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A NEW THREE-STEP ITERATIVE METHOD WITH HIGH ORDER OF CONVERGENCE FOR NON-LINEAR EQUATIONS

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Abstract. This research presents new three-step iterative method with order of convergence at last seventh for solving nonlinear equations. Per iteration, the new method requires three functions and one first derivative evaluations. The efficiency index of this method is $7^{\frac{1}{4}} \approx 1.627$. Numerical examples are given to illustrate the performance of the presented method and this is considered another method for solving nonlinear equations.

Keywords: nonlinear equations; iterative method; order of convergence.

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1. Introduction

Many problems in applied mathematics and engineering fields are reduced to finding the solution of a nonlinear equation $f(x) = 0$ and required the employment of an iterative method. In this work we are concerned with iterative methods to find a simple root x^*, i.e., $f(x^*) = 0$ and $f'(x^*) \neq 0$ of a nonlinear equation $f(x) = 0$ that uses f and f' and divided difference method but not the higher derivatives of f. There are many iterative methods such as Newton's method and its variants, Secant method, Halley's method, Euler-Chebyshev method, Jarratt's method

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and modified Jaratt's method. Among these methods Newton's method is the most widely used method for the calculation of x^*, which is defined by

$$x_{n+1} = \frac{f(x_n)}{f'(x_n)} \tag{1}$$

where x_0 is an initial approximation sufficiently close to x^*. It is well known that this method is quadratically convergent [1].

Many researchers developed efficient modifications of existing iterative methods such as those mentioned above in a number of ways to improve their order of convergence at the expense of additional evaluations of functions and/or derivatives mostly at the point iterated by the method, see [2]-[10] and the references therein. All these modifications are targeted at increasing the order of convergence with a view of increasing the efficiency of the method. Most of these focused on modifications of Newton's method [2]-[8], the others focused on modifications of Jaratt's method [2]

In 1994, Argyros et al. [2] studied the Jaratt's method which was fourth order iterative method defined by:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} \\ J_f(x_n) &= \frac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)} \\ x_{n+1} &= x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)} \end{aligned} \tag{2}$$

In 2007, Chun [5] modified Jaratt's method which was sixth order iterative method defined by:

$$\begin{aligned} y_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)} \\ J_f(x_n) &= \frac{3f'(y_n)+f'(x_n)}{6f'(y_n)-2f'(x_n)} \\ z_n &= x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)} \\ x_{n+1} &= z_n - \frac{f(z_n)}{f'(z_n)} \end{aligned} \tag{3}$$

In 2011, Heydari [10] studied three step iterative method which was eight order iterative method defined by:

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\
z_n &= y_n - \frac{f(y_n)}{f'(x_n)}\left[1+2\frac{f(y_n)}{f(x_n)}+4\left(\frac{f(y_n)}{f(x_n)}\right)^2\right] \\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}\left[\left(\frac{f(y_n)}{f(x_n)}\right)^4+\left(\frac{f(y_n)}{f(x_n)}\right)^5\right] \\
&\quad - \frac{\left[1+\frac{f(z_n)}{f(x_n)}+\left(\frac{f(z_n)}{f(x_n)}\right)^2\right] f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}
\end{aligned} \tag{4}
$$

In 2012, Babajee et al. [3] died three step iterative method which was eight order iterative method defined by:

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}A(\delta) \\
z_n &= y_n - \frac{f(y_n)}{f'(x_n)}\left(1-\frac{f(y_n)}{f'(x_n)}\right)^{-2} \\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}\left(1-\frac{f(y_n)}{f'(x_n)}-\frac{f(z_n)}{f'(x_n)}\right)^{-2} G(\gamma)+H(\mu)
\end{aligned} \tag{5}
$$

where $\delta = \frac{f(x_n)}{f'(x_n)}, \gamma = \frac{f(y_n)}{f(x_n)}$ and $\mu = \frac{f(z_n)}{f(y_n)}$ and $A(\delta), G(\gamma)$ and $H(\mu)$ were functions.

In this paper, we consider the algorithm of three-step iterative as follows:

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)}\left[1+\left(\frac{f(x_n)}{f'(x_n)}\right)^5\right] \\
z_n &= y_n - \frac{f(y_n)}{f'(x_n)}\left[\frac{1+\beta\left(\frac{f(y_n)}{f(x_n)}\right)^2}{1-2\frac{f(y_n)}{f(x_n)}}\right] \\
x_{n+1} &= z_n - \frac{f(z_n)}{f'(x_n)}\left[\left(\frac{f(y_n)}{f(x_n)}\right)^4+\left(\frac{f(y_n)}{f(x_n)}\right)^5\right] \\
&\quad - \frac{\left[1+\frac{f(z_n)}{f(x_n)}+\left(\frac{f(z_n)}{f(x_n)}\right)^2\right] f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}
\end{aligned} \tag{6}
$$

2. Iterative method and Convergence Analysis

Consider the three-step iterative algorithm defined by equation (6). The order of convergence could be analyzed as following theorem.

Theorem 2.1. *Let α be a simple zero of sufficient differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. If x_0 is sufficient closed to α, then the three-step iterative method defined by Algorithm* (6) *has seventh order of convergence and satisfy error equation:*

$$\begin{aligned} e_{n+1} &= (\beta c_2^6 - \beta^2 c_2^6 - \beta c_3 c_2^4) e_n^7 \\ &\quad + (-12\beta^2 c_3 - c_2^5 - c_3 c_4 c_2^2 - 8\beta c_3^2 c_2^3 + 28\beta c_3 c_2^5 \\ &\quad + 18\beta^2 c_2^7 + c_3 c_2^4 + 4c_3^2 c_2^3 - 6c_3 c_2^5 + 2c_2^7 - 3\beta c_4 c_2^7 \\ &\quad - 3\beta c_4 c_2^4 - 16\beta c_2^7) e_n^8 + O(e_n^9) \end{aligned} \tag{7}$$

where $e_n = x_n - \alpha, c_k = \dfrac{f^{(k)}(\alpha)}{k! f'(\alpha)}$ and $k = 2, 3, \ldots$

Proof. Let α be a simple zero of sufficient differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$. Since f is sufficient differentiable function, by Taylor's expanding of f about α, we get

$$\begin{aligned} f(x) &= f(\alpha) + f'(\alpha)(x-\alpha) + \tfrac{f''(\alpha)}{2!}(x-\alpha)^2 \\ &\quad + \frac{f^{(3)}(\alpha)}{3!}(x-\alpha)^3 + \frac{f^{(4)}(\alpha)}{4!}(x-\alpha)^4 + \ldots \end{aligned} \tag{8}$$

Substituting $x = x_n$, we get

$$\begin{aligned} f(x_n) &= f(\alpha) + f'(\alpha)(x_n-\alpha) + \tfrac{f''(\alpha)}{2!}(x_n-\alpha)^2 \\ &\quad + \frac{f^{(3)}(\alpha)}{3!}(x_n-\alpha)^3 + \frac{f^{(4)}(\alpha)}{4!}(x_n-\alpha)^4 + \ldots \\ &= f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 \\ &\quad + c_7 e_n^7 + c_8 e_n^8 + c_9 e_n^9 + c_{10} e_n^{10} + \cdots] \end{aligned} \tag{9}$$

where $e_n = x_n - \alpha, c_k = \dfrac{f^{(k)}(\alpha)}{k!f'(\alpha)}$ and $k = 2,3,\ldots$. Calculating first derivative of f with respect to x, and substituting $x = x_n$ we have

$$
\begin{aligned}
f'(x_n) &= f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 \\
&\quad +7c_7e_n^6 + 8c_8e_n^7 + 9c_9e_n^8 + 10c_{10}e_n^9 + \cdots]
\end{aligned} \tag{10}
$$

Substituting $f(x_n)$ and $f'(x_n)$ into $\dfrac{f(x_n)}{f'(x_n)}$ we get that

$$
\begin{aligned}
\frac{f(x_n)}{f'(x_n)} &= e_n - c_2e_n^2 + (2c_2^2 - 2c_3)e_n^3 + (-4c_2^3 + 7c_2c_3 - 3c_4)e_n^4 + (8c_2^4 - 20c_2^2c_3 \\
&\quad +10c_2c_4 + 6c_2^3 - 4c_5)e_n^5 + (-16c_2^5 + 52c_2^3c_3 - 28c_2^2c_4 - 33c_2c_3^2 \\
&\quad +13c_2c_5 + 17c_3c_4 - 5c_6)e_n^6 + (22c_3c_5 - 92c_2c_3c_4 - 18c_3^3 + 126c_3^2c_2^2 \\
&\quad -128c_3c_2^4 + 12c_4^2 + 72c_4c_2^3 - 36c_5c_2^2 - 6c_7 + 16c_2c_6 + 32c_2^6)e_n^7 \\
&\quad +(348c_3c_4c_2^2 - 118c_2c_3c_5 - 7c_8 + 19c_2c_7 + 31c_4c_5 - 64c_2c_4^2 - 75c_4c_3^2 \\
&\quad -176c_4c_2^4 + 92c_5c_2^3 + 27c_3c_6 - 44c_6c_2^2 + 135c_2c_3^3 - 408c_3^2c_2^3 + 304c_3c_2^5 \\
&\quad -64c_2^7)e_n^8 + \ldots
\end{aligned}
$$

Substituting $\dfrac{f(x_n)}{f'(x_n)}$ into $y_n = x_n - \dfrac{f(x_n)}{f'(x_n)}\left[1 + \left(\dfrac{f(x_n)}{f'(x_n)}\right)^5\right]$, we get that

$$
\begin{aligned}
y_n &= \Big(e_n + \alpha\Big) - e_n + c_2e_n^2 + (-2c_2^2 + 2c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 + (1 - 8c_2^4 \\
&\quad +20c_2^2c_3 - 10c_2c_4 - 6c_2^3 - 4c_5)e_n^5 + (16c_2^5 - 52c_2^3c_3 + 28c_2^2c_4 + 33c_2c_3^2 \\
&\quad -13c_2c_5 - 17c_3c_4 + 5c_6)e_n^6 + (-22c_3c_5 + 92c_2c_3c_4 + 18c_3^3 - 126c_3^2c_2^2 \\
&\quad +128c_3c_2^4 - 12c_4^2 - 72c_4c_2^3 + 36c_5c_2^2 + 6c_7 - 16c_2c_6 - 32c_2^6)e_n^7 \\
&\quad +(-348c_3c_4c_2^2 + 118c_2c_3c_5 + 7c_8 - 19c_2c_7 - 31c_4c_5 + 64c_2c_4^2 + 75c_4c_3^2 \\
&\quad +176c_4c_2^4 - 92c_5c_2^3 - 27c_3c_6 + 44c_6c_2^2 - 135c_2c_3^3 + 408c_3^2c_2^3 - 304c_3c_2^5 \\
&\quad +64c_2^7)e_n^8 + \ldots
\end{aligned}
$$

Substituting $x = y_n$ into equation (8), we have

$$\begin{aligned} f(y_n) &= f'(\alpha)\Big[-c_2e_n^2 + (2c_2^2 - 2c_3)e_n^3 + (-4c_2^3 + 7c_2c_3 - 3c_4)e_n^4 + (8c_2^4 - 20c_2^2c_3 \\ &+10c_2c_4 + 6c_3^2 - 4c_5)e_n^5 + (-16c_2^5 + 52c_2^3c_3 - 28c_2^2c_4 - 33c_2c_3^2 \\ &+13c_2c_5 + 17c_3c_4 - 5c_6)e_n^6 + (22c_3c_5 - 92c_2c_3c_4 - 18c_3^3 + 126c_3^2c_2^2 \\ &-128c_3c_2^4 + 12c_4^2 + 72c_4c_2^3 - 36c_5c_2^2 - 6c_7 + 16c_2c_6 + 32c_2^6)e_n^7 \\ &+(348c_3c_4c_2^2 - 118c_2c_3c_5 - 7c_8 + 19c_2c_7 + 31c_4c_5 - 64c_2c_4^2 - 75c_4c_3^2 \\ &-176c_4c_2^4 + 92c_5c_2^3 + 27c_3c_6 - 44c_6c_2^2 + 135c_2c_3^3 - 408c_3^2c_2^3 + 304c_3c_2^5 \\ &-64c_2^7)e_n^8 + \dots\Big] \end{aligned}$$

and we have that

$$\begin{aligned} \frac{f(y_n)}{f'(x_n)} &= \frac{f'(\alpha)[-c_2e_n^2 + (2c_2^2 - 2c_3)e_n^3 + (7c_2c_3 - 3c_4 - 4c_2^3)e_n^4 + \cdots]}{f'(\alpha)[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + 6c_6e_n^5 + 7c_7e_n^6 + \cdots]} \\ &= c_2e_n^2 + (2c_3 - 4c_2^2)e_n^3 + (13c_2^3 + 3c_4 - 14c_2c_3)e_n^4 + (64c_3c_2^2 - 38c_2^4 + 4c_5 \\ &-20c_2c_4 - 12c_3^2)e_n^5 + (-1 + 5c_6 - 26c_2c_5 - 34c_3c_4 + 90c_4c_2^2 + 103c_2c_3^2 \\ &-240c_3c_2^3 + 104c_2^5)e_n^6 + (288c_2c_3c_4 - 32c_2c_6 + 8c_2 + 6c_7 - 44c_3c_5 \\ &-558c_2^2c_3^2 + 800c_3c_2^4 - 24c_4^2 - 336c_4c_2^3 + 116c_5c_2^2 + 54c_3^3 - 272c_2^6)e_n^7 \\ &+(3700c_2c_3c_5 - 1557c_3c_4c_2^2 + 15c_3 + 7c_8 - 45c_2^2 - 38c_2c_7 - 62c_4c_5 \\ &+201c_2c_4^2 + 225c_4c_3^2 + 1121c_4c_2^4 + 6888c_2^7 - 431c_5c_2^3 - 54c_3c_6 + 142c_6c_2^2 \\ &-564c_2c_3^3 + 2418c_3^3c_2^3 - 2464c_3c_2^5)e_n^8 + \dots \end{aligned}$$

and

$$\begin{aligned} \frac{f(y_n)}{f(x_n)} &= \frac{f'(\alpha)[-c_2e_n^2 + (2c_2^2 - 2c_3)e_n^3 + (7c_2c_3 - 3c_4 - 4c_2^3)e_n^4 + \cdots]}{f'(\alpha)[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + c_6e_n^6 + c_7e_n^7 + \cdots]} \\ &= c_2e_n + (2c_3 - 3c_2^2)e_n^2 + (8c_2^3 + 3c_4 - 10c_2c_3)e_n^3 + (37c_3c_2^2 - 20c_2^4 + 4c_5 \\ &- 14c_2c_4 - 8c_3^2)e_n^4 + (-1 + 5c_6 - 18c_2c_5 - 22c_3c_4 + 51c_4c_2^2 + 55c_2c_3^2 \\ &- 118c_3c_2^3 + 48c_2^5)e_n^5 + (150c_2c_3c_4 - 22c_2c_6 + 7c_2 + 6c_7 - 22c_3c_5 - 252c_3^2c_2^2 \\ &+ 344c_3c_2^4 - 15c_4^2 - 163c_4c_2^3 + 65c_5c_2^2 + 26c_3^3 - 112c_2^6)e_n^6 + (190c_2c_3c_5 \qquad (11) \\ &- 693c_3c_4c_2^2 + 13c_3 + 7c_8 - 36c_2^2 - 26c_2c_7 - 38c_4c_5 + 102c_2c_4^2 + 105c_4c_3^2 \\ &+ 408c_4c_2^4 + 256c_2^7 - 207c_5c_2^3 - 34c_3c_6 + 79c_6c_2^2 - 228c_2c_3^3 + 952c_3^2c_2^3 \end{aligned}$$

$$+944c_3c_2^5)e_n^7+(2660c_3c_4c_2^3-936c_2c_4c_3^2-876c_3c_5c_2^2+258c_2c_4c_5$$
$$+230c_2c_3c_6-126c_2c_3-24c_5^2+19c_4+8c_9-30c_2c_8+156c_2^3-72c_3^4-576c_2^8$$
$$+132c_5c_3^2+607c_5c_2^4-40c_3c_7+93c_7c_2^2-46c_4c_6-251c_6c_2^3+141c_3c_4^2$$
$$-477c_2^2c_4^2-1336c_4c_2^5+1254c_2^2c_3^3-3200c_3^2c_2^4+2480c_3c_2^6)e_n^8+\dots$$

Since $z_n = y_n - \dfrac{f(y_n)}{f'(x_n)}\left[\dfrac{1+\beta\left(\frac{f(y_n)}{f(x_n)}\right)^2}{1-2\left(\frac{f(y_n)}{f(x_n)}\right)}\right]$, then

$$\begin{aligned} z_n &= \Big(e_n+\alpha\Big)-e_n+(-\beta c_2^3+c_2^3-c_2c_3)e_n^4+(-6\beta c_3c_2^2+8\beta c_2^4+8c_3c_2^2-4c_2^4 \\ &-2c_2c_4-2c_3^2)e_n^5+(-12\beta c_2c_3^2+58\beta c_3c_2^3-9\beta c_4c_2^2-40\beta c_2^5-3c_2c_5-7c_3c_4 \\ &+12c_4c_2^2+18c_2c_3^2-30c_3c_2^3+10c_2^5)e_n^6+(-36\beta c_2c_3c_4-8\beta c_3^3+160\beta c_2^6 \\ &+156\beta c_2^2c_3^2+84\beta c_4c_2^3-12\beta c_5c_2^2-4c_2c_6-340\beta c_3c_2^4-10c_3c_5-80c_3^2c_2^2 \\ &+80c_3c_2^4-6c_4^2-40\beta c_4c_2^3+16c_5c_2^2+12c_3^3-20c\beta c_2^6+52c_2c_3c_4)e_n^7 \\ &+(68c_2c_3c_5-209c_3c_4c_2^2-1141\beta c_3^2c_2^3+1562\beta c_3c_2^5+184\beta c_2c3^3-36\beta c_4c_3^2 \\ &+110\beta c_5c_2^3-483\beta c_4c_2^4-15\beta c_6c_2^2-27\beta c_2c_4^2-516\beta c_2^7-5c_2c_7+450\beta c_3c_4c_2^2 \\ &-48\beta c_2c_3c_5-17c_4c_5+37c_2c_4^2+50c_4c_3^2+101c_4c_2^4-51c_5c_2^3-13c_3c_6+20c_6c_2^2 \\ &-91c_2c_3^3+252c_3^2c_2^3-178c_3c_2^5+36c_2^7)e_n^8+\dots. \end{aligned}$$

So

$$\begin{aligned} z_n-\alpha &= (-\beta c_2^3+c_2^3-c_2c_3)e_n^4+(-6\beta c_3c_2^2+8\beta c_2^4+8c_3c_2^2-4c_2^4-2c_2c_4 \\ &-2c_3^2)e_n^5+(-12\beta c_2c_3^2+58\beta c_3c_2^3-9\beta c_4c_2^2-40\beta c_2^5-3c_2c_5-7c_3c_4 \\ &+12c_4c_2^2+18c_2c_3^2-30c_3c_2^3+10c_2^5)e_n^6+(-36\beta c_2c_3c_4-8\beta c_3^3+160\beta c_2^6 \\ &+156\beta c_2^2c_3^2+84\beta c_4c_2^3-12\beta c_5c_2^2-4c_2c_6-340\beta c_3c_2^4-10c_3c_5-80c_3^2c_2^2 \\ &+80c_3c_2^4-6c_4^2-40\beta c_4c_2^3+16c_5c_2^2+12c_3^3-20c\beta c_2^6+52c_2c_3c_4)e_n^7 \\ &+(68c_2c_3c_5-209c_3c_4c_2^2-1141\beta c_3^2c_2^3+1562\beta c_3c_2^5+184\beta c_2c3^3-36\beta c_4c_3^2 \\ &+110\beta c_5c_2^3-483\beta c_4c_2^4-15\beta c_6c_2^2-27\beta c_2c_4^2-516\beta c_2^7-5c_2c_7+450\beta c_3c_4c_2^2 \\ &-48\beta c_2c_3c_5-17c_4c_5+37c_2c_4^2+50c_4c_3^2+101c_4c_2^4-51c_5c_2^3-13c_3c_6+20c_6c_2^2 \\ &-91c_2c_3^3+252c_3^2c_2^3-178c_3c_2^5+36c_2^7)e_n^8+\dots. \end{aligned} \tag{12}$$

Substituting $x = z_n$ into equation (8), we have

$$
\begin{aligned}
f(z_n) &= f'(\alpha)[(z_n-\alpha)+c_2(z_n-\alpha)^2+c_3(y_n-\alpha)^3+c_4(z_n-\alpha)^4 \\
&\quad +c_5(z_n-\alpha)^5+c_6(z_n-\alpha)^6+c_7(z_n-\alpha)^7+c_8(z_n-\alpha)^8 \\
&\quad +c_9(z_n-\alpha)^9+c_{10}(z_n-\alpha)^{10}+\cdots].
\end{aligned} \tag{13}
$$

Substituting $z_n - \alpha$ from equation (12) into equation (13), we have

$$
\begin{aligned}
f(z_n) \; &= f'(\alpha)\Big[(-\beta c_2^3+c_2^3-c_2c_3)e_n^4+(-6\beta c_3c_2^2+8\beta c_2^4+8c_3c_2^2-4c_2^4-2c_2c_4 \\
&-2c_3^2)e_n^5+(-12\beta c_2c_3^2+58\beta c_3c_2^3-9\beta c_4c_2^2-40\beta c_2^5-3c_2c_5-7c_3c_4 \\
&+12c_4c_2^2+18c_2c_3^2-30c_3c_2^3+10c_2^5)e_n^6+(-36\beta c_2c_3c_4-8\beta c_3^3+160\beta c_2^6 \\
&+156\beta c_2^2c_3^2+84\beta c_4c_2^3-12\beta c_5c_2^2-4c_2c_6-340\beta c_3c_2^4-10c_3c_5-80c_3^2c_2^2 \\
&+80c_3c_2^4-6c_4^2-40\beta c_4c_2^3+16c_5c_2^2+12c_3^3-20c\beta c_2^6+52c_2c_3c_4)e_n^7 \\
&+(68c_2c_3c_5-209c_3c_4c_2^2-1141\beta c_3^2c_2^3+1562\beta c_3c_2^5+184\beta c_2c3^3-36\beta c_4c_3^2 \\
&+110\beta c_5c_2^3-483\beta c_4c_2^4-15\beta c_6c_2^2-27\beta c_2c_4^2-516\beta c_2^7-5c_2c_7+450\beta c_3c_4c_2^2 \\
&-48\beta c_2c_3c_5-17c_4c_5+37c_2c_4^2+50c_4c_3^2+101c_4c_2^4-51c_5c_2^3-13c_3c_6+20c_6c_2^2 \\
&-91c_2c_3^3+252c_3^2c_2^3-178c_3c_2^5+36c_2^7)e_n^8+\dots\Big].
\end{aligned}
$$

Then we get that

$$
\begin{aligned}
\frac{f(z_n)}{f'(x_n)} \; &= (-\beta c_2^3+c_2^3-c_2c_3)e_n^4+(-6\beta c_3c_2^2+10\beta c_3c_2^4+10c_3c_2^2-6c_2^4-2c_2c_4-2c_3^2)e_n^5 \\
&+(-9\beta c_4c_2^2+73\beta c_3c_2^3-12\beta c_2c_3^2-60\beta c_2^5+22c_2^5-3c_2c_5-7c_3c_4+16c_4c_2^2 \\
&+25c_2c_3^2-53c_3c_2^3)e_n^6+(198\beta c_2^2c_3^2+106\beta c_4c_2^3-12\beta c_5c_2^2-10c_3c_5+2c_2+18c_3^3 \\
&-64c_2^6-4c_2c_6+22c_5c_2^2-76c_4c_2^3-160c_3^2c_2^2+204c_3c_2^4-6c_4^2+76c_2c_3c_4 \\
&-36\beta c_2c_3c_4-516\beta c_3c_2^4+280\beta c_2^6-8\beta c_3^3)e_n^7+(-449c_3c_4c_2^2-15\beta c_6c_2^2 \\
&+139\beta c_5c_2^3+236\beta c_2c_3^3-36\beta c_4c_3^2-27\beta c_2c_4^2-1756\beta c_3^2c_2^3+2776\beta c_3c_2^5 \\
&+102c_2c_3c_5-17c_4c_5+\beta^2c_2^7+5c_3-18c_2^2-5c_2c_7+57c_2v_4^2+79c_4c_3^2+277c_4c_2^4 \\
&-100c_5c_2^3-13c_3c_6+28c_6c_2^2-202c_2c_3^3+732c_3^2c_2^3-654c_3c_2^5+165c_2^7+3\beta c_2^2 \\
&+573\beta c_3c_4c_2^2-48\beta c_2c_3c_5-735\beta c_4c_2^4-1123\beta c_2^7)e_n^8+\dots
\end{aligned}
$$

and

$$\begin{aligned}
\frac{f(z_n)}{f(x_n)} &= (-\beta c_2^3 + c_2^3 - c_2c_3)e_n^3 + (-6\beta c_3c_2^2 + 9\beta c_2^4 + 9c_3c_2^2 - 5c_2^4 - 2c_2c_4 - 2c_3^2)e_n^4 \\
&+(-9\beta c_4c_2^2 + 65\beta c_3c_2^3 - 12\beta c_2c_3^2 - 49\beta c_2^5 + 15c_2^5 - 3c_2c_5 - 7c_3c_4 + 14c_4c_2^2 \\
&+21c_2c_3^2 - 40c_3c_2^3)e_n^5 + (174\beta c_2^2c_3^2 + 94\beta c_4c_2^3 - 12\beta c_5c_2^2 - 10c_3c_5 + 2c_2 \\
&+14c_3^3 - 35c_2^6 - 4c_2c_6 + 19c_5c_2^2 - 55c_4c_2^3 - 110c_3^2c_2^2 + 125c_3c_2^4 - 6c_4^2 + 62c_2c_3c_4 \\
&-36\beta c_2c_3c_4 - 414\beta c_3c_2^4 + 209\beta c_2^6 - 8\beta c_3^3)e_n^6 + (-294c_3c_4c_2^2 - 15\beta c_6c_2^2 \\
&+123\beta c_5c_2^3 + 204\beta c_2c_3^3 - 36\beta c_4c_3^2 - 27\beta c_2c_4^2 - 1380\beta c_3^2c_2^3 + 2027\beta c_3c_2^5 \\
&+82c_2c_3c_5 - 17c_4c_5 + \beta^2c_2^7 + 5c_3 - 16c_2^2 - 5c_2c_7 + 45c_2c_4^2 + 59c_4c_3^2 + 161c_4c_2^4 \\
&-71c_5c_2^3 - 13c_3c_6 + 24c_6c_2^2 - 126c_2c_3^3 + 403c_3^2c_2^3 - 320c_3c_2^5 + 72c_2^7 + 3\beta c_2^2 \\
&+501\beta c_3c_4c_2^2 - 48\beta c_2c_3c_5 - 586\beta c_4c_2^4 - 772\beta c_2^7)e_n^7 + (1021c_3c_4c_2^3 + 12\beta^2c_3c_2^6 \\
&+118c_2c_4c_5 - 492c_2c_4c_3^2 - 2264\beta c_3^3c_2^2 + 102c_2c_3c_6 - 54\beta c_3c_4^2 + 8080\beta c_3^2c_2^4 \\
&-8474\beta c_3c_2^6 - 18\beta c_7c_2^2 + 12\beta c_2c_3 + 152\beta c_6c_2^3 - 758\beta c_5c_2^4 + 360\beta c_4^2c_2^2 \\
&+2837\beta c_4c_2^5 - 48\beta c_5c_3^2 - 374c_3c_5c_2^2 - 12c_5^2 - 17\beta^2c_2^8 + 8c_4 + 82c_2^3 - 65c_2c_3 \\
&-6c_2c_8 + 76c_5c_3^2 + 204c_5c_2^4 - 16c_3c_7 + 29c_7c_2^2 - 22c_4c_6 - 87c_6c_2^3 + 81c_3c_4^2 \\
&-193c_4^2c_2^2 - 402c_4c_2^5 + 636c + 3^3c_2^2 - 1214c_3^2c_2^4 + 747c_3c_2^6 - 52c_3^4 - 144c_2^8 \\
&-37\beta c_2^3 + 876\beta c_2c_4c_3^2 + 654\beta c_3c_5c_2^2 - 60\beta c_2c_3c_6 - 72\beta c_2c_4c_5 + 88\beta c_3^4 \\
&+2600\beta c_2^8 - 3888\beta c_3c_4c_2^3)e_n^8 + \ldots.
\end{aligned}$$

Since $f[x_n, y_n] = \dfrac{f(y_n) - f(x_n)}{y_n - x_n}$, we have

$$\begin{aligned}
f[x_n, y_n] &= 1 + c_2e_n + (c_3 + c_2^2)e_n^2 + (c_4 - 2c_2^3 + 3c_2c_3)e_n^3 + (c_5 - 8c_3c_2^2 + 4c_2^4 + 4c_2c_4 \\
&+2c_3^2)e_n^4 + (c_6 - 8c_2^5 + 5c_2c_5 + 5c_3c_4 - 11c_4c_2^2 - 9c_2c_3^2 + 20c_3c_2^3)e_n^5 \\
&+(6c_3c_5 + c_7 - c_2 - 2c_3^3 + 16c_2^6 + 6c_2c_6 - 14c_5c_2^2 + 29c_4c_2^3 + 31c_3^2c_2^2 \\
&-48c_3c_2^4 + 3c_4^2 - 24c_2c_3c_4)e_n^6 + (92c_3c_4c_2^2 - 30c_2c_3c_5 + c_8 + 7c_4c_5 - c_8 \\
&+6c_2^2 + 7c_2c_7 - 16c_2c_4^2 - 7c_4c_3^2 - 74c_4c_2^4 + 37c_5c_2^3 + 7c_3c_6 - 17c_6c_2^2 \\
&+11c_2c_3^3 - 94c_3^2c_2^3 + 112c_3c_2^5 - 32c_2^7)e_n^7 + (-313c_3c_4c_2^3 - 40c_2c_4c_5 \\
&+56c_2c_4c_3^2 - 36c_2c_3c_6 + 116c_3c_5c_2^2 + 4c_5^2 - c_4 + c_9 - 27c_2^3 + 16c_2c_3 + 8c_2c_8 \\
&-8c_5c_3^2 - 93c_5c_2^4 + 8c_3c_7 - 20c_7c_2^2 + 8c_4c_6 + 45c_6c_2^3 - 8c_3c_4^2 + 64c_4^2c_2^2 \\
&+184c_4c_2^5 - 42c_3^3c_2^2 + 264c_3^2c_2^4 - 256c_3c_2^6 - 6c_3^4 + 64c_2^8)e_n^8 + \ldots.
\end{aligned}$$

Since $f[x_n,z_n]=\dfrac{f(z_n)-f(x_n)}{z_n-x_n}$ and $f[y_n,z_n]=\dfrac{f(z_n)-f(y_n)}{z_n-y_n}$, we have

$$\begin{aligned}
f[x_n,z_n] &= 1+c_2e_n+c_3e_n^2+c_4e_n^3+(c_5-\beta c_2^4+c_2^4-c_3c_2^2)e_n^4\\
&+(c_6-7\beta c_3c_2^3+8\beta c_2^5+9c_3c_2^3-4c_2^5-2c_4c_2^2-3c_2c_3^2)e_n^5\\
&+(-18\beta c_2^2c_3^2+66\beta c_3c_2^4+26c_3^2c_2^2-34c_3c_2^4-10c_2c_3c_4\\
&-2c_3^3-10\beta c_4c_2^3-40\beta c_6+10c_2^6-3c_5c_2^2+13c_4c_2^3+c_7)e_n^6\\
&+(72c_3c_4c_2^2-13\beta c_5c_2^3-20\beta c_2c_3^3+214\beta c_3^2c_2^3-380\beta c_3c_2^5\\
&-14c_2c_3c_5+c_8+2c_2^2-8c_2c_4^2-9c_4c_3^2-44c_4c_2^4+17c_5c_2^3\\
&-4c_6c_2^2+30c_2c_3^3-110c_3^2c_2^3+90c_3c_2^5-20c_2^7-51\beta c_3c_4c_2^2\\
&+92\beta c_4c_2^4+160\beta c_2^7)e_n^7+(-279c_3c_4c_2^3+\beta^2c_3c_2^6-22c_2c_4c_5\\
&+120c_2c_4c_3^2+340\beta c_3^3c_2^2-18c_2c_3c_6-1479\beta c_3^2c_2^4+1720\beta c_3c_2^6\\
&-16\beta c_6c_2^3+118\beta c_5c_2^4-36\beta c_4^2c_2^2-523\beta c_4c_2^5+92c_3c_5c_2^2\\
&+c_9-14c_2^3+7c_2c_3-12c_5c_3^2-55c_5c_2^4-5c_7c_2^2+21c_6c_2^3\\
&-13c_3c_4^3+49c_4^2c_2^2+11c_4c_2^5-170c_3^3c_2^2+330c_3^2c_2^4\\
&-197c_3c_2^6+12c_3^4+36c_2^8+3\beta c_2^3-84\beta c_2c_4c_3^2\\
&-66\beta c_3c_5c_2^2-8\beta c_3^4-561\beta c_2^8+592\beta c_3c_4c_2^3)e_n^8+\ldots.
\end{aligned}\tag{14}$$

and

$$\begin{aligned}
f[y_n,z_n] &= 1+c_2^2e_n^2+(2c_2c_3-2c_2c_2^2)e_n^3+(5c_2c_2^3+c_2c_4\\
&-7c_3c_2^2-\beta c_2^4)e_n^4+(-6\beta c_3c_2^3+8c_2^5-12c_2^5+4c_2c_5\\
&-6c_4c_2^2-4c_2c_3^2+24c_3c_2^3)e_n^5+(-12\beta c_2^2c_3^2-9\beta c_4c_2^3\\
&-c_2+4c_3^3+26c_2^6+5c_2c_6-16c_5c_2^2+41c_4c_2^3\\
&+28c_3^2c_2^2-69c_3c_2^4-18c_2c_3c_4+57\beta c_3c_2^4-40\beta c_2^6)e_n^6\\
&+(116c_3c_4c_2^2+12c_4c_3^2-8\beta c_2c_3^3+148\beta c_3^2c_2^3\\
&-330\beta c_3c_2^5-12\beta c_5c_2^3+84\beta c_4c_2^4-24c_2c_3c_5\\
&+8c_2^2+52c_5c_2^3+6c_2c_7-20c_6c_2^2\\
&-18c_2c_4^2-118c_4c_2^4-14c_2c_3^3-110c_3^2c_2^3+170c_3c_2^5\\
&-52c_2^7+160\beta c_2^7-36\beta c_4c_2^2)e_n^7+\ldots,
\end{aligned}\tag{15}$$

respectively. Since $x_{n+1}=z_n-\frac{f(z_n)}{f'(x_n)}\left[\left(\frac{f(y_n)}{f(x_n)}\right)^4+\left(\frac{f(y_n)}{f(x_n)}\right)^5\right]-\frac{\left[1+\frac{f(z_n)}{f(x_n)}+\left(\frac{f(z_n)}{f(x_n)}\right)^2\right]f[x_n,y_n]f(z_n)}{f[x_n,z_n]f[y_n,z_n]}$, then we have

$$\begin{aligned} x_{n+1} &= \alpha+(-\beta^2c_2^6-\beta c_3c_2^4+\beta c_2^6)e_n^7+(-12\beta^2c_3-c_2^5-c_3c_4c_2^2 \\ &\quad -8\beta c_3^2c_2^3+28\beta c_3c_2^5+18\beta^2c_2^7+c_4c_2^4+4c_3^2c_2^3-6c_3c_2^5+2c_2^7 \\ &\quad -3\beta c_4c_2^7-3\beta c_4c_2^4-16\beta c_2^7)e_n^8+O(e_n^9) \end{aligned} \tag{16}$$

From (16) and $e_{n+1}=x_{n+1}-\alpha$, then we have the error equation

$$\begin{aligned} e_{n+1} &= (-\beta^2c_2^6-\beta c_3c_2^4+\beta c_2^6)e_n^7+(-12\beta^2c_3 \\ &\quad -c_2^5-c_3c_4c_2^2-8\beta c_3^2c_2^3+28\beta c_3c_2^5+18\beta^2c_2^7 \\ &\quad +c_4c_2^4+4c_3^2c_2^3-6c_3c_2^5+2c_2^7-3\beta c_4c_2^7 \\ &\quad -3\beta c_4c_2^4-16\beta c_2^7)e_n^8+O(e_n^9). \end{aligned} \tag{17}$$

This shows that a new algorithm defined by equation (6) has seventh order of convergence. □

Remark 2.1. *If $\beta=0$, then the algorithm defined by equation* (6) *has eighth order of convergence.*

3. Main Results

All computations were completed using Maple 7 by using 350 digit floating arithmetic. The criteria $|x_n-x_{n-1}|\leq\varepsilon$ and $|f(x_n)|\leq\varepsilon$ are used for stopping the computer program.
The following functions are used to commare numerical solutions, x_* is an exact solution and $\varepsilon=10^{-64}$

$f_1(x)=10xe^{-x^2}-1$	$x_*=1.6796306104284499$
$f_2(x)=\cos(x)-x$	$x_*=0.7390851332151606$
$f_3(x)=(x-1)^3-2$	$x_*=2.2599210498948731$
$f_4(x)=(x+2)e^x-1$	$x_*=-0.4428544010023885$
$f_5(x)=\sin^2(x)-x^2+1$	$x_*=1.4044916482153412$
$f_6(x)=\sin^2(x)-0.5x$	$x_*=1.8954942670339809$
$f_7(x)=\ln(x^2+x+2)-x+1$	$x_*=4.1525907367571582$

Numerical results are compared with the other methods. All of them have same seventh order of convergence and efficiency index 1.627. For example:

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\
z_n &= y_n - \frac{f(x_n)+\beta f(y_n)}{f(x_n)+(\beta-2)f(y_n)}\frac{f(y_n)}{f'(x_n)} \\
x_{n+1} &= z_n - \frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}
\end{aligned} \tag{18}
$$

defined by Bi, W. et.al. (G_7) ([4]).

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\
H_2(x_n,y_n) &= \frac{f(y_n)}{f(x_n)-2f(y_n)} \\
z_n &= y_n - H_2(x_n,y_n)(x_n-y_n) \\
x_{n+1} &= z_n - \left[(1+H_2(x_n,y_n))^2+\frac{f(z_n)}{f(y_n)}\right]\frac{f'(z_n)}{f(y_n)}
\end{aligned} \tag{19}
$$

defined by Kou, J. et.al .(GK_7) ([7]) and

$$
\begin{aligned}
y_n &= x_n - \frac{f(x_n)}{f'(x_n)} \\
z_n &= x_n - \frac{f(x_n)+f(y_n)}{f'(x_n)} - 2\frac{f(x_n)}{f'(x_n)}\frac{f(x_n)}{f(x_n)-f(y_n)} \\
x_{n+1} &= z_n - \frac{f(z_n)}{f[z_n,y_n]+f[z_n,x_n,x_n](z_n-y_n)}
\end{aligned} \tag{20}
$$

defined by Cordero, A. et.al. (MK_7) ([6]).

Table 1 Comparison of the iteative methods

function	method	IT	$\lvert f(x_n)\rvert$	$\lvert x_n - x_{n-1}\rvert$
$f_1(x) = 10xe^{-x^2} - 1, x_0 = 1$	G_7	4	0	0.851473e-157
	GK_7	4	0	0.315155e-143
	MK_7	4	0	0.114856e-152
	New	4	0	0.166845e-173
$f_2(x) = \cos(x) - x, x_0 = 1$	G_7	div	-	-
	GK_7	div	-	-
	MK_7	div	-	-
	New	3	0	0.686307e-64
$f_3(x) = (x-1)^3 - 2, x_0 = 3$	G_7	4	0.1e-348	0.660602e-157
	GK_7	4	0.1e-348	0.140013e-120
	MK_7	4	0.1e-348	0.300896e-202
	New	4	0.1e-348	0.6693345e-152
$f_4(x) = (x+2)e^x - 1, x_0 = 1$	G_7	5	0.1e-349	0.297211e-228
	GK_7	4	0.1e-349	0.124825e-66
	MK_7	div	-	-
	New	4	0.1e-349	0.302019e-75
$f_5(x) = \sin^2(x) - x^2 + 1, x_0 = 1$	G_7	4	0.1e-348	0.893253e-152
	GK_7	4	0.1e-348	0.124825e-66
	MK_7	4	0.1e-348	0.647245e-78
	New	4	0.1e-349	0.243985e-68

Table 1 (Cont.)

function	method	IT	$\lvert f(x_n)\rvert$	$\lvert x_n - x_{n-1}\rvert$
$f_6(x) = \sin^2(x) - 0.5x, x_0 = 2$	G_7	div	-	-
	GK_7	4	0.3e-349	0
	MK_7	div	-	-
	New	4	0.3e-349	0.177148e-64
$f_7(x) = ln(x^2 + x + 2) - x + 1, x_0 = 4$	G_7	3	0	0.115469e-89
	GK_7	3	0	0.114168e-85
	MK_7	3	0	0.986371e-88
	New	3	0	0.260709e-99

4. Conclusion and Discussion

In this paper, a three-step iterative method for solving nonlinear equations has been developed. It has been proved that the method is of the seventh order of convergence. Based on calculations, the proposed method is also compared with various other iterative methods of the same order of convergence and efficiency index 1.627. From Table 1, the performance of the new method can be seen and it is better than the other methods.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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