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FIXED POINTS OF ALMOST GENERALIZED $\mathscr{Z}_s$-CONTRACTIONS WITH RATIONAL EXPRESSIONS IN S-METRIC SPACES

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Abstract. In this paper, we introduce the notion of almost generalized $\mathscr{Z}_s$-contraction with rational expressions and α-admissible almost generalized $\mathscr{Z}_s$-contraction with rational expressions using simulation functions in S-metric spaces. We prove the existence of fixed points of such mappings in complete S-metric spaces. We give examples in support of our results.

Keywords: S-metric space; $\mathscr{Z}$-contraction; simulation function; $\mathscr{Z}_s$-contraction; almost generalized $\mathscr{Z}_s$-contraction with rational expressions; α-admissible almost generalized $\mathscr{Z}_s$-contraction with rational expressions.

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1. Introduction

In 1975, Dass and Gupta [14] introduced a contraction condition involving rational expressions and established the existence of fixed points in complete metric spaces. Later, in 1977,

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Jaggi [19] introduced another contraction condition involving rational expressions which is different from Dass and Gupta's contraction and proved fixed point results. For more works on contraction conditions involving rational expressions, we refer [4], [8], [9], [13], [18]. In 2012, Samet, Vetro, Vetro [25] introduced α-admissible maps to develop fixed results. After this, various α-admissible contraction conditions were developed specifying its significance in developing fixed point results. Some of the references in this direction are [2], [5], [20], [21]. On the other hand, in 2012, Sedghi, Shobe and Aliouche [26] introduced S-metric space and studied its properties. Later, various fixed point results on S-metric spaces were developed ([6], [10], [15], [16], [24], [27]).

Recently, in 2015, Khojasteh, Shukla, and Radenović [22] introduced $\mathscr{Z}$-contraction by using a new class of simulation functions which generalizes the Banach contraction. Following this domain of research, many authors introduced $\mathscr{Z}$-contractions involving simulation functions and proved fixed point results in various metric spaces, for more works we refer to [1], [3], [7], [12], [17]. Recently, in 2019, Mlaiki, Özgür, and Nihal Taş [23] introduced $\mathscr{Z}_s$-contraction by using the simulation function and proved the existence and uniqueness of fixed points of such mapping in complete S-metric spaces.

In Section 2, we present preliminaries that are required to develop our main results. Inspired by the works of Khojasteh, Shukla, and Radenović [22], Mlaiki, Özgür, and Nihal Taş [23], in Section 3, we introduce almost generalized $\mathscr{Z}_s$-contraction with rational expressions and prove the existence and uniqueness of fixed points of such mappings. In Section 4, we introduce α-admissible almost generalized $\mathscr{Z}_s$-contraction with rational expressions and prove the existence of fixed points of such mappings in complete S-metric spaces. We draw some corollaries and give examples in support of our results.

2. PRELIMINARIES

Khojasteh, Shukla and Radenović [22] introduced simulation functions and defined $\mathscr{Z}$- contraction with respect to a simulation function as follows.

Definition 2.1. [22] Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be a mapping, then ζ is called a simulation function if it satisfies the following conditions:

(ζ_1) $\zeta(0,0) = 0$;

(ζ_2) $\zeta(t,s) < s - t$ for all $t, s > 0$;

(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

We denote the set of all simulation functions by $\mathscr{Z}$. The following are examples of simulation functions.

Example 2.2. ([7],[22],[23]) Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by

(i) $\zeta(t,s) = \lambda s - t$ for all $t, s \in [0,\infty)$, where $\lambda \in [0,1)$.

(ii) $\zeta(t,s) = \frac{s}{1+s} - t$ for all $t, s \in [0,\infty)$.

(iii) $\zeta(t,s) = s - kt$ for all $t, s \in [0,\infty)$, where $k > 1$.

(iv) $\zeta(s,t) = \frac{s}{1+s} - te^t$ for all $t, s \in [0,\infty)$.

(v) $\zeta(t,s) = s - \varphi(s) - t$ for all $s, t \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.

Definition 2.3. [26] Let X be a nonempty set. An S-metric on X is a function $S : X^3 \to [0,\infty)$ that satisfies the following conditions: for each $x, y, z, a \in X$

(S1) $S(x,y,z) \geq 0$,

(S2) $S(x,y,z) = 0$ if and only if $x = y = z$ and

(S3) $S(x,y,z) \leq S(x,x,a) + S(y,y,a) + S(z,z,a)$.

The pair (X,S) is called an S-metric space.

Throughout this paper, we denote the set of all reals by $\mathbb{R}$, the set of all natural numbers by $\mathbb{N}$.

Example 2.4. [26] Let (X,d) be a metric space. Define $S : X^3 \to [0,\infty)$ by $S(x,y,z) = d(x,y) + d(x,z) + d(y,z)$ for all $x, y, z \in X$. Then S is an S-metric on X and S is called the S-metric induced by the metric d.

Example 2.5. [16] Let $X = \mathbb{R}$ and let $S(x,y,z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in X$. Then (X,S) is an S-metric space.

Example 2.6. [27] Let $\mathbb{R}$ be the real line. Then $S(x,y,z) = |x-z| + |y-z|$ for all $x,y,z \in \mathbb{R}$ is an S-metric on $\mathbb{R}$. This S-metric is called the usual S-metric on $\mathbb{R}$.

Lemma 2.7. [26] In an S-metric space, we have $S(x,x,y) = S(y,y,x)$.

Lemma 2.8. [16] Let (X,S) be an S-metric space. Then $S(x,x,z) \leq 2S(x,x,y) + S(y,y,z)$.

Definition 2.9. [26] Let (X,S) be an S-metric space.

(i) A sequence $\{x_n\} \subseteq X$ is said to converge to a point $x \in X$ if $S(x_n,x_n,x) \to 0$ as $n \to \infty$. That is, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n,x_n,x) < \varepsilon$ and we denote it by $\lim_{n\to\infty} x_n = x$.

(ii) A sequence $\{x_n\} \subseteq X$ is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n,x_n,x_m) < \varepsilon$ for all $n,m \geq n_0$.

(iii) An S-metric space (X,S) is said to be complete if each Cauchy sequence in X is convergent.

Lemma 2.10. [26] Let (X,S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x, then x is unique.

Lemma 2.11. [26] Let (X,S) be an S-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$, then $\lim_{n\to\infty} S(x_n,x_n,y_n) = S(x,x,y)$.

Lemma 2.12. [10] Let (X,S) be an S-metric space. Let $\{x_n\}$, $\{y_n\}$ be two sequences in X and $\{x_n\}$ converges to x in X. Then $\lim_{n\to\infty} S(x_n,x_n,y_n) = \lim_{n\to\infty} S(x,x,y_n)$.

Definition 2.13. [26] Let (X,S) be an S-metric space. A mapping $F: X \to X$ is said to be a contraction if there exists a constant $0 \leq K < 1$ such that

$$S(Fx,Fx,Fy) \leq KS(x,x,y), \text{ for all } x,y \in X. \tag{2.1}$$

Theorem 2.14. [26] *Let (X,S) be a complete S-metric space and $F: X \to X$ a contraction. Then F has a unique fixed point in X.*

Lemma 2.15. ([6], [15]) *Let (X,S) be an S-metric space and $\{x_n\}$ a sequence in X such that*

$$\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = 0.$$

If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and two sequences $\{m_k\}$ and $\{n_k\}$ of positive integers with $m_k > n_k > k$ such that

$S(x_{m_k},x_{m_k},x_{n_k}) \geq \varepsilon$ with $S(x_{m_k-1},x_{m_k-1},x_{n_k}) < \varepsilon$.

Also, we have the following:

$(i) \lim_{k\to\infty} S(x_{m_k},x_{m_k},x_{n_k}) = \varepsilon \qquad (ii) \lim_{k\to\infty} S(x_{m_k-1},x_{m_k-1},x_{n_k}) = \varepsilon$

$(iii) \lim_{k\to\infty} S(x_{m_k},x_{m_k},x_{n_k-1}) = \varepsilon \qquad (iv) \lim_{k\to\infty} S(x_{m_k-1},x_{m_k-1},x_{n_k-1}) = \varepsilon.$

3. Almost Generalized $\mathscr{Z}_s$-Contractions with Rational Expressions

Definition 3.1. Let (X,S) be an S-metric space. Let $T: X \to X$ be a mapping. If there exist a $\zeta \in \mathscr{Z}$ and $L \geq 0$ such that

$$\zeta(S(Tx,Ty,Tz), M(x,y,z) + LN(x,y,z)) \geq 0 \tag{3.1}$$

for all $x,y,z \in X$, where $M(x,y,z) = \max\{S(x,y,z), \frac{S(y,y,Ty)[1+S(x,x,Tx)]}{1+S(x,y,z)}, \frac{S(z,z,Tz)[1+S(x,x,Tx)]}{1+S(x,y,z)}, \frac{S(z,z,Tz)[1+S(y,y,Ty)]}{1+S(x,y,z)}, \frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)}, \frac{1}{3}\frac{[S(z,z,Ty)+S(y,y,Tz)][1+S(z,z,Tx)]}{1+S(x,y,z)}\}$ and $N(x,y,z) = \min\{S(x,x,Tx), S(y,y,Tx), S(z,z,Tx), \frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)}\}$. Then T is called an almost generalized $\mathscr{Z}_s$-contraction with rational expressions.

Lemma 3.2. *Let (X,S) be an S-metric space. If T is an almost generalized $\mathscr{Z}_s$-contraction with rational expressions and T has a fixed point, then the fixed point is unique.*

Proof. Suppose that $x,y \in X$ are two fixed points of T such that $x \neq y$. By using (3.1), we get

$$0 \leq \zeta(S(Tx,Tx,Ty), M(x,x,y) + LN(x,x,y)), \tag{3.2}$$

where

$$M(x,x,y) = \max\{S(x,x,y), \frac{S(x,x,Tx)[1+S(x,x,Tx)]}{1+S(x,x,y)}, \frac{S(y,y,Ty)[1+S(x,x,Tx)]}{1+S(x,x,y)}, \frac{1}{3}\frac{[S(y,y,Tx)+S(x,x,Ty)][1+S(y,y,Tx)]}{1+S(x,x,y)}\}$$
$$= \max\{S(x,x,y), 0, \frac{1}{3}\frac{[S(y,y,x)+S(x,x,y)][1+S(y,y,x)]}{1+S(x,x,y)}\} = S(x,x,y)$$

and $N(x,x,y) = \min\{S(x,x,Tx), S(y,y,Tx), \frac{S(x,x,Tx)[1+S(x,x,Tx)]}{1+S(x,x,y)}\} = 0.$

Now from (3.2) and by using (ζ_2), we get

$0 \leq \zeta(S(x,x,y),S(x,x,y)) < S(x,x,y) - S(x,x,y) = 0,$

a contradiction. Therefore $x = y$. □

Theorem 3.3. *Let (X,S) be a complete S-metric space and $T : X \to X$ be an almost generalized $\mathscr{Z}_s$-contraction with rational expressions with respect to ζ, then T has a unique fixed point in X, and the sequence $\{x_n\}$ defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$ is Cauchy in X, $\lim\limits_{n\to\infty} x_n = u$ (say) in X and u is a fixed point of T in X.*

Proof. Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined as $x_n = Tx_{n-1}$ for all $n \in N$.
If $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ for some n_0, then x_{n_0} is a fixed point of T.
Therefore we assume that $x_n \neq x_{n+1}$ i.e., $S(x_n,x_n,x_{n+1}) > 0$ for all $n \geq 0$.
STEP 1: We now prove that $\lim\limits_{n\to\infty} S(x_n,x_n,x_{n+1}) = 0$.
From (3.1), we have

$$0 \leq \zeta(S(Tx_{n-1},Tx_{n-1},Tx_n), M(x_{n-1},x_{n-1},x_n) + LN(x_{n-1},x_{n-1},x_n)), \tag{3.3}$$

where

$$\begin{aligned} M(x_{n-1},x_{n-1},x_n) &= \max\{S(x_{n-1},x_{n-1},x_n), \tfrac{S(x_{n-1},x_{n-1},Tx_{n-1})[1+S(x_{n-1},x_{n-1},Tx_{n-1})]}{1+S(x_{n-1},x_{n-1},x_n)}, \\ &\qquad \tfrac{S(x_n,x_n,Tx_n)[1+S(x_{n-1},x_{n-1},Tx_{n-1})]}{1+S(x_{n-1},x_{n-1},x_n)}, \\ &\qquad \tfrac{1}{3}\tfrac{[S(x_n,x_n,Tx_{n-1})+S(x_{n-1},x_{n-1},Tx_n)][1+S(x_n,x_n,Tx_{n-1})]}{1+S(x_{n-1},x_{n-1},x_n)}\} \\ &= \max\{S(x_{n-1},x_{n-1},x_n), S(x_n,x_n,x_{n+1}), \\ &\qquad \tfrac{1}{3}\tfrac{[S(x_n,x_n,x_n)+S(x_{n-1},x_{n-1},x_{n+1})][1+S(x_n,x_n,x_n)]}{1+S(x_{n-1},x_{n-1},x_n)}\} \\ &= \max\{S(x_{n-1},x_{n-1},x_n), S(x_n,x_n,x_{n+1}), \tfrac{1}{3}S(x_{n-1},x_{n-1},x_{n+1})\} \\ &\leq \max\{S(x_{n-1},x_{n-1},x_n), S(x_n,x_n,x_{n+1}), \\ &\qquad \tfrac{1}{3}[2S(x_{n-1},x_{n-1},x_n) + S(x_n,x_n,x_{n+1})]\} \\ &= \max\{S(x_{n-1},x_{n-1},x_n), S(x_n,x_n,x_{n+1})\} \end{aligned}$$

and

$$\begin{aligned} N(x_{n-1},x_{n-1},x_n) &= \min\{S(x_{n-1},x_{n-1},Tx_{n-1}), S(x_n,x_n,Tx_{n-1}), \\ &\qquad \tfrac{S(x_{n-1},x_{n-1},Tx_{n-1})[1+S(x_{n-1},x_{n-1},Tx_{n-1})]}{1+S(x_{n-1},x_{n-1},x_n)}\} \\ &= \min\{S(x_n,x_n,x_{n-1}), S(x_n,x_n,x_n), S(x_{n-1},x_{n-1},x_n)\} = 0. \end{aligned}$$

If $M(x_{n-1},x_{n-1},x_n) = S(x_n,x_n,x_{n+1})$ for some n, then from (3.3) and by using (ζ_2), we get
$0 \leq \zeta(S(x_n,x_n,x_{n+1}),S(x_n,x_n,x_{n+1})) < S(x_n,x_n,x_{n+1}) - S(x_n,x_n,x_{n+1}) = 0,$

a contradiction. Therefore $M(x_{n-1},x_{n-1},x_n) = S(x_{n-1},x_{n-1},x_n)$ for all $n \in \mathbb{N}$. Then from (3.3) and by using (ζ_2), we get

$0 \le \zeta(S(x_n,x_n,x_{n+1}),S(x_{n-1},x_{n-1},x_n)) < S(x_{n-1},x_{n-1},x_n) - S(x_n,x_n,x_{n+1})$

which implies that

$$S(x_n,x_n,x_{n+1}) < S(x_{n-1},x_{n-1},x_n) \text{ for all } n \in \mathbb{N}. \tag{3.4}$$

Therefore the sequence $\{S(x_n,x_n,x_{n+1})\}$ is decreasing and converges to some $r \ge 0$. Assume that $r > 0$. Let $t_n = S(x_n,x_n,x_{n+1})$ and $s_n = S(x_{n-1},x_{n-1},x_n)$.

Since $\lim\limits_{n\to\infty} t_n = \lim\limits_{n\to\infty} s_n = r > 0$, by using (3.1) and the condition (ζ_3),

we get $0 \le \limsup\limits_{n\to\infty} \zeta(S(x_n,x_n,x_{n+1}),S(x_{n-1},x_{n-1},x_n)) < 0$,

a contradiction. Therefore $r = 0$. That is

$$\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = 0. \tag{3.5}$$

STEP 2: We now prove that $\{x_n\}$ is a Cauchy sequence.

On the contrary, suppose that $\{x_n\}$ is not Cauchy. Then there exist an $\varepsilon > 0$ and sequence of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k \ge k$ such that $S(x_{m_k},x_{m_k},x_{n_k}) \ge \varepsilon$ and $S(x_{m_k-1},x_{m_k-1},x_{n_k}) < \varepsilon$. Then by Lemma 2.15, we have

$$\lim_{k\to\infty} S(x_{m_k},x_{m_k},x_{n_k}) = \varepsilon \tag{3.6}$$

and

$$\lim_{k\to\infty} S(x_{m_k-1},x_{m_k-1},x_{n_k-1}) = \varepsilon. \tag{3.7}$$

Now, we have

$$\begin{aligned}
S(x_{m_k-1},x_{m_k-1},x_{n_k-1}) &\le M(x_{m_k-1},x_{m_k-1},x_{n_k-1}) \\
&= \max\{S(x_{m_k-1},x_{m_k-1},x_{n_k-1}), \\
&\qquad \tfrac{S(x_{m_k-1},x_{m_k-1},Tx_{m_k-1})[1+S(x_{m_k-1},x_{m_k-1},Tx_{m_k-1})]}{1+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})}, \\
&\qquad \tfrac{S(x_{n_k-1},x_{n_k-1},Tx_{n_k-1})[1+S(x_{m_k-1},x_{m_k-1},Tx_{m_k-1})]}{1+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})}, \\
&\qquad \tfrac{1}{3}[S(x_{n_k-1},x_{n_k-1},Tx_{m_k-1}) \\
&\qquad + S(x_{m_k-1},x_{m_k-1},Tx_{n_k-1})]\tfrac{[1+S(x_{n_k-1},x_{n_k-1},Tx_{m_k-1})]}{1+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})}\} \\
&= \max\{S(x_{m_k-1},x_{m_k-1},x_{n_k-1}), \tfrac{S(x_{m_k-1},x_{m_k-1},x_{m_k})[1+S(x_{m_k-1},x_{m_k-1},x_{m_k})]}{1+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})},
\end{aligned}$$

$$\frac{S(x_{n_k-1},x_{n_k-1},x_{n_k})[1+S(x_{m_k-1},x_{m_k-1},x_{m_k})]}{1+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})},$$
$$\frac{1}{3}[S(x_{n_k-1},x_{n_k-1},x_{m_k})+S(x_{m_k-1},x_{m_k-1},x_{n_k})]\frac{[1+S(x_{n_k-1},x_{n_k-1},x_{m_k})]}{1+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})}\}$$
$$\leq \max\{S(x_{m_k-1},x_{m_k-1},x_{n_k-1}),\frac{S(x_{m_k-1},x_{m_k-1},x_{m_k})[1+S(x_{m_k-1},x_{m_k-1},x_{m_k})]}{1+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})},$$
$$\frac{S(x_{n_k-1},x_{n_k-1},x_{n_k})[1+S(x_{m_k-1},x_{m_k-1},x_{m_k})]}{1+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})},$$
$$\frac{1}{3}[2S(x_{n_k-1},x_{n_k-1},x_{n_k})+S(x_{n_k},x_{n_k},x_{m_k})+2S(x_{m_k-1},x_{m_k-1},x_{m_k})$$
$$+S(x_{m_k},x_{m_k},x_{n_k})]\frac{[1+2S(x_{m_k},x_{m_k},x_{m_k-1})+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})]}{1+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})}\}.$$

On letting $k\to\infty$, and by using (3.5), (3.6) and (3.7), we have

$\varepsilon \leq \lim_{k\to\infty} M(x_{m_k-1},x_{m_k-1},x_{n_k-1}) \leq \varepsilon$. That is

$$\lim_{k\to\infty} M(x_{m_k-1},x_{m_k-1},x_{n_k-1}) = \varepsilon. \tag{3.8}$$

We have

$$N(x_{m_k-1},x_{m_k-1},x_{n_k-1}) = \min\{S(x_{m_k-1},x_{m_k-1},Tx_{m_k-1}),S(x_{n_k-1},x_{n_k-1},Tx_{m_k-1}),$$
$$\frac{S(x_{m_k-1},x_{m_k-1},Tx_{m_k-1})[1+S(x_{m_k-1},x_{m_k-1},Tx_{m_k-1})]}{1+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})}\}$$
$$= \min\{S(x_{m_k-1},x_{m_k-1},x_{m_k}),S(x_{n_k-1},x_{n_k-1},x_{m_k}),$$
$$\frac{S(x_{m_k-1},x_{m_k-1},x_{m_k})[1+S(x_{m_k-1},x_{m_k-1},x_{m_k})]}{1+S(x_{m_k-1},x_{m_k-1},x_{n_k-1})}\}.$$

On letting $k\to\infty$ and by using (3.5), we get

$$\lim_{k\to\infty} N(x_{m_k-1},x_{m_k-1},x_{n_k-1}) = 0. \tag{3.9}$$

Let $t'_k = S(x_{m_k},x_{m_k},x_{n_k})$ and $s'_k = M(x_{m_k-1},x_{m_k-1},x_{n_k-1})+LN(x_{m_k-1},x_{m_k-1},x_{n_k-1})$ and by using (3.5)-(3.9), we obtain that $\lim_{k\to\infty} t'_k = \lim_{k\to\infty} s'_k = \varepsilon > 0$ for all k.

Now, by (3.1) and by (ζ_3), we have

$$0 \leq \limsup_{k\to\infty} \zeta(S(Tx_{m_k-1},Tx_{m_k-1},Tx_{n_k-1}),M(x_{m_k-1},x_{m_k-1},x_{n_k-1})+LN(x_{m_k-1},x_{m_k-1},x_{n_k-1}))$$
< 0, a contradiction.

Therefore $\{x_n\}$ is a Cauchy sequence. Since (X,S) is a complete S-metric space, there exists a $u\in X$ such that $\lim_{n\to\infty} x_n = u$.

STEP 3: We now prove that u is a fixed point of T. Suppose that $Tu\neq u$. Then $S(u,u,Tu)>0$.

Let $t'_n = S(Tx_n,Tx_n,Tu) = S(x_{n+1},x_{n+1},Tu)$ and $s'_n = M(x_n,x_n,u)+LN(x_n,x_n,u)$, where

$$M(x_n,x_n,u) = \max\{S(x_n,x_n,u),\frac{S(x_n,x_n,Tx_n)[1+S(x_n,x_n,Tx_n)]}{1+S(x_n,x_n,u)},\frac{S(u,u,Tu)[1+S(x_n,x_n,Tx_n)]}{1+S(x_n,x_n,u)},$$
$$\frac{1}{3}\frac{[S(u,u,Tx_n)+S(x_n,x_n,Tu)][1+S(u,u,Tx_n)]}{1+S(x_n,x_n,u)}\}$$

$$= \max\{S(x_n,x_n,u), \tfrac{S(x_n,x_n,x_{n+1})[1+S(x_n,x_n,x_{n+1})]}{1+S(x_n,x_n,u)}, \tfrac{S(u,u,Tu)[1+S(x_n,x_n,x_{n+1})]}{1+S(x_n,x_n,u)}, \tfrac{1}{3}\tfrac{[S(u,u,x_{n+1})+S(x_n,x_n,Tu)][1+S(u,u,x_{n+1})]}{1+S(x_n,x_n,u)}\}.$$

On letting $n \to \infty$, we have $\lim_{n\to\infty} M(x_n,x_n,u) = S(u,u,Tu)$.

We have

$$N(x_n,x_n,u) = \min\{S(x_n,x_n,Tx_n), S(u,u,Tx_n), \tfrac{S(x_n,x_n,Tx_n)[1+S(x_n,x_n,Tx_n)]}{1+S(x_n,x_n,u)}\}$$
$$= \min\{S(x_n,x_n,x_{n+1}), S(u,u,x_{n+1}), \tfrac{S(x_n,x_n,x_{n+1})[1+S(x_n,x_n,x_{n+1})]}{1+S(x_n,x_n,u)}\}.$$

On letting $n \to \infty$, we have $\lim_{n\to\infty} N(x_n,x_n,u) = 0$.

Therefore we have $\lim_{n\to\infty} t'_n = \lim_{n\to\infty} s'_n = S(u,u,Tu) > 0$ and by (ζ_2) and (ζ_3),

$0 \le \limsup_{k\to\infty} \zeta(S(Tx_n,Tx_n,Tu), M(x_n,x_n,u)+LN(x_n,x_n,u)) < 0$,

a contradiction. Therefore $u = Tu$. By Lemma 3.2, this fixed point u is unique. □

Corollary 3.4. *Let (X,S) be a complete S-metric space and $\zeta \in \mathscr{Z}$. Suppose that there exists $L \ge 0$ such that*

$$\zeta(S(Tx,Tx,Ty), M(x,x,y)+LN(x,x,y)) \ge 0 \tag{3.10}$$

for all $x,y \in X$, where $M(x,x,y) = \max\{S(x,x,y), \frac{S(x,x,Tx)[1+S(x,x,Tx)]}{1+S(x,x,y)}, \frac{S(y,y,Ty)[1+S(x,x,Tx)]}{1+S(x,x,y)}, \frac{1}{3}\frac{[S(y,y,Tx)+S(x,x,Ty)][1+S(y,y,Tx)]}{1+S(x,x,y)}\}$ *and* $N(x,x,y) = \min\{S(x,x,Tx), S(y,y,Tx), \frac{S(x,x,Tx)[1+S(x,x,Tx)]}{1+S(x,x,y)}\}$. *Then T has a unique fixed point in X.*

Proof. By choosing $y = x$ and $z = y$ in the inequality (3.1), proof of this corollary follows from Theorem 3.3. □

Corollary 3.5. *Let (X,S) be a complete S-metric space and $T : X \times X \to X$ be a mapping satisfying the following condition:*

$$S(Tx,Ty,Tz) \le \lambda M(x,y,z) \tag{3.11}$$

for all $x,y,z \in X$, where $M(x,y,z)$ is defined as in the inequality (3.1). *Then T has a unique fixed point in X.*

Proof. If we choose simulation function ζ as $\zeta(t,s) = \lambda s - t$ for all $s,t \ge 0$, where $\lambda \in [0,1)$, then the inequality (3.11) is a special case of the inequality (3.1) so that from Theorem 3.3, the conclusion of this corollary follows.

□

Remark 3.6. Theorem 2.14 follows as a corollary to Corollary 3.5.

Corollary 3.7. *Let (X,S) be a complete S-metric space and $T : X \times X \to X$ be a mapping satisfying*

$$S(Tx,Ty,Tz) \leq M(x,y,z) - \varphi(M(x,y,z)) \tag{3.12}$$

for all $x,y,z \in X$, where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi continuous function with $\varphi(t) = 0$ if and only if $t = 0$, and $M(x,y,z)$ is defined as in the inequality (3.1)*. Then T has a unique fixed point in X.*

Proof. Follows by choosing $\zeta(t,s)$ is as in the Example 2.2 (v), $L = 0$ in the inequality (3.1) and by applying Theorem 3.3. □

The following example is in support of Theorem 3.3.

Example 3.8. Let $X = [\frac{1}{4}, \frac{1}{2}]$. We define $S : X^3 \to [0,\infty)$ by

$$S(x,y,z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x,y,z\} & \text{otherwise.} \end{cases}$$

We define $T : X \to X$ by

$$Tx = \begin{cases} 4x^2 & \text{if } x \in [\frac{1}{4}, \frac{1}{3}] \\ \frac{1}{3} & \text{if } x \in (\frac{1}{3}, \frac{1}{2}]. \end{cases}$$

We define $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\zeta(t,s) = \frac{1}{2}s - t$, $s,t \geq 0$. Then ζ is a simulation function.

Let $x,y,z \in X$. We now verify the inequality (3.1).

i.e., $\zeta(S(Tx,Ty,Tz), M(x,y,z) + LN(x,y,z)) \geq 0$.

<u>Case</u> (i): Let $x,y,z \in [\frac{1}{4}, \frac{1}{3}]$.

We assume without loss of generality, we assume that $x > y > z$.

We have $S(Tx,Ty,Tz) = 4x^2$, $S(x,y,z) = x$, $S(x,x,Tx) = 4x^2$, $S(y,y,Tx) = 4x^2$, $S(z,z,Tx) = 4x^2$, $\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)} = \frac{4x^2[1+S(x,x,4y^2)]}{1+x} = \frac{4x^2[1+\max\{x,4y^2\}]}{1+x}$ and

$N(x,y,z) = \min\{4x^2, \frac{4x^2[1+\max\{x,4y^2\}]}{1+x}\} = 4x^2$.

We consider

$$\begin{aligned}\zeta(S(Tx,Ty,Tz),M(x,y,z)+LN(x,y,z)) &= \tfrac{1}{2}(M(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)\\ &\geq \tfrac{1}{2}(LN(x,y,z))-S(Tx,Ty,Tz)\\ &= \tfrac{1}{2}(L(4x^2))-4x^2\geq 0 \text{ for any } L\geq 2.\end{aligned}$$

In this case, the inequality (3.1) holds for any $L\geq 2$.

<u>*Case*</u> (ii): Let $x,y,z\in(\frac{1}{3},\frac{1}{2}]$. We assume that $x>y>z$.

$S(Tx,Ty,Tz)=0$ so that the inequality (3.1) holds trivially for any $L\geq 0$ in this case.

<u>*Case*</u> (iii): Let $x\in[\frac{1}{4},\frac{1}{3}]$ and $y,z\in(\frac{1}{3},\frac{1}{2}]$. We assume that $y>z$.

We have $S(Tx,Ty,Tz)=\max\{4x^2,\frac{1}{3}\}$, $S(x,y,z)=y$, $S(x,x,Tx)=4x^2$, $S(y,y,Tx)=\max\{y,4x^2\}$, $S(z,z,Tx)=\max\{z,4x^2\}$ and $\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)}=\max\{y,4x^2\}\frac{[1+\max\{x,\frac{1}{3}\}]}{1+y}=\frac{4}{3(1+y)}\max\{y,4x^2\}$.

<u>*Subcase*</u> (i):If $y>z\geq 4x^2$ then we have

$S(y,y,Tx)=y$, $S(z,z,Tx)=z$, $\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)}=\frac{4y}{3(1+y)}$ and

$N(x,y,z)=\min\{4x^2,y,z,\frac{4y}{3(1+y)}\}=\min\{4x^2,\frac{4y}{3(1+y)}\}$.

If $N(x,y,z)=4x^2$ and $S(Tx,Ty,Tz)=4x^2$ then the inequality (3.1) holds for any $L\geq 2$. (similar as in Case (i)).

If $N(x,y,z)=4x^2$ and $S(Tx,Ty,Tz)=\frac{1}{3}$, then we have

$$\begin{aligned}\zeta(S(Tx,Ty,Tz),M(x,y,z)+LN(x,y,z)) &= \tfrac{1}{2}(M(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)\\ &\geq \tfrac{1}{2}(LN(x,y,z))-S(Tx,Ty,Tz)\\ &= \tfrac{1}{2}(L(4x^2))-\tfrac{1}{3}>0 \text{ for any } L\geq 4.\end{aligned}$$

If $N(x,y,z)=\frac{4y}{3(1+y)}$ and $S(Tx,Ty,Tz)=4x^2$ then we have

$$\begin{aligned}\zeta(S(Tx,Ty,Tz),M(x,y,z)+LN(x,y,z)) &= \tfrac{1}{2}(M(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)\\ &\geq \tfrac{1}{2}(S(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)\\ &= \tfrac{1}{2}(y+L(\tfrac{4y}{3(1+y)}))-4x^2>0 \text{ for any } L\geq 4.\end{aligned}$$

If $N(x,y,z)=\frac{4y}{3(1+y)}$ and $S(Tx,Ty,Tz)=\frac{1}{3}$ then we have

$$\begin{aligned}\zeta(S(Tx,Ty,Tz),M(x,y,z)+LN(x,y,z)) &= \tfrac{1}{2}(M(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)\\ &\geq \tfrac{1}{2}(S(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)\\ &= \tfrac{1}{2}(y+L(\tfrac{4y}{3(1+y)}))-\tfrac{1}{3}>0 \text{ for any } L\geq 4.\end{aligned}$$

Therefore in this case the inequality (3.1) holds for any $L\geq 4$.

<u>*Subcase*</u> (ii): If $z<y\leq 4x^2$ then we have

$S(Tx,Ty,Tz)=4x^2$, $S(y,y,Tx)=4x^2$, $S(z,z,Tx)=4x^2$, $\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)}=4x^2(\frac{4}{3(1+y)})$, and

$N(x,y,z) = \min\{4x^2, 4x^2(\frac{4}{3(1+y)})\} = 4x^2(\frac{4}{3(1+y)})$.

We have

$$\begin{aligned}\zeta(S(Tx,Ty,Tz), M(x,y,z)+LN(x,y,z)) &= \tfrac{1}{2}(M(x,y,z)+LN(x,y,z)) - S(Tx,Ty,Tz)\\ &\geq \tfrac{1}{2}(LN(x,y,z)) - S(Tx,Ty,Tz)\\ &= \tfrac{1}{2}(L(4x^2(\tfrac{4}{3(1+y)}))) - 4x^2 > 0 \text{ for any } L \geq 4.\end{aligned}$$

<u>*Subcase*</u> (iii): If $z \leq 4x^2 < y$ then we have

$S(Tx,Ty,Tz) = 4x^2$, $S(y,y,Tx) = y$, $S(z,z,Tx) = 4x^2$, $\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)} = \frac{4y}{3(1+y)}$

and $N(x,y,z) = \min\{4x^2, y, \frac{4y}{3(1+y)}\} = \min\{4x^2, \frac{4y}{3(1+y)}\}$.

In this case the inequality (3.1) holds for any $L \geq 4$ as in Subcase(i) of Case (iii).

<u>*Case*</u> (iv): Let $x,y \in [\frac{1}{4}, \frac{1}{3}]$ and $z \in (\frac{1}{3}, \frac{1}{2}]$. We assume that $x > y$.

We have $S(Tx,Ty,Tz) = S(4x^2, 4y^2, \frac{1}{3}) = \max\{4x^2, \frac{1}{3}\}$, $S(x,y,z) = z$, $S(x,x,Tx) = 4x^2$,

$S(y,y,Tx) = 4x^2$, $S(z,z,Tx) = \max\{z, 4x^2\}$, and $\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)} = 4x^2\frac{[1+\max\{x,4y^2\}]}{1+z}$.

<u>*Subcase*</u> (i): If $x \geq 4y^2$ and $z \leq 4x^2$ then

we have $S(Tx,Ty,Tz) = 4x^2$, $S(z,z,Tx) = 4x^2$, $\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)} = 4x^2(\frac{1+x}{1+z})$

and $N(x,y,z) = \min\{4x^2, 4x^2(\frac{1+x}{1+z})\} = 4x^2(\frac{1+x}{1+z})$.

In this case, we have

$$\begin{aligned}\zeta(S(Tx,Ty,Tz), M(x,y,z)+LN(x,y,z)) &= \tfrac{1}{2}(M(x,y,z)+LN(x,y,z)) - S(Tx,Ty,Tz)\\ &\geq \tfrac{1}{2}(LN(x,y,z)) - S(Tx,Ty,Tz)\\ &= \tfrac{1}{2}(L(4x^2(\tfrac{1+x}{1+z}))) - 4x^2 > 0 \text{ for any } L \geq 4.\end{aligned}$$

<u>*Subcase*</u> (ii): If $x \geq 4y^2$ and $z \geq 4x^2$ then

we have $S(z,z,Tx) = z$, $\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)} = \frac{4x^2[1+x]}{(1+z)}$

and $N(x,y,z) = \min\{4x^2, z, 4x^2(\frac{1+x}{1+z})\} = 4x^2(\frac{1+x}{1+z})$.

If $S(Tx,Ty,Tz) = 4x^2$ then the inequality holds as in Subcase(i) of Case (iv).

If $S(Tx,Ty,Tz) = \frac{1}{3}$ then we have

$$\begin{aligned}\zeta(S(Tx,Ty,Tz), M(x,y,z)+LN(x,y,z)) &= \tfrac{1}{2}(M(x,y,z)+LN(x,y,z)) - S(Tx,Ty,Tz)\\ &\geq \tfrac{1}{2}(S(x,y,z)+LN(x,y,z)) - S(Tx,Ty,Tz)\\ &= \tfrac{1}{2}(z+L(4x^2(\tfrac{1+x}{1+z}))) - \tfrac{1}{3} > 0 \text{ for any } L \geq 5.\end{aligned}$$

In this case the inequality (3.1) holds for any $L \geq 5$.

<u>*Subcase*</u> (iii): If $x \leq 4y^2$ and $z \geq 4x^2$

then $S(z,z,Tx) = z$, $\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)} = 4x^2(\frac{1+4y^2}{1+z})$
and $N(x,y,z) = \min\{4x^2, z, 4x^2(\frac{1+4y^2}{1+z})\} = 4x^2(\frac{1+4y^2}{1+z})$.

If $S(Tx,Ty,Tz) = 4x^2$ then we have

$$\begin{aligned}\zeta(S(Tx,Ty,Tz), M(x,y,z)+LN(x,y,z)) &= \tfrac{1}{2}(M(x,y,z)+LN(x,y,z)) - S(Tx,Ty,Tz)\\ &\geq \tfrac{1}{2}(LN(x,y,z)) - S(Tx,Ty,Tz)\\ &= \tfrac{1}{2}(L(4x^2(\tfrac{1+4y^2}{1+z}))) - 4x^2 > 0 \text{ for any } L \geq 4.\end{aligned}$$

If $S(Tx,Ty,Tz) = \frac{1}{3}$ then we have

$$\begin{aligned}\zeta(S(Tx,Ty,Tz), M(x,y,z)+LN(x,y,z)) &= \tfrac{1}{2}(M(x,y,z)+LN(x,y,z)) - S(Tx,Ty,Tz)\\ &\geq \tfrac{1}{2}(LN(x,y,z)) - S(Tx,Ty,Tz)\\ &= \tfrac{1}{2}(L(4x^2(\tfrac{1+4y^2}{1+z}))) - \tfrac{1}{3} > 0 \text{ for any } L \geq 4.\end{aligned}$$

Subcase (iv): If $x \leq 4y^2$ and $z \leq 4x^2$,

we have $S(Tx,Ty,Tz) = 4x^2$, $S(z,z,Tx) = 4x^2$, $\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)} = 4x^2(\frac{1+4y^2}{1+z})$
and $N(x,y,z) = \min\{4x^2, 4x^2(\frac{1+4y^2}{1+z})\}$.

If $N(x,y,z) = 4x^2$ then we have

$$\begin{aligned}\zeta(S(Tx,Ty,Tz), M(x,y,z)+LN(x,y,z)) &= \tfrac{1}{2}(M(x,y,z)+LN(x,y,z)) - S(Tx,Ty,Tz)\\ &\geq \tfrac{1}{2}(LN(x,y,z)) - S(Tx,Ty,Tz) \geq 0 \text{ for any } L \geq 2.\end{aligned}$$

If $N(x,y,z) = 4x^2(\frac{1+4y^2}{1+z})$ then we have

$$\begin{aligned}\zeta(S(Tx,Ty,Tz), M(x,y,z)+LN(x,y,z)) &= \tfrac{1}{2}(M(x,y,z)+LN(x,y,z)) - S(Tx,Ty,Tz)\\ &\geq \tfrac{1}{2}(LN(x,y,z)) - S(Tx,Ty,Tz)\\ &= \tfrac{1}{2}(L(4x^2(\tfrac{1+4y^2}{1+z}))) - 4x^2 > 0 \text{ for any } L \geq 4.\end{aligned}$$

In this case the inequality (3.1) holds for any $L \geq 4$.

Case (v): Let $x, z \in [\frac{1}{4}, \frac{1}{3}]$ and $y \in (\frac{1}{3}, \frac{1}{2}]$. We assume that $x < z$.

In this case, we have $S(Tx,Ty,Tz) = S(4x^2, \frac{1}{3}, 4z^2) = \max\{\frac{1}{3}, 4z^2\}$,

$S(x,x,Tx) = 4x^2$, $S(y,y,Tx) = \max\{y, 4x^2\}$, $S(z,z,Tx) = \max\{z, 4x^2\}$ and
$\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)} = \max\{y, 4x^2\}\frac{[1+\max\{x,\frac{1}{3}\}]}{1+y} = \max\{y, 4x^2\}\frac{4}{3(1+y)}$.

Subcase (i): Let $y \leq 4x^2$ and $\frac{1}{3} \leq 4z^2$.

In this case, we have $S(Tx,Ty,Tz) = 4z^2$, $S(y,y,Tx) = 4x^2$, $S(z,z,Tx) = 4x^2$,
$\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)} = 4x^2(\frac{4}{3(1+y)})$, and $N(x,y,z) = \min\{4x^2, 4x^2(\frac{4}{3(1+y)})\} = \frac{16x^2}{3(1+y)}$.

We consider

$$\begin{aligned}\zeta(S(Tx,Ty,Tz),M(x,y,z)+LN(x,y,z)) &= \tfrac{1}{2}(M(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)\\ &\geq \tfrac{1}{2}(LN(x,y,z))-S(Tx,Ty,Tz)\\ &= \tfrac{1}{2}(L(\tfrac{16x^2}{3(1+y)}))-4z^2 \geq 0 \text{ for any } L\geq 4.\end{aligned}$$

Subcase (ii): Let $4x^2 \leq z < y$ and $\frac{1}{3} \geq 4z^2$.

In this case, we have $S(Tx,Ty,Tz)=\frac{1}{3}$, $S(y,y,Tx)=y$, $S(z,z,Tx)=z$,
$\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)} = \frac{4y}{3(1+y)}$ and $N(x,y,z)=\min\{4x^2,y,z,\frac{4y}{3(1+y)}\}=\min\{4x^2,\frac{4y}{3(1+y)}\}$.
If $N(x,y,z)=4x^2$ then we have

$$\begin{aligned}\zeta(S(Tx,Ty,Tz),M(x,y,z)+LN(x,y,z)) &= \tfrac{1}{2}(M(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)\\ &\geq \tfrac{1}{2}(S(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)\\ &= \tfrac{1}{2}(y+L(4x^2))-\tfrac{1}{3} > 0 \text{ for any } L\geq 4.\end{aligned}$$

If $N(x,y,z)=\frac{4y}{3(1+y)}$ then we have

$$\begin{aligned}\zeta(S(Tx,Ty,Tz),M(x,y,z)+LN(x,y,z)) &= \tfrac{1}{2}(M(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)\\ &\geq \tfrac{1}{2}(S(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)\\ &= \tfrac{1}{2}(y+L(\tfrac{4y}{3(1+y)}))-\tfrac{1}{3} > 0 \text{ for any } L\geq 4.\end{aligned}$$

Subcase (iii): Let $z \leq 4x^2 < y$ and $4z^2 \leq \frac{1}{3}$.

In this case, we have $S(Tx,Ty,Tz)=\frac{1}{3}$, $S(y,y,Tx)=y$, $S(z,z,Tx)=4x^2$,
$\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)} = \frac{4y}{3(1+y)}$ and $N(x,y,z)=\min\{4x^2,y,\frac{4y}{3(1+y)}\}=\min\{4x^2,\frac{4y}{3(1+y)}\}$. We have $\zeta(S(Tx,Ty,Tz),M(x,y,z)+LN(x,y,z))>0$ (similar as in Subcase (ii) of Case (v)).

Subcase (iv): Let $y \geq 4x^2$ and $\frac{1}{3} \leq 4z^2$.

In this case, we have $S(Tx,Ty,Tz)=4z^2$ and $N(x,y,z)=\min\{4x^2,\frac{4y}{3(1+y)}\}$.

We consider

$$\begin{aligned}\zeta(S(Tx,Ty,Tz),M(x,y,z)+LN(x,y,z)) &= \tfrac{1}{2}(M(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)\\ &\geq \tfrac{1}{2}(S(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)\\ &= \tfrac{1}{2}(y+LN(x,y,z))-4z^2 > 0 \text{ for any } L\geq 4.\end{aligned}$$

Case (vi): Let $z\in[\frac{1}{4},\frac{1}{3}]$ and $x,y\in(\frac{1}{3},\frac{1}{2}]$. We assume that $x>y$.

$S(Tx,Ty,Tz)=\max\{\frac{1}{3},4z^2\}$, $S(x,y,z)=x$; $S(x,x,Tx)=x$, $S(y,y,Tx)=y$, $S(z,z,Tx)=\frac{1}{3}$,
$\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)} = \frac{y[1+\max\{x,\frac{1}{3}\}]}{1+x} = y$, and $N(x,y,z)=\min\{x,y,\frac{1}{3}\}=\frac{1}{3}$.

In this case, we have

$$\zeta(S(Tx,Ty,Tz),M(x,y,z)+LN(x,y,z)) = \tfrac{1}{2}(M(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)$$

$$\geq \frac{1}{2}(S(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)$$
$$= \frac{1}{2}(y+L(\frac{1}{3}))-S(Tx,Ty,Tz) > 0 \text{ for any } L \geq 4.$$

<u>*Case*</u> (vii): Let $y \in [\frac{1}{4}, \frac{1}{3}]$ and $x, z \in (\frac{1}{3}, \frac{1}{2}]$. We assume that $z > x$.

$S(Tx,Ty,Tz) = \max\{\frac{1}{3}, 4y^2\}$, $S(x,y,z) = z$, $S(x,x,Tx) = x$,

$S(y,y,Tx) = \frac{1}{3}$, $S(z,z,Tx) = z$ and $\frac{S(y,y,Tx)[1+S(x,x,Ty)]}{1+S(x,y,z)} = \frac{1}{3}(\frac{[1+\max\{x,4y^2\}]}{1+z})$.

<u>*Subcase*</u> (i): If $\frac{1}{3} < 4y^2 \leq x < z$ then

we have $S(Tx,Ty,Tz) = 4y^2$ and $N(x,y,z) = \min\{x, \frac{1}{3}, z, \frac{1}{3}(\frac{1+x}{1+z})\} = \frac{1}{3}(\frac{1+x}{1+z})$.

In this case,

$$\zeta(S(Tx,Ty,Tz), M(x,y,z)+LN(x,y,z)) = \frac{1}{2}(M(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)$$
$$\geq \frac{1}{2}(S(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)$$
$$= \frac{1}{2}(z+L(\frac{1}{3}(\frac{1+x}{1+z})))-4y^2 > 0 \text{ for any } L \geq 4.$$

<u>*Subcase*</u> (ii): If $\frac{1}{3} < x \leq 4y^2 < z$ then

$S(Tx,Ty,Tz) = 4y^2$ and $N(x,y,z) = \min\{x, \frac{1}{3}, z, \frac{1}{3}(\frac{1+4y^2}{1+z})\} = \frac{1}{3}(\frac{1+4y^2}{1+z})$.

We consider

$$\zeta(S(Tx,Ty,Tz), M(x,y,z)+LN(x,y,z)) = \frac{1}{2}(M(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)$$
$$\geq \frac{1}{2}(S(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)$$
$$= \frac{1}{2}(z+L(\frac{1}{3}(\frac{1+4y^2}{1+z})))-4y^2 > 0 \text{ for any } L \geq 4.$$

<u>*Subcase*</u> (iii): If $\frac{1}{3} < x < z \leq 4y^2$ then

we have $S(Tx,Ty,Tz) = 4y^2$ and $N(x,y,z) = \min\{x, \frac{1}{3}, z, \frac{1}{3}(\frac{1+4y^2}{1+z})\} = \frac{1}{3}$.

We consider

$$\zeta(S(Tx,Ty,Tz), M(x,y,z)+LN(x,y,z)) = \frac{1}{2}(M(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)$$
$$\geq \frac{1}{2}(S(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)$$
$$= \frac{1}{2}(z+L(\frac{1}{3}))-4y^2 > 0 \text{ for any } L \geq 4.$$

<u>*Subcase*</u> (iv): If $4y^2 \leq \frac{1}{3} < x < z$ then

we have $S(Tx,Ty,Tz) = \frac{1}{3}$ and $N(x,y,z) = \min\{x, \frac{1}{3}, z, \frac{1}{3}(\frac{1+x}{1+z})\} = \frac{1}{3}(\frac{1+x}{1+z})$.

We consider

$$\zeta(S(Tx,Ty,Tz), M(x,y,z)+LN(x,y,z)) = \frac{1}{2}(M(x,y,z)+LN(x,y,z))-S(Tx,Ty,Tz)$$
$$\geq \frac{1}{2}(LN(x,y,z))-S(Tx,Ty,Tz)$$

$$= \frac{1}{2}(L(\frac{1+x}{1+z})) - \frac{1}{3} > 0 \text{ for any } L \geq 4.$$

In this case the inequality (3.1) holds for any $L \geq 4$.

Therefore T is an almost generalized $\mathscr{Z}_s$-contraction with rational expressions and T satisfies all the hypothesis of Theorem 3.3 for any $L \geq 5$ and T has a unique fixed point $\frac{1}{4}$.

Here we prove that the inequality (2.1) fails to hold, for $x = \frac{1}{4}$, $y = \frac{1}{3}$ in (2.1), we have $S(Tx,Tx,Ty) = S(\frac{1}{4},\frac{1}{4},\frac{4}{9}) = \frac{4}{9}$ and $S(x,x,y) = S(\frac{1}{4},\frac{1}{4},\frac{1}{3}) = \frac{1}{3}$.

In this case, we have

$S(Tx,Tx,Ty) = \frac{4}{9} \nleq \lambda(\frac{1}{3}) = \lambda S(x,x,y)$ for any $0 \leq \lambda < 1$. Hence by Remark 3.6, we have Theorem 3.3 is a generalization of Theorem 2.14.

4. α-ADMISSIBLE ALMOST GENERALIZED $\mathscr{Z}_s$-CONTRACTIONS WITH RATIONAL EXPRESSIONS

Definition 4.1. [11] Let (X,S) be an S-metric space. Let $T : X \to X$ and $\alpha : X \times X \times X \to [0,\infty)$. We say that T is α-admissible, if $x,y,z \in X$, $\alpha(x,y,z) \geq 1 \implies \alpha(Tx,Ty,Tz) \geq 1$.

Definition 4.2. Let (X,S) be an S-metric space. An α-admissible mapping T on X is said to be triangular α-admissible if

$\alpha(x,x,z) \geq 1$ and $\alpha(z,z,y) \geq 1$ implies $\alpha(x,x,y) \geq 1$.

Lemma 4.3. *Let $T : X \to X$ be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0,x_0,Tx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for each $n \in \mathbb{N}$. Then we have $\alpha(x_m,x_m,x_n) \geq 1$ for all $m,n \in \mathbb{N}$ with $m < n$.*

Proof. Let $x_0 \in X$ such that $\alpha(x_0,x_0,Tx_0) \geq 1$. Since T is α-admissible, we have

$\alpha(Tx_0,Tx_0,Tx_1) \geq 1$. That is $\alpha(x_1,x_1,x_2) \geq 1$.

On continuing this process, we get $\alpha(x_n,x_n,x_{n+1}) \geq 1$, for $n = 0,1,2,\ldots$.

Now suppose that $m < n$. We have $\alpha(x_m,x_m,x_{m+1}) \geq 1$.

Again since T is α-admissible, we have $\alpha(Tx_m,Tx_m,Tx_{m+1}) \geq 1$.

That is $\alpha(x_{m+1},x_{m+1},x_{m+2}) \geq 1$.

Since T is triangular α-admissible, we have $\alpha(x_m,x_m,x_{m+2}) \geq 1$.

Also, we have $\alpha(x_{m+2},x_{m+2},x_{m+3}) \geq 1$.

Since T is triangular α-admissible, we have $\alpha(x_m,x_m,x_{m+3}) \geq 1$.

Now $\alpha(x_{m+2},x_{m+2},x_{m+3}) \geq 1$ and since T is triangular α-admissible, we have $\alpha(x_m,x_m,x_{m+3}) \geq 1$. On continuing this process, we get $\alpha(x_m,x_m,x_n) \geq 1$, for all $m,n \in \mathbb{N}$ with $m < n$. □

Definition 4.4. Let (X,S) be an S-metric space. Let $T : X \to X$ be an α-admissible mapping. If there exists a $\zeta \in \mathscr{Z}$ and $L \geq 0$ such that

$$\zeta(\alpha(x,y,z)S(Tx,Ty,Tz),M(x,y,z)+LN(x,y,z)) \geq 0, \tag{4.1}$$

for all $x,y,z \in X$ with $x \neq y \neq z$, where

$M(x,y,z) = max\{S(x,y,z), \frac{S(x,x,Tx)S(y,y,Ty)}{S(x,y,z)}, \frac{S(x,x,Tx)S(z,z,Tz)}{S(x,y,z)}, \frac{S(y,y,Ty)S(z,z,Tz)}{S(x,y,z)}, \frac{S(x,x,Ty)S(y,y,Tx)}{S(x,y,z)}, \frac{S(y,y,Tz)S(z,z,Ty)}{S(x,y,z)}, \frac{S(z,z,Tx)S(x,x,Tz)}{S(x,y,z)}, \frac{S(y,y,Ty)S(x,x,Ty)}{S(x,y,z)}, \frac{S(x,x,Tx)S(x,x,Ty)}{S(x,y,z)}, \frac{S(z,z,Tz)S(z,z,Tx)}{S(x,y,z)}\}$

and $N(x,y,z) = \min\{S(x,x,Tx), S(y,y,Tx), S(z,z,Tx)\}$. Then T is called an α-admissible almost generalized $\mathscr{Z}_s$-contraction with rational expressions.

Theorem 4.5. *Let (X,S) be a complete S-metric space and $T : X \to X$ be an α-admissible almost generalized $\mathscr{Z}_s$-contraction with rational expressions. Suppose that*

(i) *T is triangular α-admissible*

(ii) *there exists $x_0 \in X$ such that $\alpha(x_0,x_0,Tx_0) \geq 1$*

(iii) *either T is continuous or*

whenever $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k},x_{n_k},x) \geq 1$ for all k.

Then there exists $u \in X$ such that $Tu = u$.

Proof. Let $x_0 \in X$ and the sequence $\{x_n\}$ be defined as $x_n = Tx_{n-1}$ for all $n \in N$.

If $x_{n_0} = x_{n_0+1} = Tx_{n_0}$ for some n_0, then x_{n_0} is a fixed point of T.

Therefore we assume that $x_n \neq x_{n+1}$, i.e., $S(x_n,x_n,x_{n+1}) > 0$ for all $n \geq 0$.

STEP 1: We now prove that $\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = 0$.

By (ii) there exists $x_0 \in X$ such that $\alpha(x_0,x_0,Tx_0) \geq 1$.

That is $\alpha(x_0,x_0,x_1) \geq 1$. Since T is α-admissible, we get $\alpha(x_1,x_1,x_2) \geq 1$. On continuing this process, we obtain

$$\alpha(x_n,x_n,x_{n+1}) \geq 1 \text{ for all n.} \tag{4.2}$$

From (4.1) and (4.2), we have

$$0 \leq \zeta(\alpha(x_n,x_n,x_{n+1})S(Tx_n,Tx_n,Tx_{n+1}),M(x_n,x_n,x_{n+1})+LN(x_n,x_n,x_{n+1})). \tag{4.3}$$

Here $M(x_n,x_n,x_{n+1}) = \max\{S(x_n,x_n,x_{n+1}),S(x_{n+1},x_{n+1},x_{n+2})\}$ and $N(x_n,x_n,x_{n+1}) = 0$.
If $M(x_n,x_n,x_{n+1}) = S(x_{n+1},x_{n+1},x_{n+2})$ for some n, then from (4.3) and by using (ζ_2), we get
$0 \leq \zeta(\alpha(x_n,x_n,x_{n+1})S(x_{n+1},x_{n+1},x_{n+2}),S(x_{n+1},x_{n+1},x_{n+2}))$
$\quad < S(x_{n+1},x_{n+1},x_{n+2}) - \alpha(x_n,x_n,x_{n+1})S(x_n,x_n,x_{n+1})$
which implies that
$S(x_{n+1},x_{n+1},x_{n+2}) \leq \alpha(x_n,x_n,x_{n+1})S(x_n,x_n,x_{n+1}) < S(x_{n+1},x_{n+1},x_{n+2})$, a contradiction.
Therefore $M(x_n,x_n,x_{n+1}) = S(x_n,x_n,x_{n+1})$ for all $n \in \mathbb{N}$. Then from (4.3) and by using (ζ_2), we get
$0 \leq \zeta(\alpha(x_n,x_n,x_{n+1})S(x_{n+1},x_{n+1},x_{n+2}),S(x_n,x_n,x_{n+1}))$
$\quad < S(x_n,x_n,x_{n+1}) - \alpha(x_n,x_n,x_{n+1})S(x_{n+1},x_{n+1},x_{n+2})$ which implies that

$$S(x_{n+1},x_{n+1},x_{n+2}) \leq \alpha(x_n,x_n,x_{n+1})S(x_{n+1},x_{n+1},x_{n+2}) < S(x_n,x_n,x_{n+1}) \tag{4.4}$$

for all $n = 0,1,2,\ldots$. Therefore the sequence $\{S(x_n,x_n,x_{n+1})\}$ is decreasing and converges to some $s \geq 0$. Assume that $s > 0$.
Let $p_n = \alpha(x_n,x_n,x_{n+1})S(x_{n+1},x_{n+1},x_{n+2})$ and $q_n = S(x_n,x_n,x_{n+1})$.
Since $\lim_{n\to\infty} p_n = \lim_{n\to\infty} q_n = s > 0$, by using (4.1) and the condition (ζ_3), we have
$0 \leq \limsup_{n\to\infty} \zeta(\alpha(x_n,x_n,x_{n+1})S(x_{n+1},x_{n+1},x_{n+2}),S(x_n,x_n,x_{n+1}) < 0$,
a contradiction. Therefore $s = 0$. That is

$$\lim_{n\to\infty} S(x_n,x_n,x_{n+1}) = 0. \tag{4.5}$$

STEP 2: We now prove that $\{x_n\}$ is Cauchy.
On the contrary, suppose that $\{x_n\}$ is not Cauchy. Then there exist an $\varepsilon > 0$ and sequence of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k \geq k$ such that $S(x_{m_k},x_{m_k},x_{n_k}) \geq \varepsilon$ and

$S(x_{m_k-1}, x_{m_k-1}, x_{n_k}) < \varepsilon$. Then by Lemma 2.15, we have

$$\lim_{k\to\infty} S(x_{m_k}, x_{m_k}, x_{n_k}) = \varepsilon \tag{4.6}$$

and

$$\lim_{k\to\infty} S(x_{m_k+1}, x_{m_k+1}, x_{n_k+1}) = \varepsilon. \tag{4.7}$$

Now, we have

$$\begin{aligned}
S(x_{n_k}, x_{n_k}, x_{m_k}) &\le M(x_{n_k}, x_{n_k}, x_{m_k}) \\
&= \max\Big\{S(x_{n_k}, x_{n_k}, x_{m_k}), \frac{S(x_{n_k},x_{n_k},Tx_{n_k})S(x_{n_k},x_{n_k},Tx_{n_k})}{S(x_{n_k},x_{n_k},x_{m_k})}, \frac{S(x_{n_k},x_{n_k},Tx_{n_k})S(x_{m_k},x_{m_k},Tx_{m_k})}{S(x_{n_k},x_{n_k},x_{m_k})}, \\
&\qquad \frac{S(x_{n_k},x_{n_k},Tx_{m_k})S(x_{m_k},x_{m_k},Tx_{n_k})}{S(x_{n_k},x_{n_k},x_{m_k})}, \frac{S(x_{m_k},x_{m_k},Tx_{n_k})S(x_{n_k},x_{n_k},Tx_{m_k})}{S(x_{n_k},x_{n_k},x_{m_k})}, \\
&\qquad \frac{S(x_{m_k},x_{m_k},Tx_{m_k})S(x_{m_k},x_{m_k},Tx_{n_k})}{S(x_{n_k},x_{n_k},x_{m_k})}\Big\} \\
&= \max\Big\{S(x_{n_k}, x_{n_k}, x_{m_k}), \frac{S(x_{n_k},x_{n_k},x_{n_k+1})S(x_{n_k},x_{n_k},x_{n_k+1})}{S(x_{n_k},x_{n_k},x_{m_k})}, \frac{S(x_{n_k},x_{n_k},x_{n_k+1})S(x_{m_k},x_{m_k},x_{m_k+1})}{S(x_{n_k},x_{n_k},x_{m_k})}, \\
&\qquad \frac{S(x_{n_k},x_{n_k},x_{m_k+1})S(x_{m_k},x_{m_k},x_{n_k+1})}{S(x_{n_k},x_{n_k},x_{m_k})}, \frac{S(x_{m_k},x_{m_k},x_{n_k+1})S(x_{n_k},x_{n_k},x_{m_k+1})}{S(x_{n_k},x_{n_k},x_{m_k})}, \\
&\qquad \frac{S(x_{m_k},x_{m_k},x_{m_k+1})S(x_{m_k},x_{m_k},x_{n_k+1})}{S(x_{n_k},x_{n_k},x_{m_k})}\Big\}.
\end{aligned}$$

On letting $k \to \infty$, and by using (4.5), (4.6) and (4.7), we have

$\varepsilon \le \lim_{k\to\infty} M(x_{n_k}, x_{n_k}, x_{m_k}) \le \varepsilon$. That is

$$\lim_{k\to\infty} M(x_{n_k}, x_{n_k}, x_{m_k}) = \varepsilon. \tag{4.8}$$

Also, $N(x_{n_k}, x_{n_k}, x_{m_k}) = \min\{S(x_{n_k}, x_{n_k}, Tx_{n_k}), S(x_{m_k}, x_{m_k}, Tx_{n_k})\}$.

On letting $k \to \infty$ and by using (4.5), we have

$$\lim_{k\to\infty} N(x_{n_k}, x_{n_k}, x_{m_k}) = 0. \tag{4.9}$$

By Lemma 4.3, we have $\alpha(x_{n_k}, x_{n_k}, x_{m_k}) \ge 1$. Now, by using (3.1), we have

$$\begin{aligned}
0 &\le \zeta(\alpha(x_{n_k}, x_{n_k}, x_{m_k})S(Tx_{n_k}, Tx_{n_k}, Tx_{m_k}), M(x_{n_k}, x_{n_k}, x_{m_k}) + LN(x_{n_k}, x_{n_k}, x_{m_k})) \\
&< M(x_{n_k}, x_{n_k}, x_{m_k}) + LN(x_{n_k}, x_{n_k}, x_{m_k}) - \alpha(x_{n_k}, x_{n_k}, x_{m_k})S(x_{n_k+1}, x_{n_k+1}, x_{m_k+1}).
\end{aligned}$$

That is

$S(x_{n_k+1}, x_{n_k+1}, x_{m_k+1}) \le \alpha(x_{n_k}, x_{n_k}, x_{m_k})S(x_{n_k+1}, x_{n_k+1}, x_{m_k+1}) < M(x_{n_k}, x_{n_k}, x_{m_k}) + LN(x_{n_k}, x_{n_k}, x_{m_k})$.

Let $p_k = \alpha(x_{n_k}, x_{n_k}, x_{m_k})S(x_{n_k+1}, x_{n_k+1}, x_{m_k+1})$ and $s_k = M(x_{n_k}, x_{n_k}, x_{m_k}) + LN(x_{n_k}, x_{n_k}, x_{m_k})$ and by using (4.5)-(4.9), we obtain that $\lim_{k\to\infty} p_k = \lim_{k\to\infty} s_k = \varepsilon > 0$ for all k.

Now, by (4.1) and by (ζ_3), we have

$0 \leq \limsup_{k\to\infty} \zeta(\alpha(x_{n_k}, x_{n_k}, x_{m_k})S(x_{n_k+1}, x_{n_k+1}, x_{m_k+1}), M(x_{n_k}, x_{n_k}, x_{m_k}) + LN(x_{n_k}, x_{n_k}, x_{m_k})) < 0,$
a contradiction. Thus $\varepsilon = 0$.

Therefore $\{x_n\}$ is a Cauchy sequence. Since (X,S) is a complete S-metric space, there exists a $u \in X$, such that $\lim_{n\to\infty} x_n = u$.

STEP 4: We now prove that u is a fixed point of T. Suppose that (i) holds. Then we have $u = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tu$.
Therefore $u = Tu$.

Now assume that (ii) holds. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x_{n_k}, u) \geq 1$ for all k. By (4.1), we have

$$0 \leq \zeta(\alpha(x_{n_k}, x_{n_k}, u)S(Tx_{n_k}, Tx_{n_k}, Tu), M(x_{n_k}, x_{n_k}, u) + LN(x_{n_k}, x_{n_k}, u)). \tag{4.10}$$

Here
$$\begin{aligned} M(x_{n_k}, x_{n_k}, u) &= \max\{S(x_{n_k}, x_{n_k}, u), \tfrac{S(x_{n_k},x_{n_k},Tx_{n_k})S(x_{n_k},x_{n_k},Tx_{n_k})}{S(x_{n_k},x_{n_k},u)}, \tfrac{S(x_{n_k},x_{n_k},Tx_{n_k})S(u,u,Tu)}{S(x_{n_k},x_{n_k},u)}, \\ &\quad \tfrac{S(x_{n_k},x_{n_k},Tu)S(u,u,Tx_{n_k})}{S(x_{n_k},x_{n_k},u)}, \tfrac{S(u,u,Tx_{n_k})S(x_{n_k},x_{n_k},Tu)}{S(x_{n_k},x_{n_k},u)}, \tfrac{S(u,u,Tu)S(u,u,Tx_{n_k})}{S(x_{n_k},x_{n_k},u)}\} \\ &= \max\{S(x_{n_k}, x_{n_k}, u), \tfrac{S(x_{n_k},x_{n_k},x_{n_k+1})S(x_{n_k},x_{n_k},x_{n_k+1})}{S(x_{n_k},x_{n_k},u)}, \tfrac{S(x_{n_k},x_{n_k},x_{n_k+1})S(u,u,Tu)}{S(x_{n_k},x_{n_k},u)}, \\ &\quad \tfrac{S(x_{n_k},x_{n_k},Tu)S(u,u,x_{n_k+1})}{S(x_{n_k},x_{n_k},u)}, \tfrac{S(u,u,x_{n_k+1})S(x_{n_k},x_{n_k},Tu)}{S(x_{n_k},x_{n_k},u)}, \tfrac{S(u,u,Tu)S(u,u,x_{n_k+1})}{S(x_{n_k},x_{n_k},u)}\}. \end{aligned}$$

On letting $n \to \infty$, we get $\lim_{n\to\infty} M(x_{n_k}, x_{n_k}, u) = 0$.
Also, $\lim_{n\to\infty} N(x_n, x_n, u) = 0$. Now, by (4.10), we have

$$\begin{aligned} 0 &\leq \zeta(\alpha(x_{n_k}, x_{n_k}, u)S(x_{n_k+1}, x_{n_k+1}, Tu), M(x_{n_k}, x_{n_k}, u) + LN(x_{n_k}, x_{n_k}, u)) \\ &< M(x_{n_k}, x_{n_k}, u) + LN(x_{n_k}, x_{n_k}, u) - \alpha(x_{n_k}, x_{n_k}, u)S(x_{n_k+1}, x_{n_k+1}, Tu) \end{aligned}$$

which implies that

$$\begin{aligned} S(x_{n_k+1}, x_{n_k+1}, Tu) &= S(Tx_{n_k}, Tx_{n_k}, Tu) \\ &\leq \alpha(x_{n_k}, x_{n_k}, u)S(x_{n_k+1}, x_{n_k+1}, Tu) \\ &< M(x_{n_k}, x_{n_k}, u) + LN(x_{n_k}, x_{n_k}, u). \end{aligned}$$

On letting $k \to \infty$, we get that $S(u, u, Tu) \leq 0$. Thus $u = Tu$. □

Corollary 4.6. *Let (X,S) be a complete S-metric space and $\zeta \in \mathscr{Z}$. Suppose that there exists $L \geq 0$ such that*

$$\zeta(\alpha(x,x,y)S(Tx,Tx,Ty), M(x,x,y) + LN(x,x,y)) \geq 0 \tag{4.11}$$

for all $x,y \in X$, where $M(x,x,y)$ and $N(x,x,y)$ are obtained from (4.1). *Assume that* (i), (ii) *and* (iii) *of* Theorem 4.5 *hold. Then T has a fixed point in X.*

Proof. By choosing $y = x$ and $z = y$ in the inequality (4.1), proof of this corollary follows from Theorem 4.5. □

Corollary 4.7. *Let (X,S) be a complete S-metric space and $T : X \to X$ be a mapping satisfying*

$$\alpha(x,y,z)S(Tx,Ty,Tz) \leq \lambda M(x,y,z) \tag{4.12}$$

for all $x,y,z \in X$, where $M(x,y,z)$ is defined as in the inequality (4.1). *Assume that* (i), (ii) *and* (iii) *of* Theorem 4.5 *hold. Then T has a fixed point in X.*

Proof. If we choose simulation function ζ as $\zeta(t,s) = \lambda s - t$ for all $s,t \geq 0$, where $\lambda \in [0,1)$, then the inequality (4.12) is a special case of the inequality (4.1) so that from Theorem 4.5, the conclusion of this corollary follows.

□

Corollary 4.8. *Let (X,S) be a complete S-metric space and $T : X \to X$ be a mapping satisfying*

$$\alpha(x,y,z)S(Tx,Ty,Tz) \leq M(x,y,z) - \varphi(M(x,y,z)) \tag{4.13}$$

for all $x,y,z \in X$, where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi continuous function with $\varphi(t) = 0$ if and only if $t = 0$ and $M(x,y,z)$ is defined as in the inequality (4.1). *Assume that* (i), (ii) *and* (iii) *of* Theorem 4.5 *hold. Then T has a fixed point in X.*

Proof. Follows by choosing $\zeta(t,s)$ is as in the Example 2.2 (v), $L = 0$ in the inequality (4.1) and by applying Theorem 4.5. □

The following example is in support of Theorem 4.5.

Example 4.9. Let $X = [\frac{1}{4}, 1]$. We define $S : X^3 \to [0,\infty)$ by

$$S(x,y,z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x,y,z\} & \text{otherwise.} \end{cases}$$

We define $T : X \to X$ by

$$Tx = \begin{cases} 1 - x^2 & \text{if } x \in [\frac{1}{4}, \frac{31}{50}) \\ 1 & \text{if } x \in [\frac{31}{50}, 1]. \end{cases}$$

We define $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ by $\zeta(t,s) = s - 2t$ for all $t,s \in [0,\infty)$. Then $\zeta \in \mathscr{Z}$. We define $\alpha : X \times X \times X \to [0,\infty)$ by

$$\alpha(x,y,z) = \begin{cases} 1 & \text{if } \frac{31}{50} \le x,y \le 1, \frac{31}{50} < z \le 1; \text{ or } x = y = z = \frac{31}{50} \\ 0 & \text{otherwise.} \end{cases}$$

Here T is α-admissible. For, assume that $\alpha(x,y,z)) \ge 1$.

Then we have either $x,y \in [\frac{31}{50}, 1]$, $z \in (\frac{31}{50}, 1]$ or $x = y = z = \frac{31}{50}$.

If $x,y \in [\frac{31}{50}, 1]$ and $z \in (\frac{31}{50}, 1]$ then $\alpha(Tx,Ty,Tz) = \alpha(1,1,1) = 1$.

If $x = y = z = \frac{31}{50}$ then $\alpha(Tx,Ty,Tz) = \alpha(1,1,1) = 1$.

Also, T is triangular α-admissible. Let $\alpha(x,x,y) \ge 1$ and $\alpha(y,y,z) \ge 1$.

Subcase (i) Let $\frac{31}{50} \le y \le 1$, $\frac{31}{50} < z \le 1$ and $x = y = \frac{31}{50}$. Then we have $\alpha(x,x,z) = 1$.

Subcase (ii) Let $\frac{31}{50} \le x \le 1$, $\frac{31}{50} < y \le 1$ and $\frac{31}{50} \le y \le 1$, $\frac{31}{50} < z \le 1$. Then $\alpha(x,x,z) = 1$.

Subcase (iii) Let $x = y = \frac{31}{50}$ and $y = z = \frac{31}{50}$ then we have $\alpha(x,x,z) = 1$.

Let $x,y,z \in X$. We now verify that

$\zeta(\alpha(x,y,z)S(Tx,Ty,Tz), M(x,y,z) + LN(x,y,z)) \ge 0$.

Let $x,y,z \in [\frac{31}{50}, 1]$. If $\frac{1}{2} < z \le 1$ then we have

$2\alpha(x,y,z)S(Tx,Ty,Tz) = 2S(1,1,1) = 0$. We consider

$$\begin{aligned} \zeta(\alpha(x,y,z)S(Tx,Ty,Tz), M(x,y,z) + LN(x,y,z)) &= M(x,y,z) + LN(x,y,z) - 2\alpha(x,y,z)S(Tx,Ty,Tz) \\ &= M(x,y,z) + LN(x,y,z) \ge 0 \text{ for any } L \ge 0.. \end{aligned}$$

In all the remaining cases, the inequality (4.1) holds trivially for any $L \ge 0$.

Therefore T is an α-admissible almost generalized $\mathscr{Z}_s$-contraction with rational expressions. For $x_0 = 1$, we have $\alpha(x_0, x_0, Tx_0) = 1$.

Suppose that $\{x_n\}$ is a sequence in $[\frac{31}{50}, 1]$ such that $\alpha(x_n, x_n, x_{n+1}) = 1$ and $x_n \to x$.

If $x \ne \frac{31}{50}$, then we have $\alpha(x_n, x_n, x) = 1$.

If $x = \frac{31}{50}$, then we choose subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} = \frac{31}{50}$ for each k. Then we have $\alpha(x_{n_k}, x_{n_k}, x) = 1$. Therefore T satisfies all the hypotheses of Theorem 4.5 for any $L \ge 0$ and T has two fixed points $1, \frac{-1+\sqrt{5}}{2}$.

Here we prove that the inequality (2.1) fails to hold, for $x = \frac{1}{4}$, $y = 1$ in (2.1), we have $S(Tx, Tx, Ty) = S(\frac{15}{16}, \frac{15}{16}, 1) = 1$ and $S(x, x, y) = S(\frac{1}{4}, \frac{1}{4}, 1) = 1$.
In this case, we have $S(Tx, Tx, Ty) = 1 \nleq \lambda(1) = \lambda S(x, x, y)$ for any $0 \leq \lambda < 1$.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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