitrary paraletrix.

In [9] and [12], rhotrices are classified as rings using Ajibade [1] and Sani [15] respectively.

The objective of this paper is to classify paraletrices as rings using Ajibade [1]. The result in this paper is an extension to the ones given in [9].

2. PRELIMINARIES

In this section we recall some definitions as well as some known results which will be useful in this paper. For notation and terminologies not mentioned in this paper, the reader is referred to [1], [15], [16], [7], [12] and [2] respectively.

Throughout this paper, we will use P to denote any paraletrix, while m and n are the number of rows and columns of an arbitrary paraletrix, where $m, n \epsilon (2k+1 : k \epsilon \mathbb{N})$.

Definition 2.1. Let m and n be the number of rows and columns of an arbitrary paraletrix, where $m, n \epsilon (2k+1 : k \epsilon \mathbb{N})$. An $m \times n$-dimentional heart-oriented paraletrix is of the form $P_{m\times n} = \langle a_{ij}, c_{lk} \rangle$, where $a_{ij}, c_{lk} \epsilon \mathbb{R}$ for $i = 1,2,\dots.,\frac{m+1}{2}$, $j = 1,2,\dots.,\frac{n+1}{2}$ and $= 1,2,\dots.,\frac{m-1}{2}$, $k = 1,2,\dots.,\frac{n-1}{2}$. For more information, the reader is referred to [2].

Definition 2.2. The cardinality or order of a paraletrix is defined to be the number of entries of an arbitrary paraletrix P with m number of rows and n number of columns. This is denoted by

$$O(P_{m\times n}) = \frac{1}{2}[(m \times n) + 1].$$

Lemma 2.3 [2]. Let P be an $m \times n$-dimensional paraletrix. If P has a heart then the heart is unique.

Theorem 2.4 [2]. A necessary and sufficient condition for the heart of an $m \times n$-dimensional paraletrix P to exist is that the order is odd.

Theorem 2.5 [2]. A necessary and sufficient condition for the heart of an $m \times n$-dimensional paraletrix P to exist is that $|m - n| = 4k$ where $k = 0,1,2,\dots$

Definition 2.6. Suppose P and I are two $m \times n$-dimensional and $r \times s$-dimensional heart-oriented paraletrices such that $P.I = P$ then I is said to be the identity paraletrix. The identity of a paraletrix with 3 rows and 7 columns is given by

$$I = \left\langle \begin{array}{ccccc} & 0 & & & \\ 0 & 0 & 0 & & \\ & 0 & 1 & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & \end{array} \right\rangle$$

Definition 2.7. A ring is defined as a non-empty set R with two compositions $+, \circ : R \times R \to R$ with the following properties;

i) $(R, +)$ is an abelian group (zero element 0)

ii) $(R, \circ)$ is a semigroup

iii) for all $a, b, c \in R$ the distributivity laws are satisfied;

$$(a + b)c = ac + bc, \quad a(b + c) = ab + ac.$$

It is worthy to note that a ring R such that for $a, b \in R$, $aba = a$ is referred to as a regular ring.

Definition 2.8. A subgroup I of $(R, +)$ is said to be a left ideal R if $RI \subset I$, and a right ideal if if $IR \subset I$. I is said to be an ideal if it is both a left and right ideal. I is a subring if $II \subset I$. It is worthy to note that every left or right ideal in R is also a subring of R. The intersection of (arbitrary many) (left, right) ideals is again a (left, right) ideal.

Definition 2.9. Let A, B be rings. A mapping $\varphi : A \to B$ is called a ring morphism or homomorphism if

i) $(a_1 + a_2)\varphi = a_1\varphi + a_2\varphi$

ii) $(a_1 a_2)\varphi = a_1\varphi a_2\varphi$

for all $a_1, a_2 \in A$.

Definition 2.10. Let R be a ring, we say that R is a differential ring if it is equipped with one or more derivations, that are homomorphisms of additive groups $d : R \to R$ such that each derivation d satisfies the Leibniz product rule $d(r_1 r_2) = (dr_1)r_2 + r_1(dr_2)$ for every $r_1, r_2 \in R$.

3. RING OF PARALETRICES

In this section, we apply the notions of rings in the development of new abstract structure of paraletrices with respect to the binary operations of addition (+) and multiplication $(\circ)$. The paraletrix under consideration will be a 3×7-dimensional heart-oriented paraletrix.

Now let $R^* = \langle P, +, \circ \rangle$ be an abstract structure consisting of the set P of all real a 3×7 paraletrices together with the operations of addition (+) and multiplication $(\circ)$. Let the identity of P be as in Definition 2.5 while the zero paraletrix be such that the elements of the paraletrix are all zero.

We have the following results:

Theorem 3.1. $R^* = \langle P, +, \circ \rangle$ is a ring.

Proof. It is obvious that R^* is an abelian group.

$$\text{Let } P = \left\langle \begin{array}{ccccc} & a_1 & & & \\ a_2 & a_3 & a_4 & & \\ & a_5 & a_6 & a_7 & \\ & & a_8 & a_9 & a_{10} \\ & & & a_{11} & \end{array} \right\rangle,$$

$$P' = \left\langle \begin{array}{ccccc} & b_1 & & & \\ b_2 & b_3 & b_4 & & \\ & b_5 & b_6 & b_7 & \\ & & b_8 & b_9 & b_{10} \\ & & & b_{11} & \end{array} \right\rangle,$$

$$P'' = \left\langle \begin{array}{ccccc} & u_1 & & & \\ u_2 & u_3 & u_4 & & \\ & u_5 & u_6 & u_7 & \\ & & u_8 & u_9 & u_{10} \\ & & & u_{11} & \end{array} \right\rangle \epsilon R^*,$$

to show that R^* is a semigroup, we have that

$(P \circ P') \circ P''$

$$= \left\langle \begin{array}{ccccc} & c_1 & & & \\ c_2 & c_3 & c_4 & & \\ & c_5 & c_6 & c_7 & \\ & & c_8 & c_9 & c_{10} \\ & & & c_{11} & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccc} & u_1 & & & \\ u_2 & u_3 & u_4 & & \\ & u_5 & u_6 & u_7 & \\ & & u_8 & u_9 & u_{10} \\ & & & u_{11} & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccc} & v_1 & & & \\ v_2 & v_3 & v_4 & & \\ & v_5 & v_6 & v_7 & \\ & & v_8 & v_9 & v_{10} \\ & & & v_{11} & \end{array} \right\rangle,$$

where $c_i = a_i h(P') + b_i h(P)$ for $i = 1,2,3,\ldots,11 \ni i \neq 6$, $c_6 = h(P')h(P)$, $h(P') = b_6$, $h(P) = a_6$. $v_i = c_i h(P'') + u_i h(P \circ P')$ for $i = 1,2,3,\ldots,11 \ni i \neq 6$, $v_6 = h(P \circ P')h(P'')$, $h(P \circ P') = c_6 = h(P')h(P)$, $h(P'') = u_6$.

Similarly,

$P \circ (P' \circ P'')$

$$= \left\langle \begin{array}{ccccc} & a_1 & & & \\ a_2 & a_3 & a_4 & & \\ & a_5 & a_6 & a_7 & \\ & & a_8 & a_9 & a_{10} \\ & & & a_{11} & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccc} & w_1 & & & \\ w_2 & w_3 & w_4 & & \\ & w_5 & w_6 & w_7 & \\ & & w_8 & w_9 & w_{10} \\ & & & w_{11} & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccc} & z_1 & & & \\ z_2 & z_3 & z_4 & & \\ & z_5 & z_6 & z_7 & \\ & & z_8 & z_9 & z_{10} \\ & & & z_{11} & \end{array} \right\rangle$$

where $w_i = b_i h(P'') + u_i h(P')$ for $i = 1,2,3,\ldots,11 \ni i \neq 6$, $w_6 = h(P'')h(P')$, $h(P'') = u_6$, $h(P') = b_6$.

$z_i = a_i h(P' \circ P'') + w_i h(P)$ for $i = 1,2,3,\ldots,11 \ni i \neq 6$, $z_6 = h(P)h(P' \circ P'')$, $h(P) = a_6$, $h(P' \circ P'') = w_6$.

Consequently,

$$v_i = c_i h(P'') + u_i h(P \circ P') = (a_i h(P') + b_i h(P))h(P'') + u_i h(P \circ P')$$
$$= a_i h(P' \circ P'') + b_i h(P'' \circ P) + u_i h(P \circ P').$$

$$z_i = a_i h(P' \circ P'') + w_i h(P) = a_i h(P' \circ P'') + \big(b_i h(P'') + u_i h(P')\big)h(P)$$
$$= a_i h(P' \circ P'') + b_i h(P'' \circ P) + u_i h(P \circ P').$$

Thus $(P \circ P') \circ P'' = P \circ (P' \circ P'')$ and so $R^* = \langle P, +, \circ \rangle$ is a semigroup.

Lastly, we show that the distributivity law is satisfied for $P, P', P'' \epsilon R^*$.

To do this, we have that

$(P + P') \circ P''$

$$= \left\langle \begin{array}{ccccc} & d_1 & & & \\ d_2 & d_3 & d_4 & & \\ & d_5 & d_6 & d_7 & \\ & & d_8 & d_9 & d_{10} \\ & & & d_{11} & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccc} & u_1 & & & \\ u_2 & u_3 & u_4 & & \\ & u_5 & u_6 & u_7 & \\ & & u_8 & u_9 & u_{10} \\ & & & u_{11} & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccc} & r_1 & & & \\ r_2 & r_3 & r_4 & & \\ & r_5 & r_6 & r_7 & \\ & & r_8 & r_9 & r_{10} \\ & & & r_{11} & \end{array} \right\rangle$$

where $d_i = a_i + b_i \ni i = 1,2,3, \dots ,11,$

$r_i = d_i h(P'') + u_i h(P + P')$ for $i = 1,2,3, \dots ,11 \ni i \neq 6, \; r_6 = h(P + P'')h(P''),$

$h(P + P') = a_6 + b_6, \; h(P'') = u_6.$

Similarly, we have that

$(P \circ P'') + (P' \circ P'')$

$$= \left\langle \begin{array}{ccccc} & g_1 & & & \\ g_2 & g_3 & g_4 & & \\ & g_5 & g_6 & g_7 & \\ & & g_8 & g_9 & g_{10} \\ & & & g_{11} & \end{array} \right\rangle + \left\langle \begin{array}{ccccc} & w_1 & & & \\ d_2 & d_3 & d_4 & & \\ & w_5 & w_6 & w_7 & \\ & & w_8 & w_9 & w_{10} \\ & & & w_{11} & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccc} & h_1 & & & \\ h_2 & h_3 & h_4 & & \\ & h_5 & h_6 & h_7 & \\ & & h_8 & h_9 & h_{10} \\ & & & h_{11} & \end{array} \right\rangle$$

where $g_i = a_i h(P'') + u_i h(P')$ for $i = 1,2,3, \dots ,11 \ni i \neq 6, \; g_6 = h(P)h(P''), \; h(P) = a_6,$ $h(P'') = u_6.$

$w_i = b_i h(P'') + u_i h(P')$ for $i = 1,2,3, \dots ,11 \ni i \neq 6, \; w_6 = h(P'')h(P'), h(P'') = u_6,$ $h(P') = b_6.$

$h_i = g_i + w_i \ni i = 1,2,3, \dots ,11.$

Consequently, we have that

$$\begin{aligned} r_i &= d_i h(P'') + u_i h(P + P'') = d_i u_6 + u_i(a_6 + b_6) \\ &= (a_i + b_i)u_6 + u_i(a_6 + b_6). \\ h_i &= g_i + w_i = a_i h(P'') + u_i h(P) + b_i h(P'') + u_i h(P') \\ &= a_i u_6 + u_i a_6 + b_i u_6 + u_i b_6 \\ &= (a_i + b_i)u_6 + u_i(a_6 + b_6)\,. \end{aligned}$$

Therefore, $R^* = \langle P, +, \circ \rangle$ is a ring.

It is important to note that R^* is a commutative ring with identity denoted by I_{R^*} . In R^*, it is obvious that the multiplication of nonzero paraletrices P and Q is equal to zero. Hence R^* has zero divisors and is not an integral domain.

Lemma 3.3. Let R^* be a ring and let L be an ideal of the ring R^*.

Then $T = \left\{ \left\langle \begin{array}{ccccc} & a_1 & & & \\ a_2 & a_3 & a_4 & & \\ & a_5 & a_6 & a_7 & \\ & & a_8 & a_9 & a_{10} \\ & & & a_{11} & \end{array} \right\rangle \text{ where } h(T) \epsilon L \right\}$ is an ideal in R^*.

Proof. That L is an ideal of R^* implies that T is a subset of R^* and $T \neq \emptyset$.

Let

$U = \left\langle \begin{array}{ccccc} & a_1 & & & \\ a_2 & a_3 & a_4 & & \\ & a_5 & a_6 & a_7 & \\ & & a_8 & a_9 & a_{10} \\ & & & a_{11} & \end{array} \right\rangle, V = \left\langle \begin{array}{ccccc} & a'_1 & & & \\ a'_2 & a'_3 & a'_4 & & \\ & a'_5 & a'_6 & a'_7 & \\ & & a'_8 & a'_9 & a'_{10} \\ & & & a'_{11} & \end{array} \right\rangle$

$\epsilon\, T$ and $W = \left\langle \begin{array}{ccccc} & a_1 & & & \\ a_2 & a_3 & a_4 & & \\ & a_5 & a_6 & a_7 & \\ & & a_8 & a_9 & a_{10} \\ & & & a_{11} & \end{array} \right\rangle \epsilon\, R^*$. Then $h(U), h(V) \epsilon L$ and

$h(W) \epsilon R^*, h(W).h(U),\ h(U).h(W) \epsilon L$. So we have that $U \circ W,\ W \circ U \epsilon T$. Therefore, T is an ideal in R^*.

Lemma 3.4. R^* is not regular

Proof. Let $P, P' \epsilon R^*$ be as defined in Theorem 3.1. We have that

$$P \circ P' \circ P = Q,$$

where $a_i \epsilon P, b_i \epsilon P', f_i \epsilon Q$ for $i = 1,2,3,\dots,11 \ni i \neq 6, a_6 = h(P), b_6 = h(P')$ and $f_6 = h(P \circ P')h(P)$.

$$c_i = a_i h(P') + b_i h(P) \text{ for } i = 1,2,3,\dots,11 \ni i \neq 6,\ c_6 = h(P)h(P').$$

$$f_i = c_i h(P) + a_i h(P \circ P') \quad \text{for} \quad i = 1,2,3,\dots,11 \ni i \neq 6,\ h(P \circ P') = c_6 = h(P)h(P').$$

In [8], polynomial equations are defined over variables and coefficients which are also rhotrices. We will show that analogous result holds for ring of paraletrices.

Theorem 3.5. Let $R^* = \langle P, +, \circ \rangle$ be a ring and $P, P', P'' \epsilon R^*$ be such that we have the equation

$$(h(P'))^2 - 4h(P)h(P'') \geq 0,$$

then there exist a paraletrix $X \epsilon R^*$ satisfying the equation $P \circ X^2 + P' \circ X + P'' = 0$.

Proof. Suppose we have that

$$P = \left\langle \begin{array}{cccccc} & a_1 & & & & \\ a_2 & a_3 & a_4 & & & \\ & a_5 & a_6 & a_7 & & \\ & & a_8 & a_9 & a_{10} & \\ & & & a_{11} & & \end{array} \right\rangle, P' = \left\langle \begin{array}{cccccc} & b_1 & & & & \\ b_2 & b_3 & b_4 & & & \\ & b_5 & b_6 & b_7 & & \\ & & b_8 & b_9 & b_{10} & \\ & & & b_{11} & & \end{array} \right\rangle$$

$$P'' = \left\langle \begin{array}{cccccc} & u_1 & & & & \\ u_2 & u_3 & u_4 & & & \\ & u_5 & u_6 & u_7 & & \\ & & u_8 & u_9 & u_{10} & \\ & & & u_{11} & & \end{array} \right\rangle.$$ Then we need to find two values for the paraletrix

$$X = \left\langle \begin{array}{cccccc} & x_1 & & & & \\ x_2 & x_3 & x_4 & & & \\ & x_5 & x_6 & x_7 & & \\ & & x_8 & x_9 & x_{10} & \\ & & & x_{11} & & \end{array} \right\rangle$$ satisfying the equation $P \circ X^2 + P' \circ X + P'' = 0.$

So we have that

$$\left\langle \begin{array}{ccccc} & a_1 & & & \\ a_2 & a_3 & a_4 & & \\ & a_5 & a_6 & a_7 & \\ & & a_8 & a_9 & a_{10} \\ & & & a_{11} & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccc} & x_1 & & & \\ x_2 & x_3 & x_4 & & \\ & x_5 & x_6 & x_7 & \\ & & x_8 & x_9 & x_{10} \\ & & & x_{11} & \end{array} \right\rangle^2 +$$

$$\left\langle \begin{array}{ccccc} & b_1 & & & \\ b_2 & b_3 & b_4 & & \\ & b_5 & b_6 & b_7 & \\ & & b_8 & b_9 & b_{10} \\ & & & b_{11} & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccc} & x_1 & & & \\ x_2 & x_3 & x_4 & & \\ & x_5 & x_6 & x_7 & \\ & & x_8 & x_9 & x_{10} \\ & & & x_{11} & \end{array} \right\rangle +$$

$$\left\langle \begin{array}{ccccc} & u_1 & & & \\ u_2 & u_3 & u_4 & & \\ & u_5 & u_6 & u_7 & \\ & & u_8 & u_9 & u_{10} \\ & & & u_{11} & \end{array} \right\rangle = \left\langle \begin{array}{ccccc} & 0 & & & \\ 0 & 0 & 0 & & \\ & 0 & 0 & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & \end{array} \right\rangle.$$

Now,

$$X^2 = \left\langle \begin{array}{ccccc} & x_1 & & & \\ x_2 & x_3 & x_4 & & \\ & x_5 & x_6 & x_7 & \\ & & x_8 & x_9 & x_{10} \\ & & & x_{11} & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccc} & x_1 & & & \\ x_2 & x_3 & x_4 & & \\ & x_5 & x_6 & x_7 & \\ & & x_8 & x_9 & x_{10} \\ & & & x_{11} & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccc} & x'_1 & & & \\ x'_2 & x'_3 & x'_4 & & \\ & x'_5 & x'_6 & x'_7 & \\ & & x'_8 & x'_9 & x'_{10} \\ & & & x'_{11} & \end{array} \right\rangle.$$

where $x'_i = x_i h(X) + x_i h(X) = 2x_i h(X)$ for $i = 1,2,3,\ldots,11 \ni i \neq 6$, $x'_6 = h(X)h(X) = (h(X))^2$.

$P' \circ X$

$$= \left\langle \begin{array}{ccccc} & b_1 & & & \\ b_2 & b_3 & b_4 & & \\ & b_5 & b_6 & b_7 & \\ & & b_8 & b_9 & b_{10} \\ & & & b_{11} & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccc} & x_1 & & & \\ x_2 & x_3 & x_4 & & \\ & x_5 & x_6 & x_7 & \\ & & x_8 & x_9 & x_{10} \\ & & & x_{11} & \end{array} \right\rangle =$$

$$Y=\left\langle \begin{array}{ccccc} & m_1 & & & \\ m_2 & m_3 & m_4 & & \\ & m_5 & m_6 & m_7 & \\ & & m_8 & m_9 & m_{10} \\ & & & m_{11} & \end{array} \right\rangle$$

where $m_i = b_i h(X) + x_i h(P')$ for $i = 1,2,3,\dots,11 \ni i \neq 6$, $m_6 = h(P')h(X)$.

$P \circ X^2$

$$=\left\langle \begin{array}{ccccc} & a_1 & & & \\ a_2 & a_3 & a_4 & & \\ & a_5 & a_6 & a_7 & \\ & & a_8 & a_9 & a_{10} \\ & & & a_{11} & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccc} & {x'}_1 & & & \\ {x'}_2 & {x'}_3 & {x'}_4 & & \\ & {x'}_5 & {x'}_6 & {x'}_7 & \\ & & {x'}_8 & {x'}_9 & {x'}_{10} \\ & & & {x'}_{11} & \end{array} \right\rangle =$$

$$H=\left\langle \begin{array}{ccccc} & w_1 & & & \\ w_2 & w_3 & w_4 & & \\ & w_5 & w_6 & w_7 & \\ & & w_8 & w_9 & w_{10} \\ & & & w_{11} & \end{array} \right\rangle$$

where $w_i = a_i(h(X))^2 + {x'}_i h(P)$ for $i = 1,2,3,\dots,11 \ni i \neq 6$, $w_6 = h(P)\big(h(X)\big)^2$.

So that $P \circ X^2 + P' \circ X + P'' = H + Y + P'' = 0$

$$\Rightarrow \left\langle \begin{array}{ccccc} & w_1 & & & \\ w_2 & w_3 & w_4 & & \\ & w_5 & w_6 & w_7 & \\ & & w_8 & w_9 & w_{10} \\ & & & w_{11} & \end{array} \right\rangle + \left\langle \begin{array}{ccccc} & m_1 & & & \\ m_2 & m_3 & m_4 & & \\ & m_5 & m_6 & m_7 & \\ & & m_8 & m_9 & m_{10} \\ & & & m_{11} & \end{array} \right\rangle$$

$$+ \left\langle \begin{array}{ccccc} & u_1 & & & \\ u_2 & u_3 & u_4 & & \\ & u_5 & u_6 & u_7 & \\ & & u_8 & u_9 & u_{10} \\ & & & u_{11} & \end{array} \right\rangle = Z = \left\langle \begin{array}{ccccc} & z_1 & & & \\ z_2 & z_3 & z_4 & & \\ & z_5 & z_6 & z_7 & \\ & & z_8 & z_9 & z_{10} \\ & & & z_{11} & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccc} & 0 & & & \\ 0 & 0 & 0 & & \\ & 0 & 0 & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & \end{array} \right\rangle$$

where $z_i = w_i + m_i + u_i$ for $i = 1,2,3,\dots,11 \ni i \neq 6$, $z_6 = h(P)\big(h(X)\big)^2 + h(P')h(X) + h(P'')$.

Consequently, we have that

$z_i = a_i(h(X))^2 + 2h(P)h(X)x_i + b_i h(X) + x_i h(P') + u_i = 0.$

We know from z_6 that we have a quadratic in terms of $h(X)$. The roots of the equation are:

$$h(X) = \frac{-h(P') \pm \sqrt{(h(P'))^2 - 4h(P)h(P'')}}{2h(P)},$$

where $h(P) \neq 0$ and $(h(P'))^2 - 4h(P)h(P'') \geq 0$.

From z_i we have that $x_i = -\frac{a_i(h(X))^2 + b_i h(X) + u_i}{2h(P)h(X) + h(P')}$.

By substituting $h(X)$ into x_i we obtain the values of paraletrix X satisfying the quadratic equation $P \circ X^2 + P' \circ X + P'' = 0$. Thus the theorem is proved.

In [2], it is known that we can extract 2×4 and 1×3 dimensional matrices from a 3×7-dimensional paraletrix. It is also known in [11] that a rhotrix semigroup can be embedded in a matrix semigroup. The type A version of the embedding was proved in [13]. [12] proved that a rhotrix ring can be embedded in a matrix ring.

Theorem 3.6. Let $R^* = \langle P, +, \circ \rangle$ be a paraletrix ring and $\mathbb{M} = (M_{m\times n}, +, .)$ be a matrix ring, then the map $\theta : R^* = \langle P, +, \circ \rangle \to \mathbb{M} = (M_{m\times n}, +, .)$ is not a ring morphism.

Proof. For each $P \epsilon R^*$ define a map $\theta : R^* = \langle P, +, \circ \rangle \to \mathbb{M} = (M_{m\times n}, +, .)$ by the rule

$$P\theta = \left\langle \begin{array}{ccccccc} & a_1 & & & & & \\ a_2 & a_3 & a_4 & & & & \\ & a_5 & a_6 & a_7 & & & \\ & & a_8 & a_9 & a_{10} & & \\ & & & a_{11} & & & \end{array} \right\rangle \theta = \left(\begin{bmatrix} a_1 & a_4 & a_7 & a_{10} \\ a_2 & a_5 & a_8 & a_{11} \end{bmatrix}, [a_3 \ \ a_6 \ \ a_9] \right).$$

That is θ maps each 3×7-dimensional paraletrix P to 2×4 and 1×3 dimensional matrices with the usual matrix multiplication. Obviously θ is a one-to-one map since for $P, P' \epsilon R^*$, $P\theta = P'\theta$ which implies $P = P'$. Now under addition (+) we have that $(P + P')\theta = P\theta + P'\theta$ while under multiplication ($\circ$) we have that $(P \circ P')\theta \neq P\theta \circ P'\theta$.

Thus θ is not an injective homomorphism and the result follows.

Example 3.7. Let $P = \left\langle \begin{array}{ccccc} & 1 & & & \\ 3 & 0 & 2 & & \\ & 6 & 4 & 5 & \\ & & 2 & 1 & 1 \\ & & & 3 & \end{array} \right\rangle, P' = \left\langle \begin{array}{ccccc} & 3 & & & \\ 2 & 1 & 6 & & \\ & 4 & 0 & 1 & \\ & & 2 & 3 & 6 \\ & & & 5 & \end{array} \right\rangle \epsilon\ R^*$.

We have that $(P + P')\theta = \left[\left\langle \begin{array}{ccccc} & 1 & & & \\ 3 & 0 & 2 & & \\ & 6 & 4 & 5 & \\ & & 2 & 1 & 1 \\ & & & 3 & \end{array} \right\rangle + \left\langle \begin{array}{ccccc} & 3 & & & \\ 2 & 1 & 6 & & \\ & 4 & 0 & 1 & \\ & & 2 & 3 & 6 \\ & & & 5 & \end{array} \right\rangle \right] \theta$

$$= \left\langle \begin{array}{ccccc} & 4 & & & \\ 5 & 1 & 8 & & \\ & 10 & 4 & 6 & \\ & & 4 & 4 & 7 \\ & & & 8 & \end{array} \right\rangle \theta = \left(\begin{bmatrix} 4 & 8 & 6 & 7 \\ 5 & 10 & 4 & 8 \end{bmatrix}, [1\ \ 4\ \ 4] \right).$$

$$P\theta + P'\theta = \left(\begin{bmatrix} 1 & 2 & 5 & 1 \\ 3 & 6 & 2 & 3 \end{bmatrix}, [0\ \ 4\ \ 1] \right) + \left(\begin{bmatrix} 3 & 6 & 1 & 6 \\ 2 & 4 & 2 & 5 \end{bmatrix}, [1\ \ 0\ \ 3] \right) = \left(\begin{bmatrix} 4 & 8 & 6 & 7 \\ 5 & 10 & 4 & 8 \end{bmatrix}, [1\ \ 4\ \ 4] \right)$$

$$\Longrightarrow (P + P')\theta = P\theta + P'\theta.$$

Similarly,

$$(P \circ P')\theta = \left[\left\langle \begin{array}{ccccc} & 1 & & & \\ 3 & 0 & 2 & & \\ & 6 & 4 & 5 & \\ & & 2 & 1 & 1 \\ & & & 3 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccc} & 3 & & & \\ 2 & 1 & 6 & & \\ & 4 & 0 & 1 & \\ & & 2 & 3 & 6 \\ & & & 5 & \end{array} \right\rangle \right] \theta$$

$$= \left[\left\langle \begin{array}{ccccc} & 12 & & & \\ 8 & 4 & 24 & & \\ & 16 & 0 & 4 & \\ & & 8 & 12 & 24 \\ & & & 20 & \end{array} \right\rangle \right] \theta = \left(\begin{bmatrix} 12 & 24 & 4 & 24 \\ 8 & 16 & 8 & 20 \end{bmatrix}, [4\ \ 0\ \ 12] \right).$$

$$P\theta\,.\,P'\theta = \left\langle \begin{array}{ccccc} & 1 & & & \\ 3 & 0 & 2 & & \\ & 6 & 4 & 5 & \\ & & 2 & 1 & 1 \\ & & & 3 & \end{array} \right\rangle \theta\,.\left\langle \begin{array}{ccccc} & 3 & & & \\ 2 & 1 & 6 & & \\ & 4 & 0 & 1 & \\ & & 2 & 3 & 6 \\ & & & 5 & \end{array} \right\rangle \theta$$

$$= \left(\begin{bmatrix} 1 & 2 & 5 & 1 \\ 3 & 6 & 2 & 3 \end{bmatrix}, [0\ \ 4\ \ 1] \right)\ .\ \left(\begin{bmatrix} 3 & 6 & 1 & 6 \\ 2 & 4 & 2 & 5 \end{bmatrix}, [1\ \ 0\ \ 3] \right)$$

$$= \left(\begin{bmatrix} 1 & 2 & 5 & 1 \\ 3 & 6 & 2 & 3 \end{bmatrix}\ .\ \begin{bmatrix} 3 & 6 & 1 & 6 \\ 2 & 4 & 2 & 5 \end{bmatrix}, [0\ \ 4\ \ 1]\ .\ [1\ \ 0\ \ 3] \right)$$

which is undefined. Thus $(P \circ P')\theta \neq P\theta \,.\, P'\theta$ so that $\theta : R^* = \langle P, +, \circ \rangle \to \mathbb{M} = (M_{m\times n}, +, .)$ is not a ring morphism.

4. DIFFERENTIATION OF R^*

In [3], differentiation and integration of rhotrices was presented. In this section, we will prove that analogous result holds for ring of paraletrices.

Let the elements of the paraletrix P be functions of a variable t, then the paraletrix is called a paraletrix functions of t. In this case we can rewrite R^* as

$$R^* = \langle P(t), +, \circ \rangle$$

$$\Longrightarrow \frac{d}{dt} R^* = \langle \frac{d}{dt} P(t), +, \circ \rangle,$$

and its nth order derivative with respect to t is defined as

$$\frac{d^n}{dt^n} R^* = \langle \frac{d^n}{dt^n} P(t), +, \circ \rangle, \quad \text{where } n = 1,2,3, \ldots$$

For a 3×7-dimensional paraletrix, we have that

$$\frac{d}{dt} P(t) = \left\langle \begin{array}{ccccc} & D_1 & & & \\ D_2 & D_3 & D_4 & & \\ & D_5 & D_6 & D_7 & \\ & & D_8 & D_9 & D_{10} \\ & & & D_{11} & \end{array} \right\rangle$$

where $D_i = \frac{d}{dt} a_i(t), \quad i = 1,2,3, \ldots, 11.$

With this we have the following results.

Lemma 4.1. Let R^* be a paraletrix ring such that $P(t), P'(t) \in R^*$. Then

$$\frac{d}{dt} \langle P(t) + P'(t) \rangle = \frac{d}{dt} P(t) + \frac{d}{dt} P'(t).$$

Proof. It follows from the definition of the sum of two paraletrices.

Lemma 4.2. Let R^* be a paraletrix ring such that $P(t), P'(t) \in R^*$. Then

$$\frac{d}{dt} \langle P(t) \circ P'(t) \rangle = P(t) \circ \frac{d}{dt} P'(t) + P'(t) \circ \frac{d}{dt} P(t).$$

Proof. It follows from the multiplication of two paraletrices as defined by Ajibade [1].

Example 4.3. Let $P(t) = \left\langle \begin{array}{ccccc} & & 2t^2 & & \\ & 4 & \sin t & \cos t & \\ & & 3t & 1 & e^{2t} \\ & & & 4t & -\cos t \quad \sin t \\ & & & & 2 \end{array} \right\rangle,$

$$P(t) = \left\langle \begin{array}{ccccc} & & t^3 & & \\ & 1 & 2 & \sin t & \\ & & e^t & \cos t & 3 \\ & & & 2t & 3t \quad 4 \\ & & & & t \end{array} \right\rangle$$ $\epsilon\ R^*$, then we have that

$$P(t) \circ P'(t) = \left\langle \begin{array}{ccccc} & & f_1 & & \\ & f_2 & f_3 & f_4 & \\ & & f_5 & f_6 & f_7 \\ & & & f_8 & f_9 \quad f_{10} \\ & & & & f_{11} \end{array} \right\rangle$$

where $f_1 = 2t^2 \cos t + t^3$, $f_2 = 4\cos t + 1$, $f_3 = \sin t \cos t + 2$, $f_4 = cos^2 t + \sin t$,

$f_5 = 3t\cos t + e^t$, $f_6 = \cos t$, $f_7 = e^{2t}\cos t + 3$, $f_8 = 4t\cos t + 2t$,

$f_9 = -cos^2 t + 3t$, $f_{10} = \sin t \cos t + 4$, $f_{11} = 2\cos t + t$.

$$\frac{d}{dt}P(t) = \left\langle \begin{array}{ccccc} & & 4t & & \\ & 0 & cost & -\sin t & \\ & & 3 & 0 & 2e^{2t} \\ & & & 4 & \sin t \quad \cos t \\ & & & & 0 \end{array} \right\rangle,$$

$$\frac{d}{dt}P'(t) = \left\langle \begin{array}{ccccc} & & 3t^2 & & \\ & 0 & 0 & \cos t & \\ & & e^t & -\sin t & 0 \\ & & & 2 & 3 \quad 0 \\ & & & & 1 \end{array} \right\rangle.$$

$$P(t) \circ \frac{d}{dt}P'(t) = \left\langle \begin{array}{ccccc} & & g_1 & & \\ & g_2 & g_3 & g_4 & \\ & & g_5 & g_6 & g_7 \\ & & & g_8 & g_9 \quad g_{10} \\ & & & & g_{11} \end{array} \right\rangle$$

where $g_1 = -2t^2 \sin t + 3t^2$, $g_2 = -4\sin t$, $g_3 = -sin^2 t$, $g_4 = -\sin t \cos t + \cos t$,

$g_5 = -3t\sin t + e^t$, $g_6 = -\sin t$, $g_7 = -e^{2t}\sin t$, $g_8 = -4t\sin t + 2$,

$g_9 = \sin t \cos t + 3$, $g_{10} = -sin^2 t$, $g_{11} = -2\sin t + 1$.

$$P'(t) \circ \frac{d}{dt} P(t) = \left\langle \begin{array}{ccccc} & v_1 & & & \\ v_2 & v_3 & v_4 & & \\ & v_5 & v_6 & v_7 & \\ & & v_8 & v_9 & v_{10} \\ & & & v_{11} & \end{array} \right\rangle$$

where $v_1 = 4t\cos t$, $v_2 = 0$, $v_3 = cos^2 t$, $v_4 = -\sin t \cos t$,

$v_5 = 3\cos t$, $v_6 = 0$, $v_7 = 2e^{2t}\cos t$, $v_8 = 4\cos t$,

$v_9 = \sin t \cos t$, $v_{10} = cos^2 t$, $v_{11} = 0$.

$$P(t) \circ \frac{d}{dt} P'(t) + P'(t) \circ \frac{d}{dt} P(t) = \frac{d}{dt} \left\langle \begin{array}{ccccc} & y_1 & & & \\ y_2 & y_3 & y_4 & & \\ & y_5 & y_6 & y_7 & \\ & & y_8 & y_9 & y_{10} \\ & & & y_{11} & \end{array} \right\rangle = \frac{d}{dt}(P \circ P'),$$

where $y_1 = 2t^2\cos t + t^3$, $y_2 = 4\cos t + 1$, $y_3 = \sin t \cos t + 2$, $y_4 = cos^2 t + \sin t$,

$y_5 = 3t\cos t + e^t$, $y_6 = \cos t$, $y_7 = e^{2t}\cos t + 3$, $y_8 = 4t\cos t + 2t$,

$y_9 = -cos^2 t + 3t$, $y_{10} = \sin t \cos t + 4$, $y_{11} = 2\cos t + t$.

Remark 4.4. From Lemma 4.2, it is clear that the ring R^* is a differential ring.

5. REPRESENTATION

In this section, we shall show that an arbitrary ring can be represented as a 3×7-dimensional heart-oriented paraletrix. In particular, it will be shown that an arbitrary ring of numbers is isomorphic to a 3×7-dimensional heart-oriented paraletrix. This result can be considered as an analogous case of Cayley's theorem in ring theorem.

Theorem 5.1. Let $R = (H, +, .)$ be an arbitrary ring of numbers under the usual operation of addition (+) and multiplication $(.)$ with $H = \{h_1, h_2, h_3, ...\}$ as underlying set of numbers in tabular form. Then $R = (H, +, .)$ can be represented as a heart-oriented paraletrix ring $R^* = \langle P(H), +, \circ \rangle$ such that

$$P(H) = \left\{ \left\langle \begin{array}{ccccc} & 0 & & & \\ 0 & 0 & 0 & & \\ & 0 & h_1 & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccccc} & 0 & & & \\ 0 & 0 & 0 & & \\ & 0 & h_2 & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & \end{array} \right\rangle, \left\langle \begin{array}{ccccc} & 0 & & & \\ 0 & 0 & 0 & & \\ & 0 & h_3 & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & \end{array} \right\rangle \right\}$$

is the underlying paraletrix set in tabular form over the same arbitrary ring R.

Proof. For $H \epsilon R$ define a map $\sigma : R \to R^*$ by the rule that

$$h_j\sigma = \left\langle \begin{array}{ccccc} & 0 & & & \\ 0 & 0 & 0 & & \\ & 0 & h_j & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & \end{array} \right\rangle, \ j = 1,2,3, \ldots.$$

It is obvious that the map is well defined. It is also one-to-one since for $h_1, h_2 \epsilon H$, $h_1\sigma = h_2\sigma$ which implies that $h_1 = h_2$. That σ is a homomorphism follows from the fact that

$$(h_1 + h_2)\sigma = \left\langle \begin{array}{ccccc} & 0 & & & \\ 0 & 0 & 0 & & \\ & 0 & h' & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccc} & 0 & & & \\ 0 & 0 & 0 & & \\ & 0 & h_1 & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & \end{array} \right\rangle + \left\langle \begin{array}{ccccc} & 0 & & & \\ 0 & 0 & 0 & & \\ & 0 & h_2 & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & \end{array} \right\rangle$$

$$= h_1\sigma + h_2\sigma \,.$$

$$(h_1 \,.\, h_2)\sigma = \left\langle \begin{array}{ccccc} & 0 & & & \\ 0 & 0 & 0 & & \\ & 0 & h'' & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccccc} & 0 & & & \\ 0 & 0 & 0 & & \\ & 0 & h_1 & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & \end{array} \right\rangle \circ \left\langle \begin{array}{ccccc} & 0 & & & \\ 0 & 0 & 0 & & \\ & 0 & h_2 & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & \end{array} \right\rangle$$

$$= h_1\sigma \circ h_2\sigma$$

where $h' = h_1 + h_2$, $h'' = h_1 \,.\, h_2$.

Furthermore, we have that

$$Im\,\sigma = \left\{ \left\langle \begin{array}{ccccc} & 0 & & & \\ 0 & 0 & 0 & & \\ & 0 & h_j & 0 & \\ & & 0 & 0 & 0 \\ & & & 0 & \end{array} \right\rangle = h_j\sigma \ \epsilon \ P(H) \right\} \subseteq P(H)\ .$$

That is, each element in H has an image in $P(H)$. So the image of σ under H is the whole of the heart-oriented paraletrix ring. Thus, $H\sigma = P(H)$. Hence, σ is an isomorphism for any arbitrary ring R to the heart-oriented paraletrix ring R^*. The proof is then complete.

Lemma 5.2. Let $R = (\mathbb{Z}, +,\ .)$ be the ring of all integer numbers under the usual operation of addition (+) and multiplication $(.)$. Then $R \cong R^* = \langle P(\mathbb{Z}), +,\ \circ \rangle$.

Proof. It follows from Theorem 5.1 with $H = \mathbb{Z}$ and $P(H) = P(\mathbb{Z})$.

Remark 5.3. Theorem 5.1 holds for ring of all residue integer numbers modulo prime p and the ring of all rational numbers.

6. SUMMARY

This paper considered classification of a paraletrix as an abstract structure as a follow up of known results on rhotrices.

This work showed that the results in rhotrix theory should necessarily hold in paraletrix theory. However, it is scholastic to examine further results of the heart-oriented paraletrices vis a vis the heart-oriented rhotices. These results will contribute greatly in paraletrix algebra. The introduction of a heartless paraletrix will be presented in another paper.

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CONFLICT OF INTEREST

The authors declare that there is no conflict of interest.

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