Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 3, 2768-2779
https://doi.org/10.28919/jmcs/5202
ISSN: 1927-5307

ANALYSIS ON PROPERTIES OF VECTOR SPACES OVER PRE A*-ALGEBRAS

JONNALAGADDA VENKATESWARA RAO[1], T. NAGESWARA RAO[2], S.R. RAVI KUMAR EMANI[3], M.N. SRINIVAS[4,*], B.J. BALAMURUGAN[5]

[1]Department of Mathematics, School of Science and Technology, United States International University, Nairobi, Kenya

[2]Department of Mathematics, Koneru Lakshmaiah Education Foundation, Green Fields, Vaddeswaram-522502, Guntur, Andhra Pradesh, India

[3]V.R.Siddhartha Engineering College, Vijawawada, Andhra Pradesh, India

[4]Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Vellore—632014, Tamilnadu, India

[5]Division of Mathematics, School of Advanced Sciences, Vellore Institute of Technology, Chennai—600127, Tamilnadu, India

Abstract. In this work the perception of vector space is initiated over Pre A*-algebras. This article discusses the basic properties of Pre A*-vector spaces, the notion of norm and their worth while representations.

Keywords: pre A*-algebra; pre A*-vector space; normed pre A*-vector space; Boolean pre A*-ring; R-module; pre A*-metric space; Boolean semiring.

2010 AMS Subject Classification: 06E05, 06E25, 06E99, 06B10, 16P60, 54E35.

*Corresponding author
E-mail address: mnsrinivaselr@gmail.com
Received November 13, 2020

1. Introduction and Preliminaries

Fernando et al. [1] originated the algebra of conditional logic and an equational 3-valued generality of Boolean algebra established on logic functions "or", "and" and "not". Manes [3] invented the Ada, in view of C-algebras. KoteswaraRao [2], started the idea of A*-algebra and contemplated its equality with [3], [1] and its connection with 3-ring. Venkateswara Rao [7] introduced the thought of Pre A*-algebra as reduct of [2], analogous to [1]. Satyanarayana et al. [4] well-thought-out the partial ordering. Venkateswara Rao, et al. [8] acknowledged the thought of Congruences. The idea of vector spaces over Boolean algebras started by Subrahmanyam [6] is the inspiration to the current examination. Further, Subrahmanyam [5] started the connection between the Boolean vector spaces with Boolean semirings. This manuscript imparts the vector spaces over Pre A*-algebra. In other words simply, the vector space here is a vector space in which scalars are elements in Pre A*-algebra.

Definition 1.1 [7]: A Pre A*-algebra is a system (A, $\wedge$, $\vee$, $(-)^{\sim}$) satisfying, for x, y z in A:

(a) $x^{\sim\sim}$ = x (double tilde rule)

(b) x $\wedge$ x = x (idempotent rule respecting $\wedge$)

(c) x $\wedge$ y = y $\wedge$ x (commutative rule respecting $\wedge$)

(d) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}$(De Morgan's rule)

(e) x $\wedge$ (y $\wedge$ z) = (x $\wedge$ y) $\wedge$ z (associative rule respecting $\wedge$)

(f) x $\wedge$ (y $\vee$ z) = (x $\wedge$ y) $\vee$ (x $\wedge$ z) ($\wedge$ is distributive over $\vee$)

(g) x $\wedge$ y = x $\wedge$ $(x^{\sim} \vee$ y) (representation).

Example 1.1 [7]: A three element Pre A* algebra (**3** = $\{0, 1, 2\}$) by means of $\wedge$, $\vee$, $(-)^{\sim}$ described as:

$\wedge$	0	1	2		$\vee$	0	1	2		x	$x^{\sim}$
0	0	0	2		0	0	1	2		0	1
1	0	1	2		1	1	1	2		1	0
2	2	2	2		2	2	2	2		2	2

Note 1.1 [7]: From the above (Example 1.1) we note the following: (a) 2 is merely the self-tilde element. (b) 1 is the $\wedge$ identity element. (c) 0 is the $\vee$ identity element. (d) 2 is the uncertain element.

Example 1.2 [7]: The two element Pre A* algebra (**2** = $\{0, 1\}$) by means of $\wedge$, $\vee$, $(-)^{\sim}$ described as:

$\wedge$	0	1		$\vee$	0	1		x	$x^{\sim}$
0	0	0		0	0	1		0	1
1	0	1		1	1	1		1	0

2. PRE A*- VECTOR SPACES (RESULTS AND DISCUSSIONS)

Definition 2.1: Let V be an abelian group under addition, also A be a Pre A*-algebra. V is named a Pre A*-vector space over A if there exists a mapping from, $A \times V \to V$ such that, $\forall$ u, v $\in$ V and a, b in A,

(i) a . (u + v) = a . u + a . v

(ii) a . (b . v) = $(a \wedge b)$. v

(iii) If $a \wedge b = 0$, then $(a \vee b)$. v = a . v + b . v

(iv) 1 . v = v for all v $\in$ V.

Note 2.1: We note the product a v from the ordered pairs of the above as scalar multiplication.

Theorem 2.1:

Let A be Pre A*-algebra. For all a, b in A, $a + b = (a \wedge b^{\sim}) \vee (a^{\sim} \wedge b)$ and $a . b = a \wedge b$.

Then (A, +, .) exists as Boolean Pre A*-ring.

Proof: By the expression,

$(a \wedge b^{\sim}) \vee (a^{\sim} \wedge b) = (a \vee (a^{\sim} \wedge b)) \wedge (b^{\sim} \vee ((b^{\sim})^{\sim} \wedge a^{\sim}))$

$= (a \vee b) \wedge (a \wedge b)^{\sim}$

Hence, a + b = b + a, follows by above

Consider, $(a + b) + c = (a \wedge b^{\sim} \wedge c^{\sim}) \vee (a^{\sim} \wedge b \wedge c^{\sim}) \vee (a^{\sim} \wedge b^{\sim} \wedge c) \vee (a \wedge b \wedge c)$

The above is symmetric in a, b, c and therefore, + is associate and commutative.

For any a $\in$ A, consider $a + 0 = (a \wedge 0^{\sim}) \vee (a^{\sim} \wedge 0)$

$= (a \wedge 1) \vee (a^{\sim} \wedge a)$ (since, $a^{\sim} \wedge 0 = a^{\sim} \wedge a$)

$= a \wedge (1 \vee a^{\sim}) = a \wedge (a^{\sim} \vee 1) = a \wedge 1$ (by representation) = a.

Similarly, we can see that 0 + a = a. Hence, 0 is the additive identity in A.

Further, note that, $a + (a \wedge a^{\sim}) = [a \wedge (a \wedge a^{\sim})^{\sim}] \vee [a^{\sim} \wedge (a \wedge a^{\sim})]$

$= [a \wedge (a^{\sim} \vee a)] \vee [(a^{\sim} \wedge a^{\sim}) \wedge a)]$

$= (a \wedge a) \vee (a^{\sim} \wedge a) = a \vee (a^{\sim} \wedge a) = a \vee a = a.$

This leads to $a + (a \wedge a^{\sim}) = a$ for each a in A.

Similarly, we can verify that $(a \wedge a^{\sim}) + a = a$ for each a in A.

By above, we conclude that $a + 0 = a = a + (a \wedge a^{\sim})$ and hence, $a \wedge a^{\sim} = 0$, the additive identity for each a in A.

To prove that every element of A has additive inverse:

Consider, $a + b = (a \wedge b^{\sim}) \vee (a^{\sim} \wedge b)$. Put b = a.

Then, $a + a = (a \wedge a^{\sim}) \vee (a^{\sim} \wedge a) = a \wedge a^{\sim} = 0$, the additive identity for each a in A(by above).

Hence, a is additive inverse of a in A. Therefore, (A, +) is an abelian group.

Clearly, the multiplication is associative in A (since, $\wedge$ is associative in A).

To prove verify the distributive laws in A.

Let $a, b, c \in A$.

Consider, $a.(b + c) = a \wedge [(b \wedge c^{\sim}) \vee (b^{\sim} \wedge c)]$

$= [a \wedge (b \wedge c^{\sim})] \vee [a \wedge (b^{\sim} \wedge c)]$

$=[(a \wedge b) \wedge c^{\sim})] \vee [(a \wedge c) \wedge b^{\sim}]$ (1)

On the other hand, let us consider,

$a . b + a . c = (a \wedge b) + (a \wedge c)$

$= [(a \wedge b) \wedge (a \wedge c)^{\sim}] \vee [(a \wedge b)^{\sim} \wedge (a \wedge c)]$

$= [(a \wedge b) \wedge (a^{\sim} \vee c^{\sim})] \vee [(a^{\sim} \vee b^{\sim}) \wedge (a \wedge c)]$

$= [(a \wedge b) \wedge a^{\sim}] \vee [(a \wedge b) \wedge c^{\sim}] \vee [(a \wedge c) \wedge a^{\sim}] \vee [(a \wedge c) \wedge b^{\sim}]$

$= [(a \wedge a^{\sim}) \wedge b] \vee [(a \wedge b) \wedge c^{\sim}] \vee [(a \wedge a^{\sim}) \wedge c] \vee [(a \wedge c) \wedge b^{\sim}]$

$= [(a \wedge 0) \wedge b] \vee [(a \wedge b) \wedge c^{\sim}] \vee [(a \wedge 0) \wedge c] \vee [(a \wedge c) \wedge b^{\sim}]$(since, $a \wedge a^{\sim} = a \wedge 0$)

$= [(a \wedge b) \wedge 0] \vee [(a \wedge b) \wedge c^{\sim}] \vee [(a \wedge c) \wedge 0] \vee [(a \wedge c) \wedge b^{\sim}]$

$= \{[(a \wedge b) \wedge (a \wedge b)^{\sim}] \vee [(a \wedge b) \wedge c^{\sim}]\} \vee \{[(a \wedge c) \wedge (a \wedge c)^{\sim}] \vee [(a \wedge c) \wedge b^{\sim}]\}$

(since, $a \wedge 0 = a \wedge a^{\sim}$)

$= [(a \wedge b) \wedge ((a \wedge b)^{\sim} \vee c^{\sim})] \vee [(a \wedge c) \wedge ((a \wedge c)^{\sim} \vee b^{\sim})]$

$= [(a \wedge b) \wedge c^{\sim}] \bigvee [(a \wedge c) \wedge b^{\sim}]$ (2)

By (1) and (2), a . (b + c) = a . c + a . c. Since, . is commutative (as $\wedge$ is so), we have the other distributive law. Thus, (A, +, .) is a Pre A*-ring with identity 1.

Since, a . a = a $\wedge$ a = a for all a in A, (A, +, .) is a Boolean Pre A*-ring in which 0 and 1 as required.

Example 2.1: Let A be any Pre A*-algebra and V be the additive group of the resultant Pre A*-ring as in the 2.1 theorem. Then V is an A-vector space if for $a \in A$ and $v \in V$, av in A.

Theorem 2.2: Let R be any ring with 1. Suppose that there is defined a subset A of R as $A = \{r \in R / r^2 = r$ and $rs = sr$ for all $s \in R\}$, set of central idempotents. Then, $(A, \vee, \wedge, (-)^{\sim})$ stands as Pre A*-algebra, through operations: $x \vee y = x + y - x.y$; $x \wedge y = x.y$ and $x^{\sim} = 1 - x$, for all $x, y \in A$.

Proof: For that entire x, y in A, we verify the postulates as required.

(i) $x^{\sim\sim} = (x^{\sim})^{\sim} = (1-x)^{\sim} = 1 - (1 - x) = 1 - 1 + x = x$.

(ii) and (iii) are clear.

(iv) $(x \wedge y)^{\sim} = (x.y)^{\sim} = 1 - x.y$.

Also consider $x^{\sim} \vee y^{\sim} = (1-x) + (1-y) - (1-x)(1-y) = 1 - x\,y$.

(v) Clearly $\wedge$ is associative.

(vi) Consider, $x \wedge (y \vee z) = x.y + x.z - x.y.z$ (I)

Also consider, $(x \wedge y) \vee (x \wedge z) = (x\,.\,y) \vee (x\,.\,z) = x\,y + x\,z - x\,y\,z$ (II)

(since $x^2 = x$).

Hence, by (I) and (II), the result follows as required.

(vii) Consider, $x \wedge (x^{\sim} \vee y) = x\,.\,(1-x) + x.y - x\,(1-x)\,y$. Hence, the result follows as required.

Therefore, $(A, \vee, \wedge, (-)^{\sim})$ is an algebra as required.

Illustration 2.2: Let us consider the Pre A*-algebra A and R as in the above 2.2 theorem. If V is the additive group of the ring R, then V is a Pre A*- vector space over $(A, \vee, \wedge, (-)^{\sim})$ with the similar scalar product as discussed above.

Illustration 2.3: Let $(A = P(S), \wedge, \vee, (-)^{\sim})$ be the Pre A*-algebra of all subsets of a set S (A = P(S), power set of S) and $V = \{v / v: S \to G\}$, the functions of S into a group G with respect to addition; any $u, v \in V$; $a \in A$ (a = subset of S), define, $(u + v)(p) = u(p) + v(p)$ for all $p \in S$

and (av) (p) = v p if $p \in a$; (av) (p) = 0 if $p \notin a$. At that juncture V is a Pre A*-vector space over A.

Illustration 2.4: An illustration of a Pre A*- vector space is $L_n(A) = A^n$, where, $A^n = A \times \cdots \times A$ (n factors). In this instance, we define, the vector addition and scalar multiplication defined as follows:

(i) $(a_1, \ldots, a_n) + (b_1, \ldots, b_n) = ((a_1 \wedge b_1^{\sim}) \vee (a_1^{\sim} \wedge b_1), \ldots, (a_n \wedge b_n^{\sim}) \vee (a_n^{\sim} \wedge b_n))$ for all $(a_1, \ldots, a_n), (b_1, \ldots, b_n) \in A^n$ and

(ii) $a \,.\, (b_1, \ldots b_n) = (a \wedge b_1, \ldots, a \wedge b_n)$, for all $a \in A$ and $(b_1, \ldots, b_n) \in A^n$.

Here, + is a binary operation on A^n and . (scalar multiplication) is a map from $A \times A^n \to A^n$.

Verification: Left to the reader as it is straight forward verification.

Theorem 2.3: Let A^n be a Pre A*- vector space over A. Then A^n is a Pre A*-algebra.

Proof: Let $u, v \in L_n(A)$.

Define, "$u \vee v = (u_1, u_2, \ldots\ldots, u_n) \vee (v_1, v_2, \ldots\ldots, v_n) = (u_1 \vee v_1, u_2 \vee v_2, \ldots\ldots, u_n \vee v_n)$;

$u \wedge v = (u_1, u_2, \ldots\ldots, u_n) \wedge (v_1, v_2, \ldots\ldots, v_n) = (u_1 \wedge v_1, u_2 \wedge v_2, \ldots\ldots, u_n \wedge v_n)$ and

$$(u)^{\sim} = (u_1, u_2, \ldots\ldots, u_n)^{\sim} = (u_1^{\sim}, u_2^{\sim}, \ldots\ldots, u_n^{\sim}).$$

(1) Consider $u^{\sim\sim} = (u^{\sim})^{\sim} = ((u_1^{\sim}, u_2^{\sim}, \ldots\ldots, u_n^{\sim}))^{\sim} = (u_1, u_2, \ldots\ldots, u_n) = u$, for all $u \in A^n$.

(2) Consider $u \wedge u = (u_1, u_2, \ldots\ldots, u_n) \wedge (u_1, u_2, \ldots\ldots, u_n) = (u_1, u_2, \ldots\ldots, u_n) = u$, for all $u \in A^n$.

(3) Let $u, v \in L_n(A)$. Consider $u \wedge v = (u_1, u_2, \ldots\ldots, u_n) \wedge (v_1, v_2, \ldots\ldots, v_n)$
$= (v_1, v_2, \ldots\ldots, v_n) \wedge (u_1, u_2, \ldots\ldots, u_n) = v \wedge u$, for all $u, v \in L_n(A)$..

(4) Consider, $(u \wedge v)^{\sim}$

$$= (u_1^{\sim}, u_2^{\sim}, \ldots\ldots, u_n^{\sim}) \vee (v_1^{\sim}, v_2^{\sim}, \ldots\ldots, v_n^{\sim})$$

$= u^{\sim} \vee v^{\sim}$, for all $u, v \in A^n$.

(5) Consider, $u \wedge (v \wedge w = ((u_1, u_2, \ldots\ldots, u_n) \wedge (v_1, v_2, \ldots\ldots, v_n)) \wedge (w_1, w_2, \ldots\ldots, w_n)$
$= (u \wedge v) \wedge w$, for all $u, v, w \in A^n$.

(6) Consider, $u \wedge (v \vee w) = (u_1, u_2, \ldots\ldots, u_n) \wedge ((v_1, v_2, \ldots\ldots, v_n) \vee (w_1, w_2, \ldots\ldots, w_n))$

$$= ((u_1, u_2, \ldots\ldots, u_n) \wedge (v_1, v_2, \ldots\ldots, v_n)) \vee ((v_1, v_2, \ldots\ldots, v_n) \wedge (w_1, w_2, \ldots\ldots, w_n))$$

$= (u \wedge v) \vee (u \wedge w)$, for all u, v, w $\in A^n$.

(7) Consider, $u \wedge (u^{\sim} \vee v) = (u_1, u_2, \ldots\ldots, u_n) \wedge ((u_1, u_2, \ldots\ldots, u_n)^{\sim} \vee (v_1, v_2, \ldots\ldots, v_n))$

$= (u_1, u_2, \ldots\ldots, u_n) \wedge (v_1, v_2, \ldots\ldots, v_n) = u \wedge v$.

Thus, $(A^n, \wedge, \vee, (-)^{\sim})$ is an algebra as required.

Lemma 2.1: Let V be an arbitrary Pre A*-vector space over a Pre A*-algebra. For all v in V and a in A, 0 v = 0 and a 0 = 0.

Proof: Let us consider v = 1 v = $(0 \vee 1)$ v = 0 v + 1 v = 0 v + v. Hence, as required.

Also the second result is obvious. Hence, a 0 = 0.

Lemma 2.2: Let V be an arbitrary Pre A*-vector space over A.

Then, a (- v) = - a v for all a in A and v in V.

Proof: Consider 0 = a 0 = a (v +(-v)) = a v + a (-v). Hence, as required.

Note 2.2 [8]: Henceforth, to enable the subsequent consequences, we consider a, b $\in$ A such that $a \vee b = 1$ (so that $a \vee a^{\sim} = 1$ and $a \wedge a^{\sim} = 0$ in A).

Lemma 2.3: Let V be an arbitrary Pre A*-vector space over A. If a, b $\in$ A such that $a \vee b = 1$ and v $\in$ V, then (i) $a^{\sim} v = v - a\,v$ and (ii) $(a \vee b)\, v = a\,v + b\,v - a\,b\,v$.

Proof: (i) Consider $v = 1v = (a \vee a^{\sim})\, v = a\,v + a^{\sim}\, v$. Hence, the result follows.

(ii) Consider, $(a \vee b)\, v = [a \vee (b \wedge a^{\sim})]\, v$ (since, $a \vee b = a \vee (b \wedge a^{\sim})$)

$= a\,v + (b \wedge a^{\sim})\, v$ (since, $a \wedge (b \wedge a^{\sim}) = 0$)

$= a\,v + b\,(a^{\sim} v) = a\,v + b\,(v + (-a\,v)) = a\,v + b\,v - a\,b\,v$.

Hence, result as required.

Theorem 2.4: Let V be a Pre A*-vector space over a A, such that $a \vee b = 1$, for all a, b in A; and let R = (R, +, .) be a Boolean Pre A*-ring corresponding to A. Then the necessary and sufficient condition for V is a module over R is v + v = 0 for all v $\in$ A.

Proof: Let a, b $\in$ R and v $\in$ V. Let us observe, $(a + b)\, v = (a\,b^{\sim} \vee a^{\sim} b)\, v = a\,b^{\sim}\, v + a^{\sim} b\, v$

$= a\,(v - b\,v) + b\,(v - a\,v) = a\,v + b\,v - 2\,a\,b\,v$.

Successively, V is an R-module equivalently 2 a b v = 0 for all a, b $\in$ A and v $\in$ V, or correspondingly, v + v = 0 for all v $\in$ A.

Definition 2.2: A Pre A*-vector space V over A is said to be Pre-A*-normed if and only if there exists a mapping $\|.\|$: $V \to A$ such that (1) $\|v\| = 0$ if and only if v = 0 and (2) $\|a\ v\| = a\ \|v\|$ for all $a \in A$ and $v \in V$.

Note 2.3: The Pre A*-vector spaces of above examples 2.1 and 2.3 are normed.

Theorem 2.5: For a Pre A*-vector space V over A (with $a \vee b = 1$ for all a, b in A), the subsequent are equivalent: (1) V is Pre A*-normed (2) To each $v \in V$, there relates an element $a_v \in A$ such that (i) $a_v\ v = v$ and (ii) if $b \in A$ and b v = v, then b $a_v = a_v$. (a_v, for a specified a, is exceptional).

Proof: Suppose that (1) holds. So V is A-normed. Let $a_v = \|v\|$.

(i) Consider, $\|v - a_v v\| = \|a_v{}^{\sim} v\| = a_v{}^{\sim}\ \|v\| = a_v{}^{\sim} a_v = 0$. Hence, $a_v v = v$.

(ii) Let $b \in A$ and b v = v. Consider,$a_v = \|v\| = \|b\ v\| = b\,\|v\| = b\ a_v$. Hence, b $a_v = a_v$.

Suppose that (2) holds.

Suppose $c \in A$, $v \in V$ and c v = 0. Then consider, $c^{\sim} v = v - c\ v = v$ (as c v = 0). Hence, $c^{\sim} v = v$. Then, $c^{\sim} a_v = a_v$ (By hypothesis). This indicates, $c\ c^{\sim} a_v = c\ a_v$. Hence, c $a_v = 0$ (asc $c^{\sim} a_v = 0$). Hence, if $b \in A$ and b (c v) = c v, then, $b^{\sim}(c\ v) = c\ v - b\ (c\ v) = c\ v - c\ v = 0$ (as b (c v) = c v). Therefore, $b^{\sim}(c\ v) = 0$ and hence, $b^{\sim} c\ a_v = 0$.

Consider $(c\ a_v)\ (c\ v) = c\ c\ a_v\ v = c\ v$. Thus, $(c\ a_v)\ (c\ v) = c\ v$ (X)

Also, consider, $(a_{c\ v})\ (c\ v) = c\ v$ (Y)

We conclude that $a_{c\ v} = c\ a_v$.

Let us define $\|v\| = a_v$. By above, $a_{c\ v} = \|c\ v\|$ and $c\ a_v = c\ \|v\|$.

So therefore, the mapping, $\|.\|$ describes as required.

Corollary 2.1: If V is a Pre A*-normed vector space (over A), then $\|u + v\| \leq \|u\| \vee \|v\|$ for all $u, v \in V$.

Proof: By above results, we are considering $\|v\| = a_v$ (so that $\|v\|\ v = a_v\ v = v$).

Observe that $(\|u\| \vee \|v\|)\ (u + v) = \|u\|\,(u + v) + \|v\|\,(u + v) - (\,\|u\| \wedge \|v\|)\ (u + v)$

$$= \|u\|\,u + \|u\|\,v + \|v\|\,u + \|v\|\,v - \|u\|\,(\,\|v\|\,(u) + \|v\|\,(v))$$

$= u + \|u\|\ v + \|v\|\ u + v - \|v\|\ u - \|u\|\ v = u + v$.

Therefore, $\|u + v\| = \|(\,\|u\| \vee \|v\|\,)\ (u + v)\ \| = (\,\|u\| \vee \|v\|\,)\ \|(u + v)\|$.

Here, by the partial order on the Pre A*-algebra A [4], we can observe as required.

Corollary 2.2: If V is a Pre A*-normed vector space, then d (u, v) = $\|u-v\|$ defines Pre A*-metric on V.

Proof: (i) Suppose that d (u, v) = 0 if and only if $\|u-v\|=0$ if and only if u – v = 0 if and only if u = v.

(ii) Consider, d (u, v) = $\|u-v\|=\|(-1)(v-u)\|=\|(v-u)-(-1)^{\sim}(v-u)\|$

(Since, a v = v $-a^{\sim}$v, for all a $\in$ A and v $\in$ V, by above lemma)

= $\|v-u\|$ = d (v, u). Hence, d (u, v) = d (v, u) for all u, v $\in$ V.

As the two expressions are symmetric in u and v. Hence, d (u, v) = d (v, u).

(iii) Consider d (u, w) = $\|u-w\| \leq \|u-v\| \vee \|v-v\|$ = d (u, v) $\vee$ d (v, w).

Thus, d becomes a metric as required.

Definition 2.3 [5]: A system (R, +, .) is called a Boolean semiring if it satisfies:

(i) (R, +) is an additive abelian group.

(ii) (R, .) is a semigroup of idempotents in the sense, a a = a, for all a $\in$ R

(iii) a . (b + c) = a . b + a . c and

(iv) a b c = b a c for all a, b, c $\in$ R.

Theorem 2.6: Let V be a normed Pre A*-vector space over A and let, for u, v in V, u v = $\|u\|$ v. Then (V, +, .) is a Boolean semiring.

Proof: (V, +, .) is a Boolean semiring because of the following:

(1) Note that (V, +) is an additive abelian group;

(2) To verify that (V, .) is a semigroup of idempotents:

For any u, v, w $\in$ V, consider (u v) w = $\|u\,v\|$ w = $\|u\|\,\|v\|$ w.

Also consider, u (v w) = $\|u\|$ (v w) = $\|u\|\,\|v\|$ w. Hence, (u v) w = u (v w) for all u, v, w $\in$ V.

For any u $\in$ V, u . u = $\|u\|$ u = u

(as by previous lemma, $a_v v = v$, and by $a_v = \|v\|$, $\|v\|\ v = v$).

(3) For any u, v, w $\in$ V, let us consider, u . (v + w) $= \|u\|\ v + \|u\|\ w$.

Also u v + u w = $\|u\|\ v + \|u\|\ w$. Hence, u.(v + w) = u v + u w for all u, v, w $\in$ V.

(4) For any u, v, v $\in$ V, consider (u v) w = $\|u\,v\|\ w = \|u\|\,\|v\|$ w.Also consider, (v u) w = $\|v\,u\|\,w = \|v\|\,\|u\|\,w = \|u\|\,\|v\|$ w (since, $\|u\|$, $\|v\| \in$A implies, $\|u\| \wedge \|v\|$ = $\|v\| \wedge \|u\|$ and hence, we follow that $\|u\|\,\|v\|$ = $\|v\|\,\|u\|$).

Theorem 2.7: If v $\in$ V, uniquely as v = $a_1v_1+a_2v_2+\ldots.+a_nv_n$, where $v_1, v_2, \ldots., v_n \in$ V and $a_1, a_2, \ldots., a_n \in$ A, then a = $a_1 \vee a_2 \vee \ldots. \vee a_n$ (where $a_i \wedge a_j = a_i$ if i = j and is 0 if i $\neq$ j) is the duplicator of v such that $a_i =$ b a_i.

Proof: To verify that a v = v. Consider, a v = $(a_1 \vee a_2 \vee \ldots. \vee a_n)$ $(a_1v_1+a_2v_2+\cdots+a_nv_n)$

$$= (a_1 \vee a_2 \vee \ldots. \vee a_n)\, a_1v_1 + \cdots + (a_1 \vee a_2 \vee \ldots. \vee a_n)\, a_nv_n$$

$=a_1(a_1v_1)+a_2(a_1v_1)+\ldots a_n(a_1v_1)+\ldots+a_1(a_nv_n)+a_2(a_nv_n)+\ldots a_n(a_nv_n)$

= $a_1v_1+a_2v_2+\ldots.+a_nv_n$ ($a_i \wedge a_j = a_i$ if i = j and is 0 if i $\neq$ j) = v. Hence, a v = v.

Suppose that b v = v for some b = $b_1 \vee b_2 \vee \ldots. \vee b_n$, similarly taken as a = $a_1 \vee a_2 \vee \ldots. \vee a_n$. Then, v = b v = b a_1v_1+b $a_2v_2+\ldots.+$b a_nv_n.

This implies, $a_i =$ b a_i for all i (by the uniqueness of v).

Definition 2.4: A finite subset of nonzero elements $\{v_1, v_2, \ldots., v_n\} \in$ V is named linearly independent over A if and only if $a_1v_1+a_2v_2+\ldots.+a_nv_n = 0$ and $a_1, a_2, \ldots., a_n \neq 0$ imply that $v_1+v_2+\ldots.+v_n = 0$. A subset of nonzero elements of V is called linearly independent over A if and only if every limited subset of S is linearly independent.

Definition 2.5: A subset S of V spans V if and only if each v $\in$ V can be written as a finite sum v = $\sum_{g \in S} a_g g$, $a_g a_h = 0$ for g different from h and $a_g = 0$ for nearly all g in S.

Definition 2.6: A basis of V is (i) linearly independent subset of V; and (ii) spans V.

Example 2.5: Let V be a Pre A*-vector space over A as in 2.3 example. Let K be the set of all nonzero constant maps in V. Then, K is a basis of V. Let K = $\{f_1, f_2, \ldots., f_n\} \subseteq$ V. To verify that $\{f_1, f_2, \ldots., f_n\}$ is linearly independent. Suppose that $f_1a_1+f_2a_2+\ldots.+f_na_n = 0$ and $f_1 . f_2 \ldots. f_n \neq 0$. Then, $a_1+a_2+\ldots.+a_n = 0$ (as each f_i is a constant function).

Hence, K = $\{f_1, f_2, \ldots., f_n\}$ is linearly independent. Let $v_1 \in$ V and $a_v \in$ A such that a_v u = v if u = v and 0 if u $\neq$ v. Then we can see that $v_1 = a_{v_1}v_1 + a_{v_2}v_1 + \ldots a_{v_n}v_1$. Therefore, K is a basis of V.

Lemma 2.4: Let V be a normed Pre A*-vector space and G^* be a basis of V. If g $\in G^*$, then, (i) –g $\in G^*$, (ii) if g, h $\in G^*$ in addition g + h $\neq$ 0, g + h $\in G^*$.

Proof: As G^* spans V, –g = $\sum_{k \in G^*} a_k$ k, where, $a_k a_h = 0$ for k $\neq$ h also $a_k = 0$, nearby all k $\in G^*$. As, g $\neq$ 0, $a_k \neq 0$ for some k (= m, say) in G^*. At that point $-a_m$ g $=a_m(-g)$ = a_mm.

Hence, $a_m(g+m)=0$. As, g, m $\in G^*$, $a_m \neq 0$, in addition to G^* is independent, g + m = 0 and therefore, $-g$ = m $\in G^*$.

If g, h $\in G^*$ in addition to g + h $\neq 0$, we similarly observe that $a_k(g+h)=a_k k$ for some k $\in G^*$ plus $a_k \neq 0$. This implies $a_k\, g + a_k\, h + a_k(-k) = 0$. As, k $\in G^*$ infers, - k $\in G^*$, g + h = k $\in G^*$.

Theorem 2.8: If G^* is a basis of V, then G^* is an additive subgroup G of V.

Lemma 2.5: If g $\in G^*$, then $\|g\| = 1$.

Proof: If $\|g\| = a$, then $a^{\sim}g = g - a\,g = g - \|g\|\,g = g - g = 0$. This implies, $a^{\sim}g = 0$. Since, g $\neq 0$, we must have $a^{\sim} = 0$. Then by above, 0 g = g – a g, so, a g = g. From this, it follows that a = 1 and hence, $\|g\| = 1$.

Lemma 2.6: If u = $\sum_{i=1}^{n} a_i u_i$, where $a_i a_j = 0$ for i $\neq$ j, then $\|u\| = \bigvee_{i=1}^{n} a_i \|u_i\|$.

Proof: If n = 1, then u = $a_1 u_1$ and $\|u\| = \|a_1 u_1\| = a_1 \|u_1\|$.

Suppose that the result is true for n-1. Let v = $\sum_{i=2}^{n} a_i u_i$ and b = $\|v\|$.

Then b = $\|\sum_{i=2}^{n} a_i u_i\| = \bigvee_{i=2}^{n} a_i \|u_i\|$ and u = $a_1 u_1 + v$ (since, u = $\sum_{i=1}^{n} a_i u_i$).

Also, $a_1 v = a_1(\sum_{i=2}^{n} a_i u_i) = a_1 a_2 u_2 + a_1 a_3 u_3 + \cdots + a_1 a_n u_n = 0$ ($a_i a_j = 0$ for i $\neq$ j).

Hence, $a_1 u = a_1 u_1$ (by above, since, a v = 0).

Then, $\|v\| = \|u - a_1 u_1\| = \|u - a_1 u\|$ (since, $a_1 u = a_1 u_1$) $= \|a_1^{\sim} u\| = a_1^{\sim}\|u\|$.

Hence, $\|v\| = a_1^{\sim}\|u\|$.

Thus, $\|u\| = 1\,\|u\| = (a_1 \vee a_1^{\sim})\|u\| = a_1\|u\| \vee a_1^{\sim}\|u\| = a_1\|u_1\| \vee b = \bigvee_{i=1}^{n} a_i \|u_i\|$.

Corollary 2.3: If u = $\sum_{i=1}^{n} a_i u_i$, where, where $a_i a_j = 0$ for i $\neq$ j and $u_1, u_2, \ldots u_n \in G^*$, then $\|u\| = \bigvee_{i=1}^{n} a_i$.

Proof: By above results, the proof is immediate.

CONCLUDING REMARKS

This work made a stand to study vector spaces over algebra and its useful characterizations as well. The Pre A*-vector space is initiated and observed its various representations. An n-factored set $L_n(A)$ (= $A^n = A \times A \times \cdots \times A$ (n-factors)) is observed as a vector space over A and such a Pre A*-vector space is identified as a Pre A*-algebra as well. The notion of normed Pre A*-vector space is initiated and studied its properties. The method of construction of a Boolean semiring from a normed Pre A*-vector space is obtained. It is noted that the basis of the Pre A*-vector space forms a subgroup of the Pre A*-vector space.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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