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A NOTE ON CONSTACYCLIC CODES OVER THE RING $\mathbb{Z}_3[u,v]/\langle u^2-u, v^2, uv, vu\rangle$

ST. TIMOTHY KOM[1,*], O. RATNABALA DEVI[1], TH. ROJITA CHANU[2]

[1]Department of Mathematics, Manipur University, Imphal, Manipur-795003, India

[2]Department of Mathematics, D.M. College of Science, Imphal, Manipur-795001, India

Abstract. In this paper, we study λ-constacyclic codes over the ring $R=\mathbb{Z}_3[u,v]/\langle u^2-u, v^2, uv, vu\rangle$ for $\lambda = (1+u), (2+2u)$ and 2. We introduce a Gray map from R to $\mathbb{Z}_3^3$ and show that the Gray image of a cyclic code is a quasi-cyclic code of index 3. It is proved that the Gray image of λ-constacyclic code over R is permutation equivalent to either quasi-cyclic or quasi-twisted code according to the value of λ. Moreover, we determine the structure of $(1+u)$-constacyclic codes for an odd length n over R and give some suitable examples.

Keywords: cyclic code; constacyclic code; quasi-cyclic code; Gray map.

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1. Introduction

Cyclic codes are an important class of linear codes in coding theory and have been studied extensively by mathematicians for the past few decades. Traditionally, cyclic codes have been studied over finite fields. The discovery of some good non-linear codes over $\mathbb{Z}_2$ via Gray map over $\mathbb{Z}_4$ in [9] had motivated the study of cyclic codes over the finite rings. Since then, there are a lot of works about cyclic codes and their generalizations over finite rings. Some of these works have been discussed in [1, 4, 15, 21].

*Corresponding author
E-mail address: timothyserto@manipuruniv.ac.in
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Constacyclic codes are one of the remarkable generalizations of cyclic codes and many times it has been seen that some linear codes with better parameters are found by using constacyclic codes. In [19], the authors studied linear $(1+u)$-constacyclic codes and cyclic codes over $\mathbb{F}_2+u\mathbb{F}_2$, $u^2=0$ and characterised codes over $\mathbb{F}_2$ which are the Gray images of $(1+u)$-constacyclic codes or cyclic codes over $\mathbb{F}_2+u\mathbb{F}_2$. Later, Karadeniz and Yildiz [13] introduced $(1+v)$-constacyclic codes over $\mathbb{F}_2+u\mathbb{F}_2+v\mathbb{F}_2+uv\mathbb{F}_2$ and constructed some optimal binary codes as the Gray images of the $(1+v)$-constacyclic codes over the ring. In [3], Bayram and Siap introduced the finite ring $\mathbb{Z}_3[v]/\langle v^3-v\rangle$ and studied the algebraic structures of cyclic and constacyclic codes over the ring. Later, Dertli, Cengellenmis and Eren [6] studied the structures of cyclic, constacyclic, quasi-cyclic and their skew codes over the finite commutative ring $\mathbb{Z}_3+v\mathbb{Z}_3+v^2\mathbb{Z}_3$, where $v^3=v$. They determined a sufficient condition for 1-generator skew quasi-constacyclic codes to be free. We can refer to [2, 5, 11, 10, 12, 16, 17, 18] for more studies on the topic.

Recently, Özkan, Dertli and Cengellenmis [17] introduced the finite commutative ring $\mathbb{F}_2+u_1\mathbb{F}_2+u_2\mathbb{F}_2$, $u_1^2=u_1, u_2^2=0, u_1u_2=u_2u_1=0$ and studied on $(1+u_2)$-constacyclic codes over the ring of odd length. It was shown that the Gray image of linear $(1+u_2)$-constacyclic codes over the ring of odd length is a quasi-cyclic code of index 4 and length $4n$ over $\mathbb{F}_2$.

Being motivated by the above listed works, in this paper we consider the commutative finite non-chain ring $R=\mathbb{Z}_3[u,v]/\langle u^2-u, v^2, uv, vu\rangle$ of order 27, which can be described as $\mathbb{Z}_3+u\mathbb{Z}_3+v\mathbb{Z}_3$ with $u^2=u$, $v^2=0$, $uv=vu=0$, $\mathbb{Z}_3=\{0,1,2\}$ and study λ-constacyclic codes over the ring, where λ is a unit in R. The paper is organised as follows. In section 2, we give some basic structures of the ring R and recall standard definitions of codes. Next, we introduce a Gray map from R to $\mathbb{Z}_3^3$ and show that the Gray image of cyclic code is a quasi-cyclic code of index 3. In section 4, we prove that the Gray image of $(1+u)$-constacyclic and $(2+2u)$-constacyclic codes over R are permutation equivalent to a quasi-cyclic and quasi-negacyclic codes over $\mathbb{Z}_3$, respectively. The structure of $(1+u)$-constacyclic code is discussed in Section 5 and provide some suitable examples. Section 6 concludes the paper.

2. Preliminaries

Let $R = \mathbb{Z}_3[u,v]/\langle u^2-u, v^2, uv, vu\rangle$ and $\mathbb{Z}_3 = \{0,1,2\}$. Then $R = \{a+ub+vc \mid a,b,c \in \mathbb{Z}_3\}$ is a commutative ring with cardinality 27 and characteristic 3. The set of units of the ring is $U = \{1,2,(1+u),(1+v),(1+2v),(2+v),(2+2u),(2+2v),(1+u+v),(1+u+2v),(2+2u+v),(2+2u+2v)\}$. These units can be seen as $U_1 = \{1,2,(1+u),(2+2u)\} = \{\lambda \in U \mid \lambda^2 = 1\}$ and $U_2 = \{(1+v),(1+2v),(2+v),(2+2v),(1+u+v),(1+u+2v),(2+2u+v),(2+2u+2v)\} = \{\lambda \in U \mid \lambda^2 \neq 1\}$. In this work, we use units - $2,(1+u)$ and $(2+2u)$ of the ring R in the following discussions. The ideals of the ring R are

$I_0 = \{0\}$, $I_1 = R$, $I_u = \{0,u,2u\}$, $I_v = \{0,v,2v\}$,

$I_{1+2u} = \{0,v,2v,1+2u,2+u,1+2u+v,1+2u+2v,2+u+v,2+u+2v\}$, and

$I_{u+v} = \{0,u,v,2u,2v,u+v,2u+v,u+2v,2u+2v\}$.

Clearly, R is a semi-local ring with two maximal ideals I_{1+2u} and I_{u+v} and it is a finite non-chain ring. The ring R is isomorphic to the ring $\mathbb{Z}_3+u\mathbb{Z}_3+v\mathbb{Z}_3$ with $u^2 = u, v^2 = 0$ and $uv = vu = 0$.

The following are some of the definitions that will be used in the sequel. For other basic terms and results not mentioned here, we refer [7, 14, 20]. A linear code C over R of length n is a R-submodule of R^n. An element of C is called a codeword. A cyclic code C of length n over R is a linear code with the property that if $c = (c_0,c_1,c_2,\ldots,c_{n-1}) \in C$, then $\sigma(c) = (c_{n-1},c_0,c_1,\ldots,c_{n-2}) \in C$. σ is called cyclic shift operator from R^n to R^n. A linear code C of length n over R is λ-constacyclic code if $c = (c_0,c_1,c_2,\ldots,c_{n-1}) \in C$, then $\gamma_\lambda(c) = (\lambda c_{n-1},c_0,c_1,\ldots,c_{n-2}) \in C$, where λ is a unit in R. γ_λ is called λ-constacyclic shift operator from R^n to R^n. If $\lambda = -1$, then the constacyclic code is called a negacyclic code. We can identify each codeword $c = (c_0,c_1,c_2,\ldots,c_{n-1}) \in C$ with a polynomial $c(x) = c_0+c_1x+c_2x^2+\ldots+c_{n-1}x^{n-1} \in R_n = R[x]/\langle x^n-1\rangle$. With the help of this one-one correspondence between C and R_n, we have the following results.

Proposition 2.1. *A subset C of R^n is a cyclic code of length n if and only if its polynomial representation is an ideal of $R_n = R[x]/\langle x^n-1\rangle$.*

Proposition 2.2. *A subset C of R^n is a constacyclic code of length n if and only if its polynomial representation is an ideal of $R_{n,\lambda} = R[x]/\langle x^n - \lambda \rangle$.*

Definition 2.3. [6] *Let $a \in \mathbb{Z}_3^{3n}$ with $a = (a_0, a_1, ..., a_n, ..., a_{2n}, ..., a_{3n-1}) = (a^{(0)}|a^{(1)}|a^{(2)})$, where $a^{(i)} \in \mathbb{Z}_3^n$ for $i = 0, 1, 2$ and "|" is the usual vector concatenation. Let ρ be a map from $\mathbb{Z}_3^{3n}$ to $\mathbb{Z}_3^{3n}$ defined by $\rho(a) = (\sigma(a^{(0)})|\sigma(a^{(1)})|\sigma(a^{(2)}))$, where σ is a cyclic shift operator from $\mathbb{Z}_3^n$ to $\mathbb{Z}_3^n$. A code C of length $3n$ over $\mathbb{Z}_3$ is called a quasi-cyclic code (or QC code) of index 3 if $\rho(C) = C$.*

Similarly, a code C of length $3n$ over $\mathbb{Z}_3$ is called a quasi-negacyclic code of index 3 if $\eta(C) = C$, where $\eta(a) = (\tau(a^{(0)})|\tau(a^{(1)})|\tau(a^{(2)}))$ and τ is a negacyclic shift operator from $\mathbb{Z}_3^n$ to $\mathbb{Z}_3^n$.

3. Gray Map and Cyclic Codes Over R

In this section, we introduce a Gray map ϕ on the ring R and consider the algebraic structures of cyclic codes over the ring R.

In order to connect the structure of the ring R with $\mathbb{Z}_3^3$, we define the Gray map ϕ

$$\phi : R \to \mathbb{Z}_3^3$$

by
$$\phi(a + ub + vc) = (a + 2b, b, c),$$

where $a + ub + vc \in R$ and $a, b, c \in \mathbb{Z}_3$.

From the definition, we observe that

$\phi(0) = (0,0,0), \phi(1) = (1,0,0), \phi(2) = (2,0,0), \phi(u) = (2,1,0), \phi(v) = (0,0,1), \phi(2u) = (1,2,0), \phi(2v) = (0,0,2), \phi(1+u) = (0,1,0), \phi(1+v) = (1,0,1), \phi(2+u) = (1,1,0), \phi(2+v) = (2,0,1), \phi(1+2u) = (2,2,0), \phi(1+2v) = (1,0,2), \phi(2+2u) = (0,2,0), \phi(2+2v) = (2,0,2), \phi(u+v) = (2,1,1), \phi(2u+v) = (1,2,1), \phi(u+2v) = (2,1,2), \phi(2u+2v) = (1,2,2), \phi(1+u+v) = (0,1,1), \phi(1+2u+v) = (2,2,1), \phi(1+u+2v) = (0,1,2), \phi(1+2u+2v) = (2,2,2), \phi(2+u+v) = (1,1,1), \phi(2+2u+v) = (0,2,1), \phi(2+u+2v) = (1,1,2)$ and $\phi(2+2u+2v) = (0,2,2)$.

It can be easily checked that ϕ is bijective. The map ϕ can be extended in a natural way to R^n component-wise. For $p = (p_0, p_1, ..., p_{n-1}) \in R^n$, ϕ can be defined as follows:

$$\phi : R^n \to \mathbb{Z}_3^{3n}$$

by

$$(1)\quad \phi(p_0,p_1,\ldots,p_{n-1}) = (a_0+2b_0,a_1+2b_1,\ldots,a_{n-1}+2b_{n-1},b_0,b_1,\ldots,b_{n-1},c_0,c_1,\ldots,c_{n-1}),$$

where $p_i = a_i+ub_i+vc_i \in R$ and $a_i,b_i,c_i \in \mathbb{Z}_3$ for $i = 0,1,\ldots,n-1$.

Let C be a linear code of length n over R. For any $r = (r_0,r_1,\ldots,r_{n-1}) \in C$ the Hamming weight $w_H(r)$ is defined as the number of non-zero components in r. The minimum Hamming weight $w_H(C)$ of a code C is the smallest weight among all its non-zero codewords. For $r = (r_0,r_1,\ldots,r_{n-1})$ and $r' = (r'_0,r'_1,\ldots,r'_{n-1})$ in C, the Hamming distance between r and r' is defined by $d_H(r,r') = w_H(r-r')$ and the Hamming distance for a code C is defined by $d_H(C) = \min\{d_H(r,r') \,|\, r,r' \in C\}$.

The Lee weight of any element $r = (r_0,r_1,\ldots,r_{n-1}) \in R^n$ is defined by $w_L(r) = \sum_{i=0}^{n-1} w_L(r_i)$, where $w_L(r_i) = w_H(a_i+2b_i,b_i,c_i)$ for $r_i = a_i+ub_i+vc_i \in R, i = 0,1,\ldots,n-1$. The Lee distance for the code C is defined by $d_L(C) = \min\{d_L(r,r') \,|\, r \neq r', \forall\, r,r' \in C\}$, where $d_L(r,r')$ is the Lee distance between r and r' defined by $d_L(r,r') = w_L(r-r')$.

Theorem 3.1. *The Gray map $\phi : R^n \to \mathbb{Z}_3^{3n}$ is a distance preserving $\mathbb{Z}_3$-linear map from R^n (with respect to Lee distance, d_L) to $\mathbb{Z}_3^{3n}$ (with respect to Hamming distance, d_H).*

Proof. Let $p = (p_0,p_1,\ldots,p_{n-1})$, $q = (q_0,q_1,\ldots,q_{n-1}) \in R^n$, where $p_i = a_i+ub_i+vc_i$, $q_i = e_i+uf_i+vg_i \in R$ for $i = 0,1,\ldots,n-1$ and $\alpha \in \mathbb{Z}_3$. Then

$$\begin{aligned}
\phi(p+q) =& \phi(p_0+q_0,p_1+q_1,\ldots,p_{n-1}+q_{n-1})\\
=&(a_0+e_0+2(b_0+f_0),\ldots,a_{n-1}+e_{n-1}+2(b_{n-1}+f_{n-1}),b_0+f_0,\\
&\ldots,b_{n-1}+f_{n-1},\, c_0+g_0,\ldots,c_{n-1}+g_{n-1})\\
=&(a_0+2b_0,\ldots,a_{n-1}+2b_{n-1},b_0,\ldots,b_{n-1},c_0,\ldots,c_{n-1})\\
&+(e_0+2f_0,\ldots,e_{n-1}+2f_{n-1},f_0,\ldots,f_{n-1},g_0,\ldots,g_{n-1})\\
=&\phi(p)+\phi(q).
\end{aligned}$$

And,

$$\begin{aligned}\alpha\phi(p) =&\alpha(a_0+2b_0,...,a_{n-1}+2b_{n-1},b_0,...,b_{n-1},c_0,...,c_{n-1})\\ =&(\alpha a_0+2\alpha b_0,...,\alpha a_{n-1}+2\alpha b_{n-1},\alpha b_0,...,\alpha b_{n-1},\alpha c_0,...,\alpha c_{n-1})\\ =&\phi(\alpha p).\end{aligned}$$

Hence, ϕ is a $\mathbb{Z}_3$-linear map.

Since ϕ is a linear map, we have $\phi(p-q)=\phi(p)-\phi(q)$, for any $p,q\in R^n$. By the definition of the Lee distance, we have $d_L(p,q)=w_L(p-q)=w_H(\phi(p-q))=w_H(\phi(p)-\phi(q))=d_H(\phi(p),\phi(q))$. This shows that ϕ is a distance preserving $\mathbb{Z}_3$-linear map. □

Theorem 3.2. *If C is a linear code of length n over R with cardinality $|C|=3^k$ and Lee distance d_L, then the Gray image $\phi(C)$ is a $[3n,k,d_H]$ linear code over $\mathbb{Z}_3$.*

Proof. Since C is a linear code of length n over R with $|C|=3^k$, $p+q\in C$ and $\alpha p\in C$ for any $p,q\in R^n$, $\alpha\in\mathbb{Z}_3$. Let $\phi(p),\phi(q)\in\phi(C)$ and $\alpha\in\mathbb{Z}_3$. Then $\phi(p)+\phi(q)=\phi(p+q)\in\phi(C)$ as $p+q\in C$ and $\alpha\phi(p)=\phi(\alpha p)\in\phi(C)$ as $\alpha p\in C$. So, $\phi(C)$ is a linear code.
From the definition of ϕ and Theorem 3.1, we observe that $\phi(C)$ is a $[3n,k,d_H]$ linear code over $\mathbb{Z}_3$ with $d_L=d_H$. □

Example 3.3. *If $C=\{(0,0,0),(v,v,v),(2v,2v,2v),(u,0,0),(2u,0,0),(u+v,v,v),(u+2v,v,v),(2u+v,v,v),(2u+2v,2v,2v)\}$, then C is a linear code of length 3 over R and $\phi(C)$ is a [9,2,2] linear code over $\mathbb{Z}_3$.*

Theorem 3.4. *Let ϕ be the Gray map from R^n to $\mathbb{Z}_3^{3n}$. Let σ be the cyclic shift operator and ρ be the quasi-cyclic shift operator as defined in the preliminaries. Then $\phi\sigma=\rho\phi$.*

Proof. Let $p=(p_0,p_1,...,p_{n-1})\in R^n$, where $p_i=a_i+ub_i+vc_i\in R$ and $a_i,b_i,c_i\in\mathbb{Z}_3$, for $i=0,1,...,n-1$. Now, $\phi(p)=(a_0+2b_0,a_1+2b_1,...,a_{n-1}+2b_{n-1},b_0,b_1,...,b_{n-1},c_0,c_1,...,c_{n-1})$. Applying ρ on both sides, we get

$$\begin{aligned}\rho\phi(p) =&\rho(a_0+2b_0,a_1+2b_1,...,a_{n-1}+2b_{n-1},b_0,b_1,...,b_{n-1},c_0,c_1,...,c_{n-1})\\ =&(a_{n-1}+2b_{n-1},a_0+2b_0,...,a_{n-2}+2b_{n-2},b_{n-1},b_0,...,b_{n-2},c_{n-1},c_0,...,c_{n-2}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\phi\sigma(p) =&\phi(p_{n-1},p_0,\dots,p_{n-2})\\ &=(a_{n-1}+2b_{n-1},a_0+2b_0,\dots,a_{n-2}+2b_{n-2},\, b_{n-1},b_0,\dots,b_{n-2},\, c_{n-1},c_0,\dots,c_{n-2}).\end{aligned}$$

$\therefore\ \ \phi\sigma = \rho\phi.$ □

Corollary 3.5. *Let C be a subset of R^n. Then C is a cyclic code of length n over R if and only if the Gray image $\phi(C)$ is a quasi-cyclic code of index 3 over $\mathbb{Z}_3$ with length $3n$.*

Proof. Suppose C is a cyclic code of length n over R. Then $\sigma(C) = C$. Applying ϕ on both sides, we get $\phi\sigma(C) = \phi(C)$. Also, by Theorem 3.4, $\rho\phi(C) = \phi\sigma(C) = \phi(C)$. This shows that $\phi(C)$ is a quasi-cyclic code of index 3 over $\mathbb{Z}_3$ with length 3n.

Conversely, let us assume that the Gray image $\phi(C)$ of C is a quasi-cyclic code of index 3 over $\mathbb{Z}_3$ with length $3n$. Then $\rho\phi(C) = \phi(C)$. By Theorem 3.4, $\phi\sigma(C) = \rho\phi(C) = \phi(C)$. Since ϕ is injective, it follows that $\sigma(C) = C$. This shows that C is a cyclic code of length n over R. □

We can consider the permutation version ϕ_π of the Gray map ϕ defined on $R^n \to \mathbb{Z}_3^{3n}$ as

$$\begin{aligned}\phi_\pi(p_0,p_1,\dots,p_{n-1}) &= (\phi(p_0),\phi(p_1),\dots,\phi(p_{n-1}))\\ &= (a_0+2b_0,b_0,c_0,a_1+2b_1,b_1,c_1,\dots,a_{n-1}+2b_{n-1},b_{n-1},c_{n-1}),\end{aligned}$$

where $p_i = a_i+ub_i+vc_i \in R$ and $a_i,b_i,c_i \in \mathbb{Z}_3$ for $i = 0,1,2,\dots,n-1$.

Using the above permutation version of the Gray map, we obtain the following results.

Theorem 3.6. *Let ϕ be the Gray map from R^n to $\mathbb{Z}_3^{3n}$, σ be the cyclic shift operator and ϕ_π be the permutation version of the Gray map ϕ as given before. Then $\phi_\pi\sigma = \sigma^3\phi_\pi$.*

Proof. For any $p = (p_0,p_1,\dots,p_{n-1}) \in R^n$, where $p_i = a_i+ub_i+vc_i \in R$ and $a_i,b_i,c_i \in \mathbb{Z}_3$ for $i = 0,1,2,\dots,n-1$. We have, $\sigma(p) = (p_{n-1},p_0,p_1,\dots,p_{n-2})$. Applying ϕ_π, we get

$$\begin{aligned}\phi_\pi\sigma(p) &= \phi_\pi(p_{n-1},p_0,p_1,\dots,p_{n-2})\\ &= (\phi(p_{n-1}),\phi(p_0),\phi(p_1),\dots,\phi(p_{n-2}))\\ &= (a_{n-1}+2b_{n-1},b_{n-1},c_{n-1},a_0+2b_0,b_0,c_0,\dots,a_{n-2}+2b_{n-2},b_{n-2},c_{n-2}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\phi_\pi(p) &= (a_0+2b_0, b_0, c_0, a_1+2b_1, b_1, c_1, ..., a_{n-1}+2b_{n-1}, b_{n-1}, c_{n-1})\\
\sigma\,\phi_\pi(p) &= (c_{n-1}, a_0+2b_0, b_0, c_0, a_1+2b_1, b_1, c_1, ..., a_{n-1}+2b_{n-1}, b_{n-1})\\
\sigma^2\,\phi_\pi(p) &= (b_{n-1}, c_{n-1}, a_0+2b_0, b_0, c_0, a_1+2b_1, b_1, c_1, ..., a_{n-1}+2b_{n-1})\\
\sigma^3\phi_\pi(p) &= (a_{n-1}+2b_{n-1}, b_{n-1}, c_{n-1}, a_0+2b_0, b_0, c_0, ..., a_{n-2}+2b_{n-2}, b_{n-2}, c_{n-2}).
\end{aligned}$$

$\therefore\ \phi_\pi\sigma = \sigma^3\phi_\pi$. □

Corollary 3.7. *Let C be a subset of R^n. Then C is a cyclic code of length n over R if and only if $\phi_\pi(C)$ is equivalent to a 3-quasi-cyclic code of length $3n$ over $\mathbb{Z}_3$.*

Proof. We recall that a linear code C over the ring R is a s-quasi-cyclic if it is invariant under the cyclic shift σ^s, *i.e.*, $\sigma^s(C) = C$, where σ is a cyclic shift on R^n.

Suppose C is a cyclic code of length n over R. Then $\sigma(C) = C$. On applying ϕ_π on both sides and using Theorem 3.6, we get, $\sigma^3\phi_\pi(C) = \phi_\pi(C)$. This shows that $\phi_\pi(C)$ is equivalent to a 3-quasi-cyclic code of length $3n$ over $\mathbb{Z}_3$.

Conversely, let $\phi_\pi(C)$ be a 3-quasi-cyclic code of length $3n$ over $\mathbb{Z}_3$. Then $\sigma^3\phi_\pi(C) = \phi_\pi(C)$. By Theorem 3.6, $\phi_\pi\sigma(C) = \phi_\pi(C)$. This shows that C is a cyclic code of length n over R. □

4. CONSTACYCLIC CODES OVER R

In this section, we discuss the algebraic properties of λ-constacyclic codes of length n over R with $\lambda = (1+u), (2+2u)$ and 2. After thorough investigation, the following results are obtained according to the value of the units in R.

Theorem 4.1. *Let ϕ be the Gray map defined in equation (1), $\gamma_{(1+u)}$ be the $(1+u)$-constacyclic shift operator and ρ be the quasi-cyclic shift operator as defined in the preliminaries. Then $\phi\gamma_{(1+u)} = \delta\rho\phi$, where δ is a permutation of $\mathbb{Z}_3^{3n}$ defined by*
$\delta(x_1, x_2, ..., x_n, ..., x_{2n}, ..., x_{3n}) = (x_{\beta(1)}, ..., x_{\beta(n)}, ..., x_{\beta(2n)}, ..., x_{\beta(3n)})$ with the permutation
$\beta = (1, n+1)$ of $1, 2, ..., 3n$.

Proof. Let $p = (p_0, p_1, ..., p_{n-1}) \in R^n$, where $p_i = a_i + ub_i + vc_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_3$, for $i = 0, 1, ..., n-1$. Then

$$\begin{aligned}\phi\gamma_{(1+u)}(p) =& \phi((1+u)p_{n-1}, p_0, ..., p_{n-2})\\ =& \phi(a_{n-1} + u(a_{n-1} + 2b_{n-1}) + vc_{n-1}, a_0 + ub_0 + vb_0, ..., a_{n-2} + ub_{n-2} + vb_{n-2})\\ =& (b_{n-1}, a_0 + 2b_0, ..., a_{n-2} + 2b_{n-2}, a_{n-1} + 2b_{n-1}, b_0, ..., b_{n-2}, c_{n-1}, c_0, ..., c_{n-2}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\rho\phi(p) =& \rho(a_0 + 2b_0, a_1 + 2b_1, ..., a_{n-1} + 2b_{n-1}, b_0, b_1, ..., b_{n-1}, c_0, c_1, ..., c_{n-1})\\ =& (a_{n-1} + 2b_{n-1}, a_0 + 2b_0, ..., a_{n-2} + 2b_{n-2}, b_{n-1}, b_0, ..., b_{n-2}, c_{n-1}, c_0, c_1, ..., c_{n-2}).\end{aligned}$$

Applying δ on both sides, we get

$$\begin{aligned}\delta\rho\phi(p) =& \delta(a_{n-1} + 2b_{n-1}, a_0 + 2b_0, ..., a_{n-2} + 2b_{n-2}, b_{n-1}, b_0, ..., b_{n-2}, c_{n-1}, c_0, ..., c_{n-2})\\ =& (b_{n-1}, a_0 + 2b_0, ..., a_{n-2} + 2b_{n-2}, a_{n-1} + 2b_{n-1}, b_0, ..., b_{n-2}, c_{n-1}, c_0, ..., c_{n-2}).\end{aligned}$$

$\therefore \ \phi\gamma_{(1+u)} = \delta\rho\phi.$ □

Corollary 4.2. *A code C is a $(1+u)$-constacyclic code of length n over R if and only if $\phi(C)$ is a permutation equivalent to a quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_3$.*

Proof. Let C be $(1+u)$-constacyclic code of length n over R. Then $\gamma_{(1+u)}(C) = C$. Applying ϕ on both sides, we get $\phi\gamma_{(1+u)}(C) = \phi(C)$. By Theorem 4.1, we have $\delta\rho\phi(C) = \phi(C)$. This shows that $\phi(C)$ is a permutation equivalent to a quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_3$.

Conversely, if $\phi(C)$ is a permutation equivalent to a quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_3$. Then $\delta\rho\phi(C) = \phi(C)$. By Theorem 4.1, we have $\phi\gamma_{(1+u)}(C) = \phi(C)$. Since ϕ is injective it follows that $\gamma_{(1+u)}(C) = C$. This shows that C is a $(1+u)$-constacyclic code of length n over R. □

Theorem 4.3. *Let ϕ be the Gray map defined in equation (1), $\gamma_{(2+2u)}$ be the $(2+2u)$-constacyclic shift operator and η be the quasi-negacyclic shift operator as given in the preliminaries. Then $\phi\gamma_{(2+2u)} = \delta\eta\phi$, where δ is the permutation of $\mathbb{Z}_3^{3n}$ as defined in the Theorem 4.1.*

Proof. Let $p=(p_0,p_1,...,p_{n-1})\in R^n$, where $p_i=a_i+ub_i+vc_i\in R$ and $a_i,b_i,c_i\in\mathbb{Z}_3$, for $i=0,1,...,n-1$. Then

$$\begin{aligned}\phi\gamma_{(2+2u)}(p)=&\phi((2+2u)p_{n-1},p_0,...,p_{n-2})\\ =&\phi(2a_{n-1}+u(2a_{n-1}+b_{n-1})+v2c_{n-1},a_0+ub_0+vc_0,...,a_{n-2}+ub_{n-2}+vc_{n-2})\\ =&(2b_{n-1},a_0+2b_0,...,a_{n-2}+2b_{n-2},2a_{n-1}+b_{n-1},b_0,...,b_{n-2},2c_{n-1},c_0,...,c_{n-2}).\end{aligned}$$

Also, we have

$$\begin{aligned}\eta\phi(p)=&\eta(a_0+2b_0,a_1+2b_1,...,a_{n-1}+2b_{n-1},\,b_0,b_1,...,b_{n-1},c_0,c_1,...,c_{n-1})\\ =&(-(a_{n-1}+2b_{n-1}),a_0+2b_0,...,a_{n-2}+2b_{n-2},\,-b_{n-1},b_0,...,b_{n-2},-c_{n-1},c_0,...,c_{n-2}).\end{aligned}$$

Applying δ on both sides, we get

$$\begin{aligned}\delta\eta\phi(p)=&\delta(-(a_{n-1}+2b_{n-1}),a_0+2b_0,...,a_{n-2}+2b_{n-2},-b_{n-1},b_0,...,b_{n-2},-c_{n-1},c_0,...,c_{n-2})\\ =&(-b_{n-1},a_0+2b_0,...,a_{n-2}+2b_{n-2},-(a_{n-1}+2b_{n-1}),b_0,...,b_{n-2},-c_{n-1},c_0,...,c_{n-2})\\ =&(2b_{n-1},a_0+2b_0,...,a_{n-2}+2b_{n-2},2a_{n-1}+b_{n-1},b_0,...,b_{n-2},2c_{n-1},c_0,...,c_{n-2}).\end{aligned}$$

$\therefore\ \phi\gamma_{(2+2u)}=\delta\eta\phi.$ □

Corollary 4.4. *A code C is a (2+2u)-constacyclic code of length n over R if and only if $\phi(C)$ is a permutation equivalent to a quasi-negacyclic code of length $3n$ and index 3 over Z_3.*

Proof. Let C be $(2+2u)$-constacyclic code of length n over R. Then $\gamma_{(2+2u)}(C)=C$. Applying ϕ on both sides, we get $\phi\gamma_{(2+2u)}(C)=\phi(C)$. From the above Theorem 4.3, $\delta\eta\phi(C)=\phi\gamma_{(2+2u)}(C)=\phi(C)$. This shows that $\phi(C)$ is a permutation equivalent to a quasi-negacyclic code of length $3n$ and index 3 over $\mathbb{Z}_3$.

Conversely, if $\phi(C)$ is a permutation equivalent to a quasi-negacyclic code of length $3n$ and index 3 over $\mathbb{Z}_3$. Then $\delta\eta\phi(C)=\phi(C)$. By using Theorem 4.3, we have $\phi\gamma_{(2+2u)}(C)=\delta\eta\phi(C)=\phi(C)$. Since ϕ is injective it follows that $\gamma_{(2+2u)}(C)=C$. This shows that C is a $(2+2u)$-constacyclic code of length n over R. □

Definition 4.5. [16] *For $a\in\mathbb{Z}_3^{3n}$ with $a=(a_0,a_1,...,a_{n-1},a_n,...,a_{2n},...,a_{3n-1})=(a^{(0)}|a^{(1)}|a^{(2)})$, where $a^{(i)}\in\mathbb{Z}_3^n$ for $i=0,1,2$, quasi-twisted shift operator on $\mathbb{Z}_3^{3n}$ is defined by*

$\nu(a) = (\gamma_2(a^{(0)})|\gamma_2(a^{(1)})|\gamma_2(a^{(2)}))$, where γ_2 is a 2-constacyclic shift operator from $\mathbb{Z}_3^n$ to $\mathbb{Z}_3^n$. A linear code C of length $3n$ over $\mathbb{Z}_3$ is called a quasi-twisted code of index 3 if $\nu(C) = C$.

Theorem 4.6. *Let γ_2 be 2-constacyclic shift operator, ϕ be the Gray map and ν be the quasi-twisted shift operator as given before. Then $\phi\gamma_2 = \nu\phi$.*

Proof. Let $p = (p_0, p_1, ..., p_{n-1}) \in R^n$, where $p_i = a_i + ub_i + vc_i \in R$ and $a_i, b_i, c_i \in \mathbb{Z}_3$, for $i = 0, 1, ..., n-1$. Then

$$\begin{aligned}\phi\gamma_2(p) =& \phi(2p_{n-1}, p_0, ..., p_{n-2})\\ =& \phi(2a_{n-1} + u(2b_{n-1}) + v(2c_{n-1}), a_0 + ub_0 + vb_0, ..., a_{n-2} + ub_{n-2} + vb_{n-2})\\ =& (2a_{n-1} + 2(2b_{n-1}), a_0 + 2b_0, ..., a_{n-2} + 2b_{n-2}, 2b_{n-1}, b_0, ..., b_{n-2}, 2c_{n-1}, c_0, ..., c_{n-2})\\ =& (2a_{n-1} + b_{n-1}, a_0 + 2b_0, ..., a_{n-2} + 2b_{n-2}, 2b_{n-1}, b_0, ..., b_{n-2}, 2c_{n-1}, c_0, ..., c_{n-2}).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\nu\phi(p) =& \nu(a_0 + 2b_0, a_1 + 2b_1, ..., a_{n-1} + 2b_{n-1}, b_0, b_1, ..., b_{n-1}, c_0, c_1, ..., c_{n-1})\\ =& (2(a_{n-1} + 2b_{n-1}), a_0 + 2b_0, ..., a_{n-2} + 2b_{n-2}, 2b_{n-1}, b_0, ..., b_{n-2}, 2c_{n-1}, c_0, c_1, ..., c_{n-2})\\ =& (2a_{n-1} + b_{n-1}, a_0 + 2b_0, ..., a_{n-2} + 2b_{n-2}, 2b_{n-1}, b_0, ..., b_{n-2}, 2c_{n-1}, c_0, c_1, ..., c_{n-2}).\end{aligned}$$

$\therefore\ \phi\gamma_2 = \nu\phi$. □

Corollary 4.7. *A code C is a 2-constacyclic code over R if and only if $\phi(C)$ is a quasi-twisted code of index 3 over $\mathbb{Z}_3$ with length $3n$.*

Proof. Suppose C is a 2-constacyclic code over R. Then $\gamma_2(C) = C$. Applying ϕ on both sides and using the above Theorem 4.6, $\nu\phi(C) = \phi(C)$. This shows that $\phi(C)$ is a quasi-twisted code of index 3 over $\mathbb{Z}_3$ with length $3n$.

Conversely, let $\phi(C)$ be a quasi-twisted code of index 3 over $\mathbb{Z}_3$ with length $3n$. From the definition of a quasi-twisted code and Theorem 4.6, we have $\phi\gamma_2(C) = \nu\phi(C) = \phi(C)$. This implies that $\gamma_2(C) = C$ as ϕ is injective. Thus, C is a 2-constacyclic code over R. □

5. Generators of Constacyclic Codes over R

In this section, we discuss λ-constacyclic codes of odd length n over R for $\lambda = (1+u)$. Note that $\lambda^n = 1$, if n is an even integer and $\lambda^n = \lambda$, if n is an odd integer. Analogous to results given in [6, 10, 12], we have the following results.

Theorem 5.1. *Let $\mu : R[x]/\langle x^n - 1\rangle \longrightarrow R[x]/\langle x^n - \lambda\rangle$ be a map defined by $\mu(a(x)) = a(\lambda x)$. If n is an odd integer, then μ is a ring isomorphism.*

Proof. For $a(x), b(x) \in R[x]$ such that

$$a(x) \equiv b(x) \bmod (x^n - 1)$$
$$\Rightarrow a(x) - b(x) = (x^n - 1)q(x), \text{ for some } q(x) \in R[x].$$

Putting $x = \lambda x$ in the above, we get

$$\begin{aligned} a(\lambda x) - b(\lambda x) &= ((\lambda x)^n - 1)q(\lambda x) \\ &= (\lambda x^n - 1)q(\lambda x) \\ &= \lambda(x^n - \lambda)q(\lambda x) \\ &= \lambda q(\lambda x)(x^n - \lambda). \end{aligned}$$

$\therefore\ a(\lambda x) \equiv b(\lambda x) \bmod (x^n - \lambda)$.

i.e., $a(x) \equiv b(x) \bmod (x^n - 1) \iff \mu(a(x)) \equiv \mu(b(x)) \bmod (\mathrm{x^n} - \lambda)$.
This shows that μ is well defined and one-one.
For any $a(x), b(x) \in R[x]/\langle x^n - 1\rangle$, we have $\mu(a(x) + b(x)) = \mu((a+b)(x)) = (a+b)(\lambda x) = a(\lambda x) + b(\lambda x) = \mu(a(x)) + \mu(b(x))$ and $\mu(a(x)b(x)) = \mu(ab(x)) = ab(\lambda x) = a(\lambda x)b(\lambda x) = \mu(a(x))\mu(b(x))$.
Hence, μ is a ring isomorphism. □

Corollary 5.2. *Let n be an odd integer. Then I is an ideal of $R_n = R[x]/\langle x^n - 1\rangle$ if and only if $\mu(I)$ is an ideal of $R_{n,\lambda} = R[x]/\langle x^n - \lambda\rangle$.*

Corollary 5.3. *Let $\overline{\mu}$ be a permutation of R^n with n odd, such that $\overline{\mu}(c_0, c_1, c_2, ..., c_{n-1}) = (c_0, \lambda c_1, \lambda^2 c_2, ..., \lambda^{n-1} c_{n-1})$ and C be a subset of R^n, then C is a cyclic code if and only if $\overline{\mu}(C)$ is a λ-constacyclic code.*

Proof. Let C be a cyclic code of odd length n over R. Then by Proposition 2.1, its polynomial representation I is an ideal of $R_n = R[x]/\langle x^n - 1\rangle$. So, the above Corollary 5.2, is satisfied and thus $\mu(I)$ is an ideal of $R_{n,\lambda} = R[x]/\langle x^n - \lambda\rangle$. Hence, $\overline{\mu}(C)$ is a λ-constacyclic code (by Proposition 2.2).

In the similar manner, the converse part can be proved. □

In [15], Liu developed the following results to determine the generators of cyclic codes over $R = \mathbb{F}_p + u\mathbb{F}_p + v\mathbb{F}_p$ of length n.

Theorem 5.4. [15] *Let C be a cyclic code over $R = \mathbb{F}_p + u\mathbb{F}_p + v\mathbb{F}_p$ of length n. Then $C = \langle g(x) + vp_1(x) + up_2(x), va_1(x) + uq_1(x), ua_2(x)\rangle$, where $g(x), p_1(x), p_2(x), q_1(x), a_1(x), a_2(x)$ are polynomials in $\mathbb{F}_p[x]/\langle x^n - 1\rangle$ with $a_2(x)|a_1(x)|g(x)|(x^n - 1)$ and $a_1(x)|p_1(x)\frac{x^n-1}{g(x)}$.*

Theorem 5.5. [15] *Let C be a cyclic code over $\mathbb{F}_p + u\mathbb{F}_p + v\mathbb{F}_p$ of length n. When $(n, p) = 1$, then C is an ideal in $R[x]/\langle x^n - 1\rangle$ and generated by $C = \langle g_1(x) + vp_1(x) + ub_1(x), ug_2(x)\rangle$, where $g_1(x), p_1(x), b_1(x), g_2(x)$ are polynomials in $\mathbb{F}_p[x]/\langle x^n - 1\rangle$ satisfying the conditions $p_1(x)|g_1(x)|(x^n - 1)$ and $g_2(x)|(x^n - 1)$.*

Using Theorem 5.4, and Theorem 5.5, we can construct the generators for λ - constacyclic codes of length n over R as follows:

Theorem 5.6. *Let C be a λ-constacyclic code of length n over R. Then, C is an ideal of $R_{n,\lambda}$ given by $C = \langle g(\widehat{x}) + vp_1(\widehat{x}) + up_2(\widehat{x}), va_1(\widehat{x}) + uq_1(\widehat{x}), ua_2(\widehat{x})\rangle$, where $g(x), p_1(x), p_2(x), q_1(x), a_1(x), a_2(x) \in \mathbb{Z}_3[x]/\langle x^n - 1\rangle$, $a_2(x)|a_1(x)|g(x)|(x^n - 1), a_1(x)|p_1(x)\frac{x^n-1}{g(x)}$ and $\widehat{x} = \lambda x$.*

Proof. The result follows from Corollary 5.3 and Theorem 5.4. □

Theorem 5.7. *Let C be a λ-constacyclic code of even length n over R. Then C is an ideal of $R_{n,\lambda}$ given by $C = \langle g_1(\widehat{x}) + vp_1(\widehat{x}) + ub_1(\widehat{x}), ug_2(\widehat{x})\rangle$, where $g_1(x), p_1(x), b_1(x), g_2(x) \in \mathbb{Z}_3[x]/\langle x^n - 1\rangle$, $p_1(x)|g_1(x)|(x^n - 1),\ g_2(x)|(x^n - 1)$ and $\widehat{x} = \lambda x$.*

Proof. The result follows from Corollary 5.3 and Theorem 5.5. □

Theorem 5.8. *Let C be a λ-constacyclic code of odd length n over R and $C = \langle a(x) + ub(x) + vc(x) \rangle$, where $a(x), b(x), c(x) \in \mathbb{Z}_3[x]$ with degree less than n. Then $\phi(C)$ is a permutation equivalent to a quasi-cyclic code of length $3n$ over $\mathbb{Z}_3$ generated by the polynomials $[a(x) + 2b(x)] + x^n[b(x)] + x^{2n}[c(x)], 2[a(x) + b(x)] + x^n[a(x) + b(x)]$ and $x^{2n}[a(x)]$.*

Proof. The polynomial corresponding to the Gray map ϕ of (1) can be defined as

$$\phi : \frac{R[x]}{\langle x^n - 1 \rangle} \to \frac{\mathbb{Z}_3[x]}{\langle x^n - 1 \rangle} \times \frac{\mathbb{Z}_3[x]}{\langle x^n - 1 \rangle} \times \frac{\mathbb{Z}_3[x]}{\langle x^n - 1 \rangle}$$

$$\phi(a(x) + ub(x) + vc(x)) = (a(x) + 2b(x), b(x), c(x)),$$

where $a(x), b(x), c(x) \in \mathbb{Z}_3[x]$.
For any $r_1(x), r_2(x), r_3(x) \in \mathbb{Z}_3[x]$, it can be shown that

$$\begin{aligned} &\phi[(r_1(x) + ur_2(x) + vr_3(x))(a(x) + ub(x) + vc(x))] \\ &= r_1(x)[a(x) + 2b(x), b(x), c(x)] + r_2(x)[2a(x) + 2b(x), a(x) + b(x), 0] + r_3(x)[0, 0, a(x)]. \end{aligned}$$

And, the vector $(a(x), b(x), c(x)) \in \frac{\mathbb{Z}_3[x]}{\langle x^n - 1 \rangle} \times \frac{\mathbb{Z}_3[x]}{\langle x^n - 1 \rangle} \times \frac{\mathbb{Z}_3[x]}{\langle x^n - 1 \rangle}$ can be identified with the element $([a(x)] + x^n[b(x)] + x^{2n}[c(x)]) \in \mathbb{Z}_3[x]/\langle x^{3n} - 1 \rangle$, which corresponds to the quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_3$.
Hence, $\phi(C)$ is generated by the polynomials $[a(x) + 2b(x)] + x^n[b(x)] + x^{2n}[c(x)], 2[a(x) + b(x)] + x^n[a(x) + b(x)]$ and $x^{2n}[a(x)]$. □

Definition 5.9. *Let n be an odd positive integer and $\tau = (1, n+1)(3, n+3)...(2i+1, n+2i+1)...(n-2, 2n-2)$ be a permutation of the set $\{0, 1, 2, ..., 3n-1\}$. Then a permutation Π of $\mathbb{Z}_3^{3n}$ is defined by*

$$\Pi(r_0, r_1, ..., r_{3n-1}) = (r_{\tau(0)}, r_{\tau(1)}, ..., r_{\tau(3n-1)}).$$

Theorem 5.10. *Let n be an odd positive integer, ϕ be the Gray map, $\overline{\mu}$ be the permutation of R^n with $\lambda = (1 + u)$ and Π be the permutation as given before. Then $\phi\overline{\mu} = \Pi\phi$.*

Proof. Let $p = (p_0, p_1, ..., p_{n-1}) \in R^n$, where $p_i = a_i + ub_i + vc_i \in R$, for $i = 0, 1, ..., n-1$. When $\lambda = 1 + u$, we have

$$\begin{aligned}
\phi\overline{\mu}(p) =& \phi(p_0, (1+u)p_1, (1+u)^2 p_2, ..., (1+u)^{n-1} p_{n-1}) \\
=& \phi(a_0 + ub_0 + vc_0, a_1 + u(a_1 + 2b_1) + vc_1, ..., a_{n-2} + u(a_{n-2} + 2b_{n-2}) + vc_{n-2}, \\
& a_{n-1} + ub_{n-1} + vc_{n-1}) \\
=& (a_0 + 2b_0, a_1 + 2(a_1 + 2b_1), a_2 + 2b_2, ..., a_{n-2} + 2(a_{n-2} + 2b_{n-2}), a_{n-1} + 2b_{n-1}, \\
& b_0, a_1 + 2b_1, b_2, ..., a_{n-2} + 2b_{n-2}, b_{n-1}, c_0, c_1, c_2, ..., c_{n-2}, c_{n-1}) \\
=& (a_0 + 2b_0, b_1, a_2 + 2b_2, ..., b_{n-2}, a_{n-1} + 2b_{n-1}, b_0, a_1 + 2b_1, b_2, ..., a_{n-2} + 2b_{n-2}, b_{n-1}, \\
& c_0, c_1, c_2, ..., c_{n-2}, c_{n-1}).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\Pi\phi(p) =& \Pi(a_0 + 2b_0, a_1 + 2b_1, ..., a_{n-2} + 2b_{n-2}, a_{n-1} + 2b_{n-1}, b_0, b_1, ..., b_{n-2}, b_{n-1}, c_0, c_1, \\
& ..., c_{n-2}, c_{n-1}) \\
=& (a_0 + 2b_0, b_1, a_2 + 2b_2, ..., b_{n-2}, a_{n-1} + 2b_{n-1}, b_0, a_1 + 2b_1, ..., a_{n-2} + 2b_{n-2}, b_{n-1}, \\
& c_0, c_1, ..., c_{n-2}, c_{n-1}).
\end{aligned}$$

$\therefore\ \phi\overline{\mu} = \Pi\phi.$ □

Corollary 5.11. *Let $\phi(C) = D$ be the Gray image of a cyclic code C of odd length n over R. Then $\Pi(D)$ is a permutation equivalent to a quasi-cyclic code of length $3n$ over $\mathbb{Z}_3$.*

Proof. By Theorem 5.3, $\overline{\mu}(C)$ is a λ-constacyclic code over R as C is a cyclic code over R. From Theorem 4.2, we see that $\phi\,\overline{\mu}(C)$ is a permutation equivalent to a quasi-cyclic code of length $3n$ and index 3 over $\mathbb{Z}_3$. By Theorem 5.10, we have $\Pi\,\phi(C) = \Pi(D) = \phi\,\overline{\mu}(C)$. This implies that $\Pi(D)$ is a permutation equivalent to a quasi-cyclic code of length $3n$ over $\mathbb{Z}_3$. □

Example 5.12. *Let $n = 6, \lambda = 1 + u$. Now in $\mathbb{Z}_3[x]$ we have $x^6 - 1 = (x+1)^3(x+2)^3$. As per Theorem 5.6, let $g(x) = (x+1)(x+2)^2 = x^3 + 2x^2 + 2x + 1$, $a_1(x) = (x+2)^2 = x^2 + x +$*

$1, a_2(x) = x+2, p_1(x) = x+2, p_2(x) = q_1(x) = 1$. *Then*

$$C = \langle (1+u)x^3 + 2x^2 + (2+2u+v)x + u + 2v + 1, vx^2 + vx + u + v, 2ux + 2u \rangle$$

is a $(1+u)$*-constacyclic code of length* 6 *over R. Therefore, the Gray image* $\phi(C)$ *is a* $[18,15,2]$ *linear code over* $\mathbb{Z}_3$*. Note that it is an optimal linear code according to the database* [8].

Example 5.13. *Let* $n = 4, \lambda = 1+u$*. We have* $x^4 - 1 = (x+1)(x+2)(x^2+1)$ *in* $\mathbb{Z}_3[x]$*. Now, following Theorem 5.7, let* $g_1(x) = (x+1)(x+2) = x^2+2$, $p_1(x) = x+1$, $p_2 = 1$, $g_2(x) = x^2+1$. *Then*

$$C = \langle x^2 + vx + 2 + u + v, ux^2 + u \rangle$$

is a $(1+u)$*-constacyclic code of length* 4 *over R. Further, its Gray image* $\phi(C)$ *is a* $[12,8,3]$ *linear code over* $\mathbb{Z}_3$*. As per online database* [8]*, it is an optimal linear code.*

6. CONCLUSION

In this paper, we have studied the algebraic structure of λ-constacyclic code of length n over $R = \mathbb{Z}_3[u,v]/\langle u^2 - u, v^2, uv, vu \rangle$ with $\mathbb{Z}_3 = \{0,1,2\}$, for $\lambda = (1+u), (2+2u)$ and 2 . By introducing a Gray map from R^n to $\mathbb{Z}_3^{3n}$, we obtained a good relation among the Gray image, cyclic, constacyclic, quasi-cyclic, quasi-negacyclic and quasi-twisted codes over $\mathbb{Z}_3$. Further, we have found that the Gray image of λ-constacyclic codes is permutation equivalent to either quasi-cyclic or quasi-negacyclic or quasi-twisted code when $\lambda = (1+u), (2+2u)$ or 2, respectively. Also, the generator for $(1+u)$-constacyclic code over R is constructed with some examples to illustrate the results.

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Conflict of Interests

The author(s) declare that there is no conflict of interests.

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