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THE PO-F (POISSON-FRECHET) DISTRIBUTION: AN OVERVIEW WITH RESPECT TO RAINFALL DATA

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Abstract: A new mix distribution has been proposed in this paper. We have mixed Poisson distribution with Frechet distribution. Various properties like Reliability analysis, moments, quantile function and order statistics of the new distribution have been discussed. Maximum–likelihood estimation method has been used to estimate the parameters. The confidence interval along with fishar information matrix have been proposed. Finally 57 years rainfall data have been applied in the data analysis part and compared with other distributions. As a result, our new distribution has been fit better than the other distributions.

Keywords: Poisson distribution; Frechet distribution; reliability analysis; order statistics; fishar information matrix; rainfall data.

2010 AMS Subject Classification: 62E15, 62G30, 62P15.

1. INTRODUCTION

In the field of Statistics theory, many generalized families of distribution have been proposed and substantially used in modeling data in various applied sciences such as economics, finance, engineering and life testing etc. For example, exponentiated-G (Exp-G) (Gupta et al, 1998),

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Beta-G (Eugene et al, 2002), transmuted-G (Shaw and Buckley, 2007), Kumaraswamy-G (Cordeiro and de Castro, 2011), Mc.Donald-G (Alexandar et al, 2012), transmuted exponentiated generalized-G (Yousof et al, 2015) and generalized transmuted-G (Nofal et al, 2017) families. In this paper, we have discussed the Poisson-G (Po-G) (Hamed and Ebrahim, 2017) family of distribution. In their paper, they have provided a useful mixture representation for the Po-G family in terms of Exp-G densities. The probability density function (pdf) and the cumulative distribution function (cdf) of Po-G family of distribution have been shown as

$$f(x)=\frac{\theta\, g(x)}{e^{\theta}-1}\exp[\theta\, G(x)] \tag{1.1}$$

$$F(x)=\frac{\exp[\theta G(x)]-1}{e^{\theta}-1} \tag{1.2}$$

Hamed and Ebrahim (2017) have proposed Po-W (Poisson-Weibull) model with pdf and cdf as

$$f(x)=\frac{\theta\alpha^{\beta}\beta x^{\beta-1}\exp\left\{-(\alpha x)^{\beta}\right\}}{e^{\theta}-1}\exp\left[\theta-\theta\exp\left\{-(\alpha x)^{\beta}\right\}\right] \tag{1.3}$$

and $$F(x)=\frac{\exp\left[\theta-\theta\exp\left\{-(\alpha x)^{\beta}\right\}\right]-1}{e^{\theta}-1} \tag{1.4}$$

In this paper, we have been discussed about Po-F (Poisson-Frechet) model. The main motivation of considering Poisson- Frechet distribution is that Frechet distribution being two-parameter distribution is very much useful for modeling data relating to rainfall, temperature and it is expected that the proposed distribution, being three-parameter distribution and based on the concept of Poisson-G distribution, would provide better fit over Frechet distribution. Some of the important properties of the proposed distribution including shapes of the pdf and cdf, asymptotic behavior, hazard rate function, reverse hazard rate function, Mills ratio have been studied. Maximum likelihood estimation has been discussed for estimating parameters of the proposed distribution. Finally, applications of the proposed distribution for modeling datasets relating to rainfall of Silchar, Assam have been discussed.

2. THE PO-F (POISSON-FRECHET) DISTRIBUTION

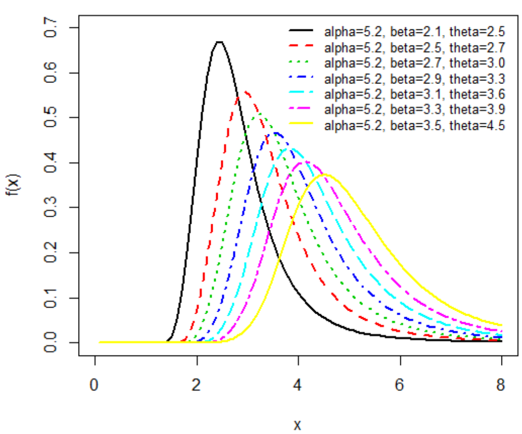
Let us consider the cdf and pdf (for $x>0$), $G(x)=e^{-\left(\frac{\beta}{x}\right)^{\alpha}}$ and $g(x)=\alpha\beta^{\alpha}x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)^{\alpha}}$ respectively of the Frechet distribution with positive parameter α and β. Then the equation (1.1) and (1.2) can be expressed as

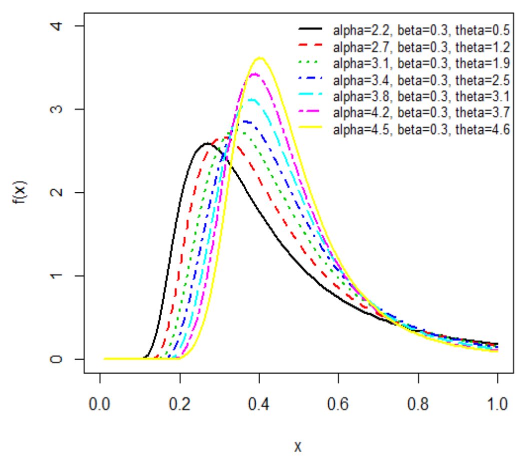
$$f(x;\alpha,\beta,\theta)=\frac{\theta\alpha\beta^{\alpha}x^{-(\alpha+1)}\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}-\left(\frac{\beta}{x}\right)^{\alpha}\right\}}{e^{\theta}-1};\ x>0 \text{ and } (\alpha,\beta,\theta)>0 \tag{2.1}$$

$$F(x;\alpha,\beta,\theta)=\frac{\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\right\}-1}{e^{\theta}-1};\ x>0 \text{ and } (\alpha,\beta,\theta)>0 \tag{2.2}$$

where α and θ are shape parameter and β is scale parameter. Further, $\lim_{x\to-\infty}F(x;\alpha,\beta,\theta)=0$ and $\lim_{x\to\infty}F(x;\alpha,\beta,\theta)=1$. This shows that the Po-F distribution has proper density function.

Graphs of the pdf and the cdf of Po-F distribution have shown in figure 1 and figure 2 for varying the value of parameters $[\alpha,\beta \text{ and } \theta]$.





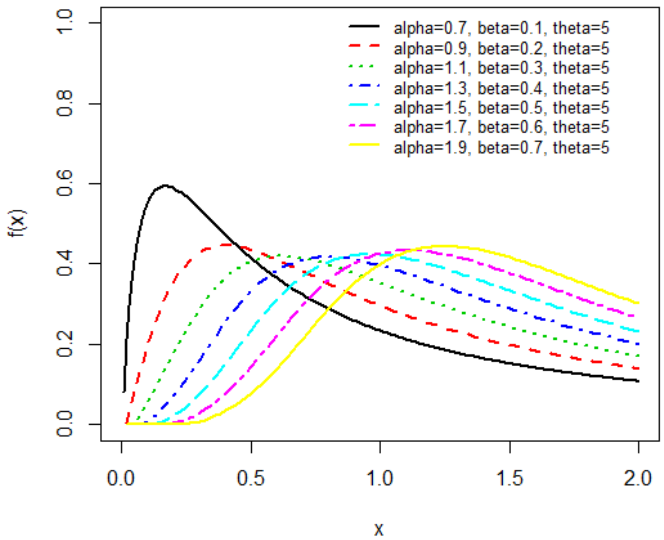


Figure 1. pdf plot of Po-F distribution for varying the value of parameters

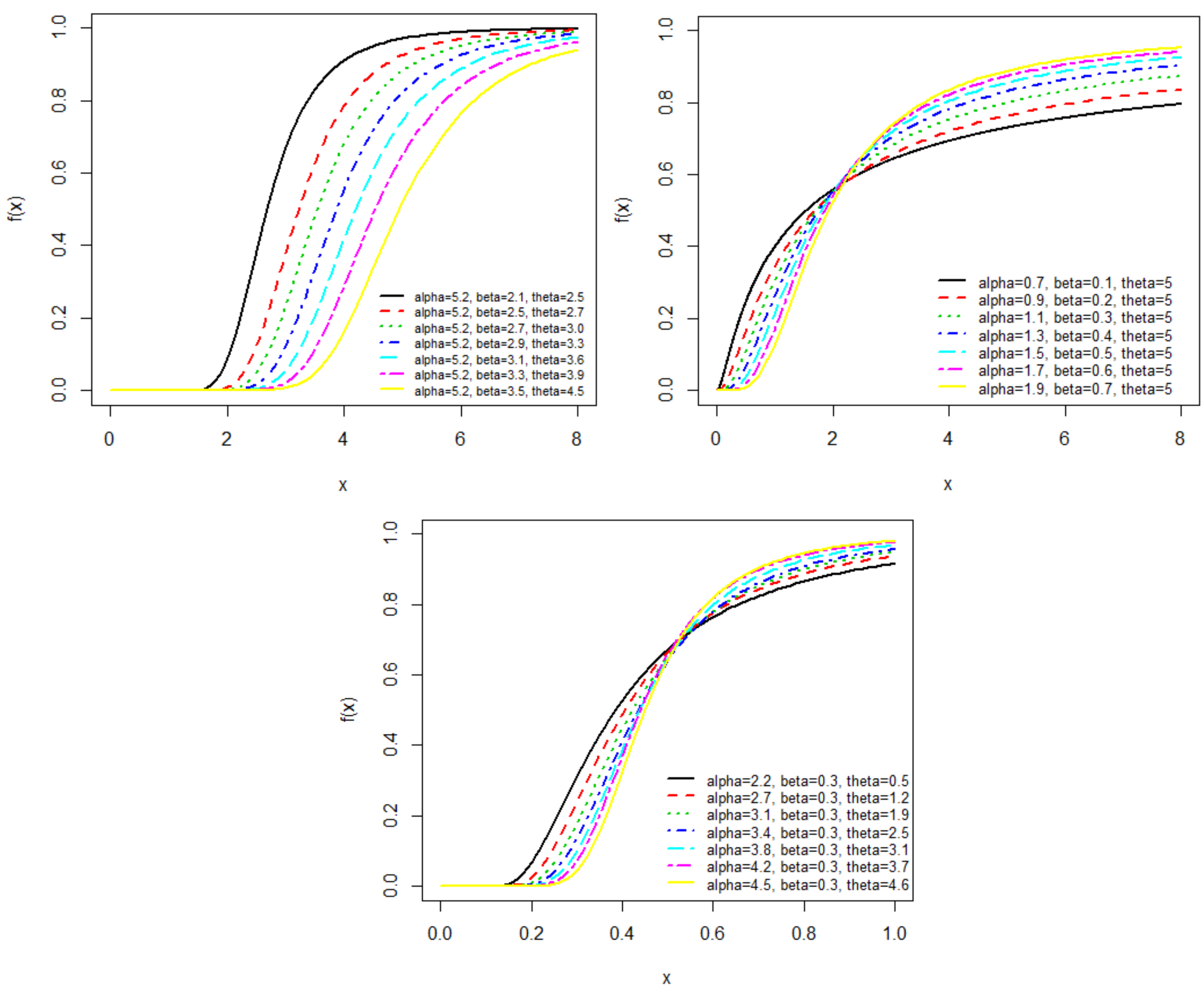


Figure 2. cdf plot of Po-F distribution for varying the value of parameters

3. STATISTICAL PROPERTIES

In this section, statistical properties including asymptotic behavior, reliability properties, quantile function of Po-F distribution has been discussed.

3.1. Asymptotic behavior

The asymptotic behavior of Po-F distribution for $x \to 0$ and $x \to \infty$ are

$$\lim_{x\to 0} f\left(x;\alpha,\beta,\theta\right)=\lim_{x\to 0}\frac{\theta\alpha\beta^{\alpha}x^{-(\alpha+1)}\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}-\left(\frac{\beta}{x}\right)^{\alpha}\right\}}{e^{\theta}-1}=0$$

$$\lim_{x\to \infty} f\left(x;\alpha,\beta,\theta\right)=\lim_{x\to \infty}\frac{\theta\alpha\beta^{\alpha}x^{-(\alpha+1)}\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}-\left(\frac{\beta}{x}\right)^{\alpha}\right\}}{e^{\theta}-1}=0$$

These results confirm that the proposed distribution has a mode.

3.2. Reliability Analysis

The survival function (or the reliability function) is the probability that a subject survives longer than the expected time. The survival function of the Po-F distribution is given by

$$S\left(x;\alpha,\beta,\theta\right)=1-F\left(x;\alpha,\beta,\theta\right)=\frac{e^{\theta}-\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\right\}}{e^{\theta}-1}$$

The hazard function (also known as the hazard rate, instantaneous failure rate or force of mortality) is the probability to measure the instant death rate of a subject. The hazard rate function of Po-F distribution is given by

$$h\left(x;\alpha,\beta,\theta\right)=\frac{\alpha\beta^{\alpha}\theta x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\right\}}{e^{\theta}-\exp\left\{\theta e^{\left(\frac{\beta}{x}\right)^{\alpha}}\right\}}$$

The reverse hazard rate is the ratio between the probability density function and its distribution function. The reverse hazard function of Po-F distribution is given by

$$h_r(x;\alpha,\beta,\theta)=\frac{\alpha\beta^{\alpha}\theta x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\right\}}{\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\right\}-1}$$

Mills ratio is the ratio between the survival function and a probability density function. The mills ratio of Po-F distribution is given by

$$m(x;\alpha,\beta,\theta)=\frac{e^{\theta}-\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\right\}}{\alpha\beta^{\alpha}\theta x^{-(\alpha+1)}e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\right\}}$$

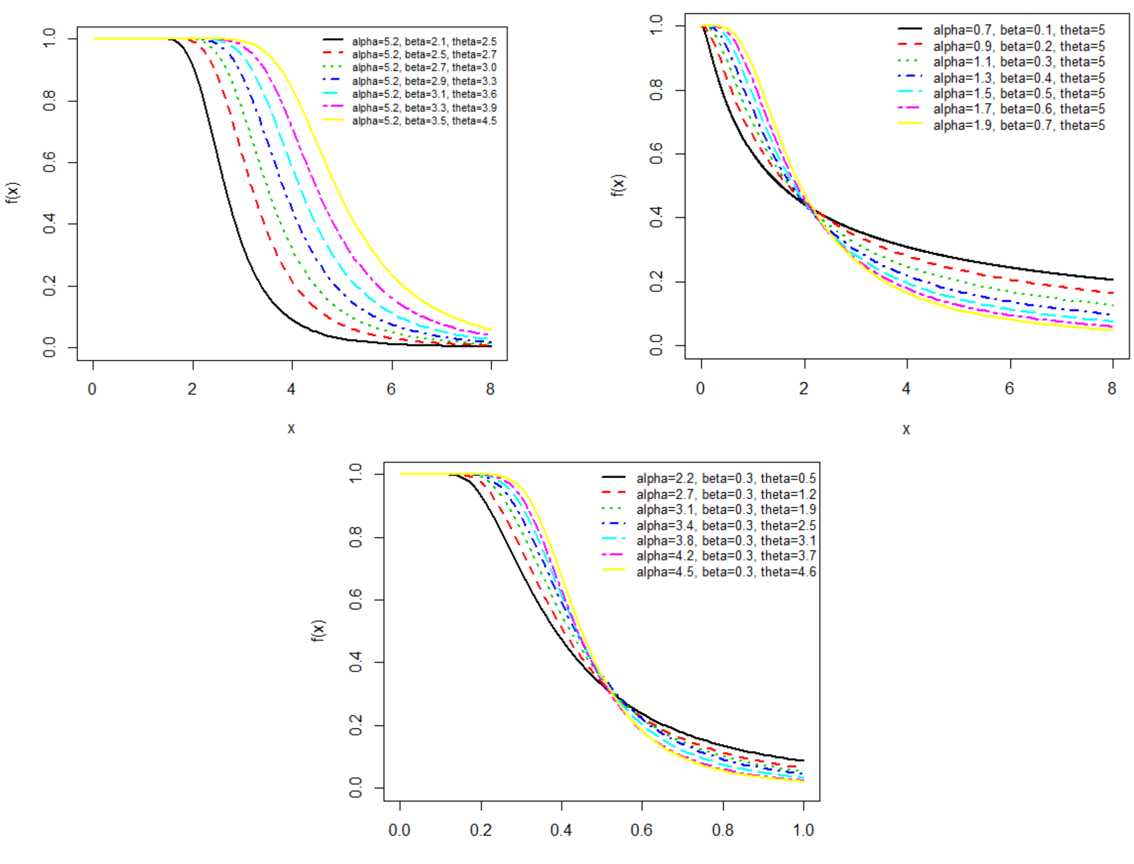


Figure 3. survival plots of Po-F distribution for varying the value of parameters

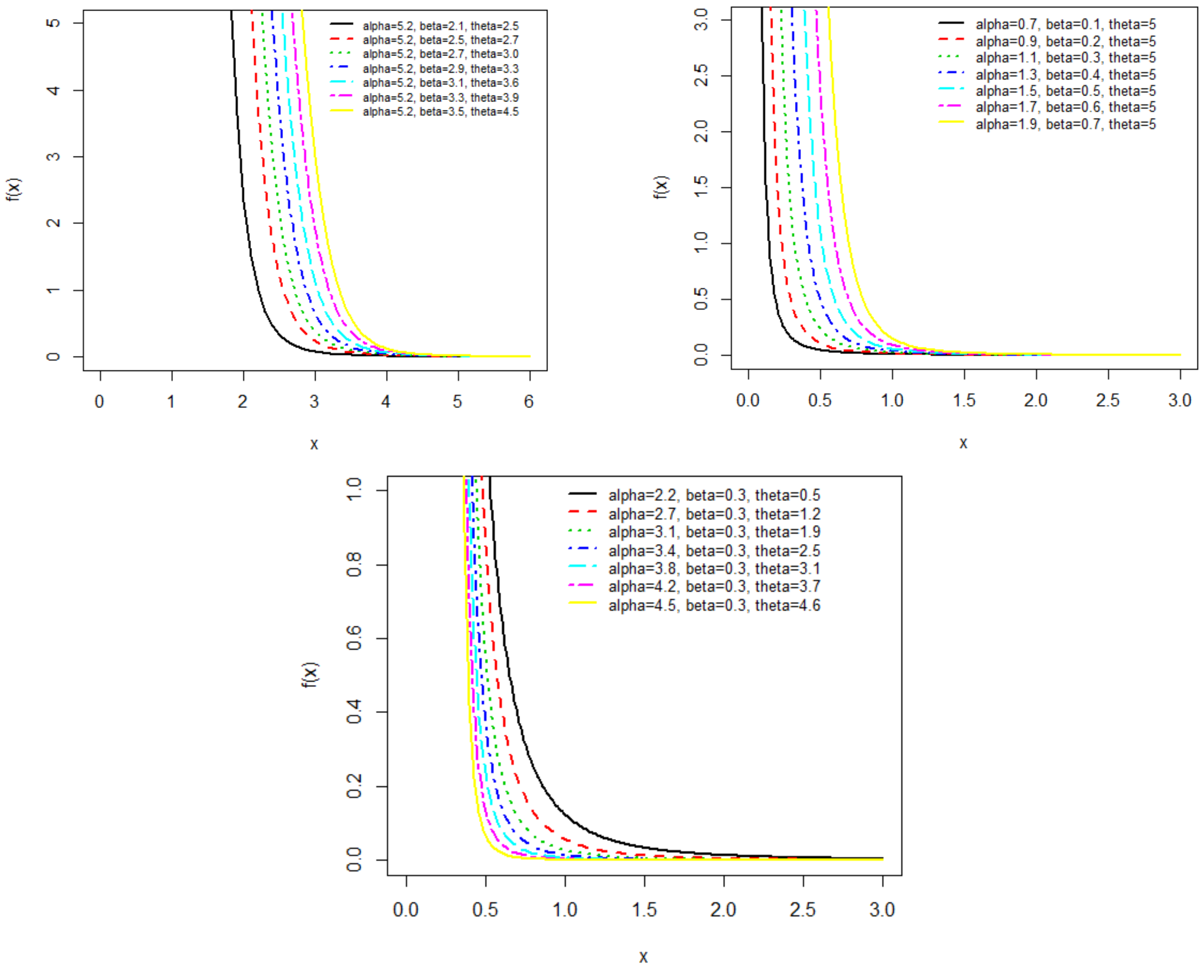


Figure 4. hazard plots of Po-F distribution for varying the value of parameters

3.3. Quantile Function

The quantile function is defined as

$$Q(u) = F^{-1}(u)$$

Therefore, the corresponding quantile function for Po-F distribution can be expressed as

$$Q(u) = \frac{\beta}{-\left[\log\left[\frac{\log\{u(e^{\theta}-1)-1\}}{\theta}\right]\right]^{\frac{1}{\alpha}}}$$

Let U has the uniform $U(0,1)$ distribution. Taking $u = 0.5$, the median of Po-F distribution can be obtained as

$$Q(0.5)=\frac{\beta}{-\left[\log\left[\frac{\log\{0.5(e^{\theta}-1)-1\}}{\theta}\right]\right]^{\frac{1}{\alpha}}}$$

Thus, the formula for generating random samples from Po-F distribution for simulating random variable X is given by

$$X=Q(u)=\frac{\beta}{-\left[\log\left[\frac{\log\{u(e^{\theta}-1)-1\}}{\theta}\right]\right]^{\frac{1}{\alpha}}}$$

4. DISTRIBUTION OF ORDER STATISTICS

Let $x_1, x_2, ..., x_n$ be the random samples from Po-F distribution (α,β,θ). The pdf of i^{th} order statistics is given by

$$f_{i:n}(x)=\frac{n!}{(i-1)!(n-i)!}f_X(x)[F_X(x)]^{i-1}[1-F_X(x)]^{n-i}$$

The pdf of i^{th} order statistics $X_{(i)}$ of Po-F distribution is given by

$$f_{i:n}(x;\alpha,\beta,\theta)=\frac{n!}{(i-1)!(n-i!)}\frac{\theta\alpha\beta^{\alpha}x^{-(\alpha+1)}\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}-\left(\frac{\beta}{x}\right)^{\alpha}\right\}}{e^{\theta}-1}\left[\frac{\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\right\}-1}{e^{\theta}-1}\right]^{i-1}\left[1-\frac{\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\right\}-1}{e^{\theta}-1}\right]^{n-i}$$

Therefore, the pdf of the first order statistic $X_{(1)}$ can be expressed as

$$f_{1:n}(x;\alpha,\beta,\theta)=n\frac{\theta\alpha\beta^{\alpha}x^{-(\alpha+1)}\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}-\left(\frac{\beta}{x}\right)^{\alpha}\right\}}{e^{\theta}-1}\left[1-\frac{\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\right\}-1}{e^{\theta}-1}\right]^{n-1}$$

The pdf of the highest order statistic $X_{(n)}$ can be expressed as

$$f_{n:n}(x;\alpha,\beta,\theta)=n\frac{\theta\alpha\beta^{\alpha}x^{-(\alpha+1)}\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}-\left(\frac{\beta}{x}\right)^{\alpha}\right\}}{e^{\theta}-1}\left[\frac{\exp\left\{\theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}\right\}-1}{e^{\theta}-1}\right]^{n-1}$$

5. MAXIMUM LIKELIHOOD ESTIMATION

Let $x_1, x_2, ..., x_n$ be a random sample of size n from a Po-F distribution (α, β, θ). The log-likelihood function can be expressed as

$$\log L = n\left[\log\alpha + \alpha\log\beta + \log\theta - \log\left(e^{\theta} - 1\right)\right] - (\alpha+1)\sum_{i=1}^{n} x_i - \left(\frac{\beta}{\sum_{i=1}^{n} x_i}\right)^{\alpha} + \theta e^{-\left(\frac{\beta}{x}\right)^{\alpha}}$$

The maximum likelihood estimate (MLE) $(\hat{\alpha}, \hat{\beta}, \hat{\theta})$ of (α, β, θ) of Po-F distribution are the solutions of the following log- likelihood equations

$$\frac{\partial}{\partial\alpha}\log L = \frac{n}{\alpha} + n\log\beta - \sum_{i=1}^{n} x_i - \alpha\left(\frac{\beta}{\sum_{i=1}^{n} x_i}\right)^{\alpha-1} + \theta e^{-\left(\frac{\beta}{\sum_{i=1}^{n} x_i}\right)^{\alpha}}$$

$$\frac{\partial}{\partial\beta}\log L = \frac{n\alpha}{\beta} - \alpha\beta^{\alpha-1}\left(\frac{1}{\sum_{i=1}^{n} x_i}\right)^{\alpha} - \theta e^{-\left(\frac{\beta}{\sum_{i=1}^{n} x_i}\right)^{\alpha}}$$

$$\frac{\partial}{\partial\theta}\log L = \frac{n}{\theta} + \frac{n}{\left(e^{\theta} - 1\right)e^{\theta}} + e^{-\left(\frac{\beta}{\sum_{i=1}^{n} x_i}\right)^{\alpha}}$$

These log-likelihood equation can't be solved analytically and required statistical software with iterative numerical techniques. These equations can be solved using R-software.

The 3×3 observed information matrix of Po-F distribution can be presented as,

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\theta}\end{pmatrix} \sim \left[\begin{pmatrix}\alpha\\ \beta\\ \theta\end{pmatrix}, \begin{pmatrix} \frac{\partial^2 \log L}{\partial\alpha^2} & \frac{\partial^2 \log L}{\partial\alpha\,\partial\beta} & \frac{\partial^2 \log L}{\partial\alpha\,\partial\theta} \\ \frac{\partial^2 \log L}{\partial\beta\,\partial\alpha} & \frac{\partial^2 \log L}{\partial\beta^2} & \frac{\partial^2 \log L}{\partial\beta\,\partial\theta} \\ \frac{\partial^2 \log L}{\partial\theta\,\partial\alpha} & \frac{\partial^2 \log L}{\partial\theta\partial\beta} & \frac{\partial^2 \log L}{\partial\theta^2} \end{pmatrix}\right]$$

The inverse of the information matrix results in the well-known variance-covariance matrix. The 3×3 approximate Fishar information matrix corresponding to the above observed information matrix is given by

$$I^{-1} = -E\begin{bmatrix} \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \beta} & \frac{\partial^2 \log L}{\partial \alpha \partial \theta} \\ \frac{\partial^2 \log L}{\partial \beta \partial \alpha} & \frac{\partial^2 \log L}{\partial \beta^2} & \frac{\partial^2 \log L}{\partial \beta \partial \theta} \\ \frac{\partial^2 \log L}{\partial \theta \partial \alpha} & \frac{\partial^2 \log L}{\partial \theta \partial \beta} & \frac{\partial^2 \log L}{\partial \theta^2} \end{bmatrix}$$

The solution of the Fisher information matrix will yield asymptotic variance and covariance of the ML estimators for $(\hat{\alpha}, \hat{\beta}, \hat{\theta})$. The approximate 100(1-α)% confidence intervals for (α, β, θ) respectively are $\hat{\alpha} \pm Z_{\frac{\alpha}{2}} \frac{\sigma_{\alpha\alpha}}{n}$, $\hat{\beta} \pm Z_{\frac{\alpha}{2}} \frac{\sigma_{\beta\beta}}{n}$ and $\hat{\theta} \pm Z_{\frac{\alpha}{2}} \frac{\sigma_{\theta\theta}}{n}$, where Z_α is the upper 100αth percentile of the standard normal distribution.

6. APPLICATION

In this study, 57 years of rainfall data of Silchar, Assam from January 1951 to December 2015 which is collected from India Meteorological Department (IMD), Pune, India has been analyzed. In order to compare the Po-F distribution with Frechet distribution (FD), Poisson-Weibull distribution and weibull distribution, we consider the criteria like Bayesian information criterion *(BIC),* Akaike Information Criterion *(AIC),* Akaike Information Criterion Corrected *(AICC)* and $-2\log L$. The better distribution corresponds to lesser values of *AIC, BIC, AICC* and $-2\log L$. The formulae for calculating *AIC, BIC* and *AICC* are as follows:

$$AIC = 2K - 2\log L,\ BIC = k\log n - 2\log L,\ AICC = AIC + \frac{2k(k+1)}{(n-k-1)},$$

where *k* is the number of parameters, *n* is the sample size and *-2 logL* is the maximized value of log likelihood function. The ML estimates of the parameters of the considered distributions along with values of $-2\log L, AIC, AICC$ and BIC for the datasets as mentioned are presented in Table 1.

Table 1: ML estimates of the parameters of the considered distributions along with values of $-2\log L, AIC, AICC, BIC$

Distribution	**ML Estimates of Parameters**			$-2\log L$	AIC	$AICC$	BIC
	α	β	θ				
Po-F	5.8739	175.7850	5.3542	589.2176	595.2176	595.6882	601.2396
Poisson-Weibull	0.10000	0.2486	0.1000	918.5444	924.5444	925.0150	930.5664
Frechet	0.9813	242.5616	-	718.2099	722.2099	722.4407	726.2245
Weibull	0.1000	0.2507	-	915.9685	919.9685	920.1992	923.9831

It is obvious from above table that Po-F distribution provides much better fit than Frechet distribution, Poisson-weibull distribution and Weibull distribution for data relating to rainfall and hence the proposed distribution can be considered an improtant distribution for modeling rainfall data.

7. CONCLUDING REMARKS

In this paper Poisson- Frechet (Po-F) distribution has been proposed. Its statistical properties including behaviour of pdf, cdf and hazard rate function have been discussed. The distribution of the order statistics has been given. The maximum likelihood estimation for estimating parameters of the proposed distribution has been discussed. The applications of the proposed distribution for modeling data relating to rainfall has been explained and the goodness of fit of the Po-F , Poisson-Weibull, Weibull and Frechet distribution have been presented for ready comparison.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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