Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 3, 2624-2649
https://doi.org/10.28919/jmcs/5214
ISSN: 1927-5307

NEIGHBORHOOD ALLIANCE IN JOIN OF A GRAPH WITH K_1

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Abstract. Let $G=(V,E)$ be a graph. A subset S of V is called a neighborhood set of G if union of induced subgraph of $N[s]$ is isomorphic to G, where union is taken over all s in S. A *defensive alliance* is a non-empty subset S of V satisfying the condition that every $v \in S$ has at most one more neighbor in $V-S$ than it has in S. The minimum cardinality of any defensive alliance of G is called the *alliance number* of G. Further, a subset of V which is both a neighborhood set of G as well as a defensive alliance of G is called a *neighborhood alliance set*, or simply an *na*-set. The minimum cardinality of an *na*-set is called neighborhood alliance number of G. The minimum cardinality (in possible cases) of various types of *na*-sets of join of a graph G with K_1, specifically when G is K_{n-1}, $\overline{K_n}$, C_n and P_n are determined in this article.

Keywords:defensive alliance, neighborhood set, neighborhood alliance set.

2010 AMS Subject Classification: 05C69, 05C76.

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Received November 18, 2020

1. Introduction

The topology of a network of multiprocessor, or local area network is usually modeled by a graph in which vertices represent nodes (processors) while undirected edges stand for 'links' or other types of connections. In the design of such networks, there are a number of features that must be taken into account. The most common ones, however, seem to be limitations on the vertex degrees and on the diameter. The network interpretation of these two parameters is obvious. The degree of a vertex is the number of the connections attached to a node, while the diameter indicates the largest number of links that must be traversed in order to transmit a message between any two nodes.

The dominating set and neighborhood sets play an important role in computer and communication networks to route the information between the nodes. The network models considered in this article are such that one node (computer/server) is directly connected to all the other nodes(computers).

P. Kristiansen *et al.* introduced the term alliances in a graph in [8] and more number of variants of alliances are studied in [5], [3], [7], [9], [13], [6]. One can find a large amount of related work being done in various fields of alliances in [2]. The neighborhood number in a graph and its properties was first defined and studied in [11] by E. Sampathkumar and Prabha S. Neeralagi. The concept of neighborhood resolving sets was introduced by B. Sooryanarayana and studied varieties of neighborhood resolving sets of paths and cycles in [15]. Similar work for the rational resolving sets is found in [10]. In [4], global defensive alliances were introduced and studied. Motivated by these, the concept of neighborhood alliance set was introduced in [14] by B.Sooryanarayana and studied the various conditioned neighborhood alliance sets for path and cycle. In this article, the minimum cardinality of different types of neighborhood alliance sets for certain graphs, in particular for Complete graphs, Wheel graphs, Star graphs and Fan graphs are determined.

All the graphs considered here are simple, finite, undirected and connected. We use the standard terminology, the terms not defined here may be found in [1]. Let $G(V,E)$ be a graph and v be a vertex of G. Let $N(v)$ be the set of vertices adjacent to v in G and the cardinality of $N(v)$ is called the degree of the vertex v denoted as $\deg(v)$. Let $N[v] = N(v) \cup \{v\}$. The *union* of two graphs G_1 and G_2, denoted as $G_1 \cup G_2$, is a graph G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. For two vertex disjoint graphs G_1 and G_2, the *join* of G_1 and G_2, denoted as $G_1 + G_2$, is a graph G such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{xy : x \in V(G_1), y \in V(G_2)\}$.

Definition 1.1. [11] *A subset S of the vertex set of G is called an neighborhood set (n-set) of G whenever $G \cong \bigcup_{v\in S}\langle N[v]\rangle$. Further, the minimum cardinality of an n-set of G is called the neighborhood number of G and is denoted by $l_n(G)$.*

Clearly, for every graph G,

$$1 \leq l_n(G) \leq |V(G)| - 1 \tag{1}$$

Remark 1.2. *In any graph $G = (V,E)$, if $\deg(v) = |V| - 1$, then every subset S of V containing v is always an n-set.*

Definition 1.3. [9] *A defensive alliance or defensive alliance set (a-set) is a non-empty set $S \subseteq V$ such that for every $v \in S$, $|N[v] \cap S| \geq |N(v) \cap (V - S)|$.*

The minimum cardinality of any defensive alliance of G is called the *alliance number* of G and is denoted as $l_a(G)$.

Clearly, for any graph G,

$$1 \leq l_a(G) \leq |V(G)| - 1 \tag{2}$$

A subset $S \subseteq V$ which is an neighborhood set of G and also a defensive alliance in G is called a *neighborhood alliance set* or simply an *na*-set [14]. A neighborhood alliance set (*na*-set) is called a minimal neighborhood alliance set if no proper subsets of it is an *na*-set. The minimum cardinality of a *na*-set is called the neighborhood alliance number of G, denoted by $l_{na}(G)$.

The vertices of a defensive alliance set S are considered as defenders and the vertices which are not in S are considered to be attackers. A *global defensive alliance* [4] S is a defensive alliance, which is also a dominating set. In a global defensive alliance, every vertex of G is either an attacker or a defender. Every n-set is a dominating set. Thus, each na-set is a global defensive alliance.

The main advantage of the neighborhood set over dominating set is that it covers all vertices, as well as edges. Thus, na-set is defined as an extension of global defensive alliances.

Definition 1.4. *For an na-set S of G,*

(1) if $\overline{S}$ is also an n-set then S is an $n^{\star}a$-set and if $\overline{S}$ is not an n-set then S is an Na-set.
(2) if $\overline{S}$ is also an a-set then S is an $na^{\star}$-set and if $\overline{S}$ is not an a-set then S is an nA-set.
(3) if $\overline{S}$ is both an a-set as well as an n-set then S is an $n^{\star}a^{\star}$-set.
(4) if $\overline{S}$ is also an n-set but not an a-set then S is an $n^{\star}A$-set.
(5) if $\overline{S}$ is also an a-set but not an n-set then S is an $Na^{\star}$-set.
(6) if $\overline{S}$ is neither an n-set nor an a-set then S is an NA-set.

The minimum cardinality of a minimal $n^{\star}a$-set, a minimal Na-set, a minimal $na^{\star}$-set, a minimal nA-set, a minimal $n^{\star}a^{\star}$-set, a minimal $n^{\star}A$-set, a minimal $Na^{\star}$-set and a minimal NA-set are denoted by $l_{n^{\star}a}(G)$, $l_{Na}(G)$, $l_{na^{\star}}(G)$, $l_{nA}(G)$, $l_{n^{\star}a^{\star}}(G)$, $l_{n^{\star}A}(G)$, $l_{Na^{\star}}(G)$ and $l_{NA}(G)$ respectively.

Definition 1.5. *Let S be a subset of V such that S and $\overline{S}$ both are not an a-sets, then S is an $A^{\star}$-set. Let S be an $A^{\star}$-set. Further,*

(1) if S is an n-set then S is an $nA^{\star}$-set.
(2) if S and $\overline{S}$ both are n-sets then S is an $n^{\star}A^{\star}$-set.
(3) if S is an n-set and $\overline{S}$ is not an n-set then S is an $NA^{\star}$-set.
(4) if S and $\overline{S}$ both are not an n-sets then S is an $N^{\star}A^{\star}$-set.

The minimum cardinality of a minimal $nA^{\star}$-set, a minimal $n^{\star}A^{\star}$-set, a minimal $NA^{\star}$-set and a minimal $N^{\star}A^{\star}$-set are denoted by $l_{nA^{\star}}(G)$, $l_{n^{\star}A^{\star}}(G)$, $l_{NA^{\star}}(G)$ and $l_{N^{\star}A^{\star}}(G)$ respectively.

Definition 1.6. *Let S be a subset of V such that S and $\overline{S}$ both are not an n-sets, then S is an $N^{\star}$-set. Let S be an $N^{\star}$-set. Further,*

(1) if S is an a-set then S is an $N^{\star}a$-set.
(2) if S and $\overline{S}$ both are a-sets then S is an $N^{\star}a^{\star}$-set.
(3) if S is an a-set and $\overline{S}$ is not an a-set then S is an $N^{\star}A$-set.

The minimum cardinality of a minimal $N^\star a$-set, a minimal $N^\star a^\star$-set and a minimal $N^\star A$-set are denoted by $l_{N^\star a}(G)$, $l_{N^\star a^\star}(G)$ and $l_{N^\star A}(G)$ respectively.

Remark 1.7. *If S is a defensive alliance of the graph G, and $v \in S$, then at least $\lfloor \frac{\deg(v)}{2} \rfloor$ neighbors of v should be in S.*

The following results are recalled for immediate reference.

Theorem 1.8 ([8])**.** *For any graph G of order n;*

(i) $l_a(G) = 1$ *if and only if there exists a pendant vertex v in G.*

(ii) $l_a(G) = 2$ *if and only if $\delta(G) \geq 2$ and G has two adjacent vertices of degree at most three.*

(iii) $l_a(G) = 3$ *if and only if $l_a(G) \neq 1$, $l_a(G) \neq 2$ and G has an induced subgraph isomorphic to either (i) P_3, with vertices, in order, u, v and w, where $\deg(u)$, $\deg(w) \leq 3$ and $\deg(v) \leq 5$ or (ii) isomorphic to K_3, each vertex of which has degree at most 5.*

Corollary 1.9 ([8])**.** *For any cycle C_k and any wheel $W_{1,k}$ with $k \geq 3$, $a(C_k) = a(W_{1,k}) = 2$.*

Theorem 1.10 ([8])**.** *For the complete graph K_k on k vertices, $a(K_k) = \lceil \frac{k}{2} \rceil$.*

Theorem 1.11 ([11])**.** *For a path P_k on k vertices with $k \geq 2$, $l_n(P_k) = \lfloor \frac{k}{2} \rfloor$.*

Theorem 1.12 ([11])**.** *For a cycle C_k of length $k \geq 4$, $l_n(C_k) = \lceil \frac{k}{2} \rceil$.*

Theorem 1.13 ([12])**.** *A set S of vertices of a graph G is an n-set if and only if every edge of $\langle V(G) - S \rangle$ belongs to a triangle one of whose vertices belongs to S.*

Corollary 1.14. *A set S is an n-set of a triangular free graph G if and only if $\overline{S}$ is totally disconnected.*

Theorem 1.15 ([12])**.** *For the complete graph K_k on k vertices, $l_n(K_k) = 1$.*

Remark 1.16. *If S is a defensive alliance and $\overline{S}$ is totally disconnected, then by the above Theorem 1.13 it is clear that S is an na-set.*

Lemma 1.17 ([14])**.** *For any two properties p, q of a graph G, $l_{pq} \geq \max\{l_p, l_q\}$.*

Throughout this article, we have considered $G+K_1$ particularly when $G \cong K_{n-1}, C_n, \overline{K}_n$ and P_n with $V(K_1)=x$. Therefore there exists a full degree vertex x in K_k, $W_{1,k}$, S_k and $F_{1,k}$ respectively. This indicates that it is not possible to have a non-empty set S such that both S and $\overline{S}$ are not n-sets(since vertex x may belong to either S or $\overline{S}$ and in either of the cases any one set will be an n-set by Remark 1.2). Hence there is no $N^{\star}$-set. Thus we conclude:

Remark 1.18. *For any integer $k \geq 3$, $l_{N^{\star}a}(W_{1,k})$, $l_{N^{\star}A}(W_{1,k})$, $l_{N^{\star}a^{\star}}(W_{1,k})$ and $l_{N^{\star}A^{\star}}(W_{1,k})$ do not exist. Also, $l_{N^{\star}a}(G_1)$, $l_{N^{\star}a^{\star}}(G_1)$, $l_{N^{\star}A}(G_1)$ and $l_{N^{\star}A^{\star}}(G_1)$ do not exist for $G_1 \cong K_k$, S_k, $F_{1,k}$, where k is an integer such that $k \geq 1$.*

2. NEIGHBORHOOD ALLIANCE SETS OF A COMPLETE GRAPH

Lemma 2.1. *For any positive integer k, every p-element subset of vertices of a complete graph K_k is an na-set if and only if $p \geq \lceil \frac{k}{2} \rceil$.*

Proof. Let $p \in Z^+$ and S be an p-element subset of vertices of a complete graph K_k. Then, for each $v_i \in S$, $|N[v_i] \cap S| = p \geq |N[v_i] \cap \overline{S}| = k-p$ if and only if $p \geq \lceil \frac{k}{2} \rceil$. Therefore, by Definition 1.3, S is an a-set if and only if $p \geq \lceil \frac{k}{2} \rceil$. Also S contains a full degree vertex and hence, by Remark 1.2, S is always an n-set of G. Hence the lemma. $\square$

Theorem 2.2. *For any integer $k \geq 1$, $l_{na}(K_k) = \lceil \frac{k}{2} \rceil$.*

Proof. Follows directly by Lemma 2.1 as $l_{na} = \min\{|S| : S \text{ is an } na\text{-set}\} = \lceil \frac{k}{2} \rceil$. $\square$

Theorem 2.3. *For any integer $k \geq 1$, $l_{na^{\star}}(K_{2k}) = k$ and $l_{na^{\star}}(K_{2k-1})$ does not exist.*

Proof. Let S be an $na^{\star}$-set of K_n. Then, S is an na-set and hence by Lemma 2.1, $|S| \geq \lceil \frac{n}{2} \rceil$. Also $\overline{S}$ is an a-set (as S being an $a^{\star}$-set), and hence by Theorem 1.10, $|\overline{S}| = n - \lceil \frac{n}{2} \rceil \geq \lceil \frac{n}{2} \rceil$. This is possible only if $n = 2k$ for some $k \in Z^+$, $|S| = |\overline{S}| = k$. Hence $l_{na^{\star}}(K_{2k}) = k$ and $l_{na^{\star}}(K_{2k-1})$ does not exist. $\square$

Theorem 2.4. *For any integer $k \geq 1$, $l_{nA}(K_k) = \lceil \frac{k+1}{2} \rceil$.*

Proof. Let S be an nA-set. Then, S is an na-set and $\overline{S}$ is not an a-set. Hence by Lemma 2.1, S is a subset of $V(G)$ with $|S| \geq \lceil \frac{k}{2} \rceil$ and the equality holds only when n is odd. Hence $l_{nA}(K_k) = \min\{|S| : S \text{ is an } nA\text{-set}\} = \lceil \frac{k+1}{2} \rceil$. $\square$

From Theorem 2.4 and Remark 1.2 it follows that every nA-set of K_k is always an $n^\star A$-set except for the cases $k = 1,2$. When $k = 1$ or 2, an $n^\star A$-set shall contain at least two vertices (as being an A-set) and hence its complement is not an n-set (being an empty set). Thus we conclude:

Theorem 2.5. *For any integer $k \geq 1$, $l_{n^\star A}(K_k) = \lceil \frac{k+1}{2} \rceil$ whenever $k \neq 1,2$ and $l_{n^\star A}(K_k)$ does not exist if $k = 1,2$.*

Let S be an $A^\star$-set of K_k. Then, by the proof of Lemma 2.1, it follows that $|S| < \lceil \frac{k}{2} \rceil$ and $|\overline{S}| < \lceil \frac{k}{2} \rceil$. But then, $k - \lfloor \frac{k}{2} \rfloor \leq |\overline{S}| < \lceil \frac{k}{2} \rceil \Rightarrow k < \lceil \frac{k}{2} \rceil + \lfloor \frac{k}{2} \rfloor = k$, a contradiction. Hence we conclude:

Theorem 2.6. *For any integer $k \geq 1$, $l_{nA^\star}(K_k)$ and $l_{n^\star A^\star}(K_k)$ do not exist.*

Let S be an na-set. So, by Lemma 2.1, $|S| \geq \lceil \frac{k}{2} \rceil \geq 1$ and hence $\overline{S}$ is non-empty for all $k \geq 2$. Thus, $\overline{S}$ has a full degree vertex. Therefore, by Remark 1.2, $\overline{S}$ is an n set. This shows that every na-set is an $n^\star a$-set for all $k \geq 2$. Further, when $k = 1$, $\overline{S} = \emptyset$ which is not an n-set. Thus we conclude:

Theorem 2.7. *For any integer $k \geq 2$, $l_{n^\star a}(K_k) = \lceil \frac{k}{2} \rceil$ and $l_{n^\star a}(K_1)$ does not exist.*

Similar argument of $n^\star a^\star$-set S with Theorem 2.3 yields the following;

Theorem 2.8. *For any integer $k \geq 1$, $l_{n^\star a^\star}(K_{2k}) = k$ and $l_{n^\star a^\star}(K_{2k-1})$ does not exist.*

If S is an N-set of K_k, then by Remark 1.2, $\overline{S} = \emptyset$ and hence we have the following;

Theorem 2.9. *For any integer $k \geq 1$, $l_{Na}(K_k) = l_{NA}(K_k) = k$. Further, $l_{Na^\star}(K_k)$ and $l_{NA^\star}(K_k)$ do not exist.*

3. Neighborhood Alliance Sets of a Wheel Graph

Throughout this section, let v be the central vertex and $v_1, v_2, \ldots, v_k$ be the k rim vertices adjacent to v of a wheel $W_{1,k}$ in order.

Theorem 3.1. *For any integer $k \geq 3$, $l_{na}(W_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$.*

Proof. Let S be an *na*-set of $W_{1,k}$.

Case 1: $v \in S$.

Since S is an a-set, by Remark 1.7, at least $\lfloor \frac{\deg(v)}{2} \rfloor$ neighbors of v should be in S and hence $|S| \geq 1 + \lfloor \frac{\deg(v)}{2} \rfloor \geq 1 + \lfloor \frac{k}{2} \rfloor$.

Case 2: $v \notin S$.

Define $S_i = \{v_{i-1}, v_i, v_{i+1}\}$ and let $T_i = S \cap S_i$, for $i = 1, 2, \ldots, k$, where $v_{k+1} = v_1$.

Claim 1: $|T_i| \geq 2$ for each $i, 1 \leq i \leq k$.

By Theorem 1.13, every edge of $\langle V(G) - S \rangle$ belongs to a triangle one of whose vertices belongs to S. Hence $\{v_{i-1}, v_{i+1}\} \subseteq T_i$ or $v_i \in T_i$. In the second case, as $\deg(v_i) = 3$, by Remark 1.7, at least one neighbor of v_i in $\{v_{i-1}, v_{i+1}\}$ to be in S. Hence the Claim 1 holds.

By Claim 1, it follows that for every three consecutive rim vertices of $W_{1,k}$ at least two of them have to be in S. Hence $|S| \geq \lceil \frac{2k}{3} \rceil$.

Therefore, by the above cases, $|S| \geq \min\{1 + \lfloor \frac{k}{2} \rfloor, \lceil \frac{2k}{3} \rceil\} = 1 + \lfloor \frac{k}{2} \rfloor$. Hence $l_{na}(W_{1,k}) \geq 1 + \lfloor \frac{k}{2} \rfloor$.

Now to prove the reverse inequality, consider the set $S = \{v, v_1, v_2, v_3, \ldots, v_{\lfloor \frac{k}{2} \rfloor}\}$. The set S is an n-set as $v \in S$ (by Remark 1.2).

Also for every vertex $u \in S$, $|N[u] \cap S| \geq 3 > 1 \geq |N(u) \cap \overline{S}|$ and hence S is also an a-set. Thus, S is an *na*-set implies that $l_{na}(W_{1,k}) \leq |S| = 1 + \lfloor \frac{k}{2} \rfloor$. Hence the theorem. □

Theorem 3.2. *For any integer $k \geq 3$, $l_{na^\star}(W_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$.*

Proof. Let S be an $na^\star$-set. Then, by Theorem 3.1, $l_{na^\star}(W_{1,k}) \geq 1 + \lfloor \frac{k}{2} \rfloor$. On the other hand, for the *na*-set $S = \{v, v_1, v_2, \ldots, v_{\lfloor \frac{k}{2} \rfloor}\}$ considered in the proof of Theorem 3.1, its complement is $\overline{S} = \{v_{\lfloor \frac{k}{2} \rfloor + 1}, v_{\lfloor \frac{k}{2} \rfloor + 2}, \ldots, v_k\}$. For every vertex $v_i \in \overline{S}$, $|N[v_i] \cap \overline{S}| \geq 2 \geq |N(v_i) \cap S|$ implies that $\overline{S}$ is an a-set. Thus, S is an $na^\star$-set. Therefore, $l_{na^\star}(W_{1,k}) \leq |S| = 1 + \lfloor \frac{k}{2} \rfloor$. Hence the theorem. □

Theorem 3.3. *For any integer $k \geq 4$, $l_{nA}(W_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$ and $l_{nA}(W_{1,3}) = 3$.*

Proof. For $k = 3$, the result follows by Theorem 2.4.

Let $k \geq 4$ and S be an nA-set of $W_{1,k}$. Then, by Theorem 3.1 it follows that $l_{nA}(W_{1,k}) \geq 1 + \lfloor \frac{k}{2} \rfloor$. Now to prove the reverse inequality, consider the set $S = \{v, v_1, v_3, v_5, \ldots, v_{k-1}\}$ for k even and $S = \{v, v_1, v_3, v_5, \ldots, v_{k-2}\}$ for k odd. Clearly, S is an n-set (as $v \in S$ and by Remark 1.2). Further, the set S is an a-set because for each $u \in S$, $|N[u] \cap S| \geq 2 \geq |N(u) \cap \overline{S}|$. And the set $\overline{S}$ is not an a-set because $v_2 \in \overline{S}$ and $1 = |N[v_2] \cap \overline{S}| \ngeq |N(v_2) \cap \overline{\overline{S}}| = 3$. Therefore, the set S is an nA-set and hence $l_{nA}(W_{1,k}) \leq |S| = 1 + \lfloor \frac{k}{2} \rfloor$. □

Theorem 3.4. *For any integer $k \geq 6$, $l_{nA^\star}(W_{1,k}) = 3$.*

Proof. Let S be an nA^*-set of $W_{1,k}$. If $|S| \leq 2$, then $v \in S$ (by Theorem 1.13 as $k \geq 6$ and S is an n-set). Then, we observe that $|N[v] \cap \overline{S}| \geq k-1 > 2 \geq |N(v) \cap S|$ and $|N[v_i] \cap \overline{S}| \geq 2 \geq |N(v_i) \cap S|$ for each $v_i \in \overline{S}$, implies that $\overline{S}$ is an a-set, a contradiction. So, $|S| \geq 3$. Thus, $l_{nA^\star}(W_{1,k}) \geq 3$.

To prove the reverse inequality, consider the set $S = \{v, v_i, v_{i+2}\}$ for $1 \leq i \leq k-2$. This set S is an n-set (since $v \in S$ and by Remark 1.2) and also not an a-set (since $|N[v] \cap S| = 3 \ngeq k-2 = |N(v) \cap \overline{S}|$). Further, for the vertex $v_{i+1} \in \overline{S}$, $|N[v_{i+1}] \cap \overline{S}| = 1 \ngeq 3 = |N(v_{i+1}) \cap S|$. This proves that S is an $nA^\star$-set. So, $l_{nA^\star}(W_{1,k}) \leq |S| = 3$. Hence the theorem. □

Lemma 3.5. *For any positive integer $k \geq 3$, if $p \geq k-1$ then every p-element subset of vertices of a wheel graph $W_{1,k}$ is an a-set.*

Proof. Let S be a p-element subset of vertices of $W_{1,k}$. If $p = k$ or $k+1$, then the result follows by Definition 1.3 as $|\overline{S}| \leq 2$. Now for the case $p = k-1$, we have for every $w \in S$, $|N[w] \cap S| \geq 2 \geq |N(w) \cap \overline{S}|$ and hence S is an a-set. □

Remark 3.6. *For any integer k, $3 \leq k \leq 5$, $l_{nA^\star}(W_{1,k})$ does not exist.*

Theorem 3.7. *For any integer $k \geq 4$, $l_{Na}(W_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$.*

Proof. Let S be an Na-set. Then, S is an na-set and $\overline{S}$ is not an n-set. Hence from Theorem 3.1, it follows that $l_{Na}(W_{1,k}) \geq 1 + \lfloor \frac{k}{2} \rfloor$.

On the other hand, for the na-set $S = \{v, v_1, v_2, \ldots, v_{\lfloor \frac{k}{2} \rfloor}\}$ considered in the proof of Theorem 3.1, its complement is not an n-set (since $v, v_1, v_2 \in S$, the edge $v_1 v_2 \notin E(\bigcup_{x \in \overline{S}} \langle N[x] \rangle)$). Thus, S is an Na-set. Hence $l_{Na}(W_{1,k}) \leq |S| = 1 + \lfloor \frac{k}{2} \rfloor$. Therefore, $l_{Na}(W_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$. □

Theorem 3.8. *For any integer $k \geq 4$, $l_{Na^\star}(W_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$.*

Proof. Let S be an $Na^\star$-set. Then, S is an Na-set and $\overline{S}$ is also an a-set. Hence from Theorem 3.7, it follows that $l_{Na^\star}(W_{1,k}) \geq 1 + \lfloor \frac{k}{2} \rfloor$.

On the other hand, for the Na-set $S = \{v, v_1, v_2, \dots, v_{\lfloor \frac{k}{2} \rfloor}\}$ considered in the proof of Theorem 3.7, its complement is $\overline{S} = \{v_{\lfloor \frac{k}{2} \rfloor+1}, v_{\lfloor \frac{k}{2} \rfloor+2}, \dots, v_k\}$. For every vertex $v_i \in \overline{S}$, $|N[v_i] \cap \overline{S}| \geq 2 \geq |N(v_i) \cap S|$ implies that $\overline{S}$ is an a-set. Therefore, S is an $Na^\star$-set. Thus, $l_{Na^\star}(W_{1,k}) \leq |S| = 1 + \lfloor \frac{k}{2} \rfloor$. Hence $l_{Na^\star}(W_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$. □

Theorem 3.9. *For any integer $k \geq 4$,* $l_{NA}(W_{1,k}) = \begin{cases} 2 + \lfloor \frac{k}{2} \rfloor & \textit{for} \quad k = 4, 5. \\ 1 + \lfloor \frac{k}{2} \rfloor & \textit{for} \quad k \geq 6. \end{cases}$

Proof. Let S be an NA-set of $W_{1,k}$. Then, S is an na-set and $\overline{S}$ is neither an n-set nor an a-set. So, $v \in S$ and $|S| \geq 1 + \lfloor \frac{k}{2} \rfloor$ for any integer $k \geq 4$ (by Theorem 3.1).

For $k = 4$, consider $S = \{v, v_1, v_2, v_3\}$, which is an n-set (since $v \in S$ and by Remark 1.2) and also an a-set (since for each vertex $v_i \in S$, $|N[v_i] \cap S| \geq 3 > 1 = |N(v_i) \cap \overline{S}|$). Moreover, $\overline{S} = \{v_4\}$ is neither an n-set nor an a-set. Hence $l_{NA}(W_{1,4}) \leq |S| = 4 = 2 + \lfloor \frac{k}{2} \rfloor$. This proves that $l_{NA}(W_{1,k}) = 2 + \lfloor \frac{k}{2} \rfloor$.

For $k = 5$, consider the set $S = \{v, v_1, v_3, v_4\}$ then $\overline{S} = \{v_2, v_5\}$. Then S is an n-set (since $v \in S$ and by Remark 1.2) and an a-set (since for every vertex $x \in S$, $|N[x] \cap S| \geq 2 > 1 \geq |N(x) \cap \overline{S}|$). Also $\overline{S}$ is not an a-set (since v_2 has no neighbors in $\overline{S}$) and $\overline{S}$ is not an n-set(since an edge $v_3v_4 \notin E(\bigcup_{x \in \overline{S}} < N[x] >))$. Therefore, S is an NA-set. Hence $l_{NA}(W_{1,k}) \leq |S| = 4 = 2 + \lfloor \frac{k}{2} \rfloor$. This proves that $l_{NA}(W_{1,k}) = 2 + \lfloor \frac{k}{2} \rfloor$.

For $k = 6, 7$, consider the set $S = \{v, v_1, v_3, v_4\}$. Then, S is an n-set (since $v \in S$ and by Remark 1.2) and an a-set(since $|N[v] \cap S| = 4 \geq |N(v) \cap \overline{S}|$ and for other rim vertex $x \in S$, $|N[x] \cap S| \geq 2 \geq |N(x) \cap \overline{S}|$). Now $\overline{S}$ is not an a-set (since v_2 has no neighbors in $\overline{S}$) and $\overline{S}$ is not an n-set(since the edge $v_3v_4 \notin E(\bigcup_{x \in \overline{S}} \langle N[x] \rangle))$. Therefore, S is an NA-set. Hence $l_{NA}(W_{1,k}) \leq |S| = 4 = 1 + \lfloor \frac{k}{2} \rfloor$. This proves that $l_{NA}(W_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$.

For $k \geq 8$, consider the set $S = \{v, v_1, v_3, v_5, v_6, v_8, \dots, v_{k-4}, v_{k-2}\}$ for k even and $S = \{v, v_1, v_3, v_5, v_6, v_8, \dots, v_{k-5}, v_{k-3}\}$ for k odd. Then, S is an n-set (since $v \in S$ and by Remark 1.2) and an a-set (since $|N[v] \cap S| = 1 + \lfloor \frac{k}{2} \rfloor = |N(v) \cap \overline{S}|$ and for other rim vertex $x \in S$, $|N[x] \cap S| \geq 2 \geq |N(x) \cap \overline{S}|$). Also $\overline{S}$ is not an a-set (since v_2 has no neighbors in $\overline{S}$) and $\overline{S}$ is not an n-set (since the edge $v_5v_6 \notin E(\bigcup_{x \in \overline{S}} \langle N[x] \rangle))$. Therefore, S is an NA-set. Hence $|S| \leq 1 + \lfloor \frac{k}{2} \rfloor$. This proves that $l_{NA}(W_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$.

Hence the theorem. □

Theorem 3.10. *For any integer $k \geq 8$, $l_{NA^\star}(W_{1,k}) = 4$.*

Proof. Let S be an $NA^\star$-set. Then, S is an n-set and $\overline{S}$ is not an n-set. So, $v \in S$ (else if $v \in \overline{S}$, then $\overline{S}$ is an n-set by Remark 1.2). Also both S and $\overline{S}$ are not a-sets. Now S is an N-set is possible if $\langle S \rangle$ contains a triangle vv_iv_{i+1} for $i = 1, 2, ..., k$, where $v_{k+1} = v_1$ as its subgraph, such that the edge $v_iv_{i+1} \notin E(\bigcup_{x \in \overline{S}} \langle N[x] \rangle))$. Also S is an $A^\star$-set, hence along with the vertices v, v_i, v_{i+1} it must also contain at least one rim vertex say v_{i+3} which is not adjacent to v_i and v_{i+1} such that v_{i+2} is not defendable in $\overline{S}$. Also $4 = |N[v] \cap S| < |N(v) \cap \overline{S}| = k-3$ which indicates S is not an a-set. Therefore, S contains at least four vertices. This proves that $|S| \geq 4$.

To prove the reverse inequality, consider the set $S = \{v, v_1, v_2, v_4\}$. Then, S is an n-set (since $v \in S$ and by Remark 1.2) and $\overline{S}$ is not an n-set (since the edge $v_1v_2 \notin E(\bigcup_{x \in \overline{S}} \langle N[x] \rangle))$. Also $4 = |N[v] \cap S| < |N(v) \cap \overline{S}| = k-1$ and hence S is not an a-set. Further, there exists a vertex v_3 in $\overline{S}$ such that $|N[v_3] \cap \overline{S}| = 1 < 3 = |N(v_3) \cap S|$ which implies that $\overline{S}$ is not an a-set. This proves that S is an $NA^\star$-set. Therefore, $l_{NA^\star}(W_{1,k}) \leq |S| = 4$. Hence $l_{NA^\star}(W_{1,k}) = 4$. □

Remark 3.11. *For $k = 3$, as $W_{1,3} \cong K_4$, every vertex of $W_{1,3}$ is a full degree vertex and hence by Remark 1.2 every non-empty set of vertices of $W_{1,3}$ is an n-set. Therefore, there is no N-set for $W_{1,3}$. Hence $l_{Na}(W_{1,k})$, $l_{Na^\star}(W_{1,k})$, $l_{NA}(W_{1,k})$ and $l_{NA^\star}(W_{1,k})$ do not exist.*

Remark 3.12. *For any integer k, $4 \leq k \leq 7$, $l_{NA^\star}(W_{1,k})$ does not exist.*

Proof. Let S be an $NA^\star$-set. Then, S is an n-set and $\overline{S}$ is not an n-set. So, $v \in S$ (else if $v \in \overline{S}$, then $\overline{S}$ is an n-set by Remark 1.2). Also both S and $\overline{S}$ are not an a-sets. By Lemma 3.5, any p-element subset of vertices of G is an a-set if $p \geq k-1$. Hence $1 \leq |S| \leq k-2$. Further, if $|S| = 1$ then $|\overline{S}| = k$ and if $|S| = 2$ then $|\overline{S}| = k-1$, this implies that $\overline{S}$ is an a-set(by Lemma 3.5),a contradiction for S being an $A^\star$-set. Hence $|S| \geq 3$. Therefore, $3 \leq |S| \leq k-2$. This clearly proves that $l_{NA^\star}(W_{1,k})$ does not exist for $k = 4$. Now let us consider the following cases.

Case 1: $|S| = 3$ for $k = 5, 6, 7$.

Then the possible case for set S is either $S = \{v, v_i, v_{i+1}\}$ or $S = \{v, v_i, v_{i+2}\}$ for $1 \leq i \leq k$ with $v_{k+1} = v_1$ and $v_{k+2} = v_2$. If $S = \{v, v_i, v_{i+1}\}$ then $\overline{S}$ contains $k-2$ rim vertices of $W_{1,k}$. Now for each vertex $w \in \overline{S}$, $|N[w] \cap \overline{S}| \geq 2 \geq |N(w) \cap S|$ and hence $\overline{S}$ is an a-set, a contradiction for S being an $A^\star$-set. If $S = \{v, v_i, v_{i+2}\}$ then $\overline{S}$ contains $k-2$ rim vertices of $W_{1,k}$. Now this set $\overline{S}$

is an n-set(since by Theorem 1.11 every edge of $\langle V(G)-\overline{S}\rangle$ belongs to a triangle one of whose vertices belongs to $\overline{S}$), a contradiction for S being an N-set. This proves that there is no $NA^{\star}$-set S with $|S|=3$.

Case 2:$|S|=4$ for $k=6,7$.

In this case S contains the vertex v and any three rim vertices of $W_{1,k}$. Now $|N[v]\cap S|=4\geq |N(v)\cap\overline{S}|$ and for any rim vertex $w\in S$, $|N[w]\cap S|\geq 2\geq |N(w)\cap\overline{S}|$, therefore S is an a-set, a contradiction for S being an $A^{\star}$-set.

Case 3: $|S|=5$ for $k=7$.

In this case S contains the vertex v and any four rim vertices of $W_{1,k}$. Now $|N[v]\cap S|=5>3=|N(v)\cap\overline{S}|$ and for any rim vertex $w\in S$, $|N[w]\cap S|\geq 2\geq |N(w)\cap\overline{S}|$, therefore S is an a-set, a contradiction for S being an $A^{\star}$-set.

Therefore, the above cases proves that there is no $NA^{\star}$-set for $W_{1,k}$ for $4\leq k\leq 7$. Hence $l_{NA^{\star}}(W_{1,k})$ does not exist for $4\leq k\leq 7$. □

Theorem 3.13. *For any integer $k\geq 3$, $l_{n^{\star}a}(W_{1,k})=1+\lfloor\frac{k}{2}\rfloor$.*

Proof. Let S be an $n^{\star}a$-set. Then, by Theorem 3.1, $|S|\geq 1+\lfloor\frac{k}{2}\rfloor$. To prove the reverse inequality, consider the set $S=\{v,v_1,v_3,\dots,v_{k-3},v_{k-1}\}$ for k even and $S=\{v,v_1,v_3,\dots,v_{k-4},\ v_{k-2}\}$ for k odd. Then S is an n-set (since $v\in S$ and by Remark 1.2) and for each vertex $u\in S$, $|N[u]\cap S|\geq 2\geq |N(u)\cap\overline{S}|$ and so S is an a-set. We have $\overline{S}=\{v_2,v_4,\dots,v_{k-2},v_k\}$ for k even and $\overline{S}=\{v_2,v_4,\dots,v_{k-2},v_k\}$ for k odd, which shows that $\overline{S}$ is an n-set (since by Theorem 1.11 every edge of $\langle V(G)-\overline{S}\rangle$ belongs to a triangle one of whose vertices belongs to $\overline{S}$). Therefore, $l_{n^{\star}a}(W_{1,k})\leq |S|=1+\lfloor\frac{k}{2}\rfloor$. Hence $l_{n^{\star}a}(W_{1,k})=1+\lfloor\frac{k}{2}\rfloor$. □

Theorem 3.14. *For $k\geq 3$, $l_{n^{\star}A}(W_{1,k})=1+\lfloor\frac{k}{2}\rfloor$.*

Proof. Let S be an $n^{\star}A$-set. Then, by Theorem 3.1, $|S|\geq 1+\lfloor\frac{k}{2}\rfloor$. To prove the reverse inequality, consider the set $S=\{v,v_1,v_3,\dots,v_{k-3},v_{k-1}\}$ for k even and $S=\{v,v_1,v_3,\dots,v_{k-4},\ v_{k-2}\}$ for k odd, which is an $n^{\star}a$-set as considered in the proof of Theorem 3.13. Now as the vertex $v_2\in\overline{S}$ has no neighbors in $\overline{S}$, $\overline{S}$ is not an a-set. Hence S is an $n^{\star}A$-set. Therefore, $l_{n^{\star}A}(W_{1,k})\leq |S|=1+\lfloor\frac{k}{2}\rfloor$. Hence $l_{n^{\star}A}(W_{1,k})=1+\lfloor\frac{k}{2}\rfloor$. □

Theorem 3.15. *For any integer $k\geq 4$, $l_{n^{\star}a^{\star}}(W_{1,k})$ does not exist whereas $l_{n^{\star}a^{\star}}(W_{1,3})=2$.*

Proof. Let S be an $n^\star a^\star$-set. Then, S and $\overline{S}$ both are na-sets. By Theorem 3.1, we have $l_{na}(W_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$ implies $|S| = 1 + \lfloor \frac{k}{2} \rfloor$. Thus, when k is even, say $k = 2m$, $m \in Z^+$, $|\overline{S}| = |V| - |S| = (k+1) - (1 + \lfloor \frac{k}{2} \rfloor) = (2m+1) - (1 + \lfloor \frac{2m}{2} \rfloor) = m = \frac{k}{2} < 1 + \lfloor \frac{k}{2} \rfloor$ and hence there is no set S such that S and $\overline{S}$ both are na-sets. When k is odd, say $k = 2m+1$, $m \in Z^+$, $|\overline{S}| = |V| - |S| = (k+1) - (1 + \lfloor \frac{k}{2} \rfloor) = (2m+2) - (1 + \lfloor \frac{2m+1}{2} \rfloor) = m+1 = \frac{k+1}{2} = 1 + \lfloor \frac{k}{2} \rfloor$. Now we have the following two possibilities for the na-set S.

Case 1: $S = \{v, v_1, v_2, v_3, \dots, v_{\lfloor \frac{k}{2} \rfloor}\}$. Then in this case $\overline{S}$ is not an n-set(since the edge $v_1v_2 \notin E(\bigcup_{x \in \overline{S}} \langle N[x] \rangle)$), a contradiction for S being an $n^\star$-set.

Case 2: $S = \{v, v_1, v_3, v_5, \dots, v_{k-2}\}$. Then in this case $\overline{S}$ is not an a-set(since $|N[v_2] \cap \overline{S}| = 1 < 3 = |N(v_2) \cap S|$), a contradiction for S being an $a^\star$-set.

Therefore, from the above cases we conclude that there is no $n^\star a^\star$-set. Hence $l_{n^\star a^\star}(W_{1,k})$ for $k \geq 4$ does not exist.

Further, as $W_{1,3} \cong K_4$, the result follows from Theorem 2.8. $\square$

Theorem 3.16. *For any integer $k \geq 6$, $l_{n^\star A^\star}(W_{1,k}) = 3$.*

Proof. Let S be an $n^\star A^\star$-set. Then, S is an $nA^\star$-set and hence by Theorem 3.4, we have $|S| \geq 3$. To prove the reverse inequality, let us consider the $nA^\star$-set $S = \{v, v_i, v_{i+2}\}$ for $1 \leq i \leq k$ with $v_{k+1} = v_1$ and $v_{k+2} = v_2$ as taken in the proof of Theorem 3.4. Now $\overline{S}$ is also an n-set(since by Theorem 1.11 every edge of $\langle V(G) - \overline{S} \rangle$ belongs to a triangle one of whose vertices belongs to $\overline{S}$). Therefore, S is an $n^\star A^\star$-set. Hence $l_{n^\star A^\star}(W_{1,k}) \leq |S| = 3$. This proves $l_{n^\star A^\star}(W_{1,k}) = 3$ $\square$

By Remark 3.6, there is no $nA^\star$-set S for $W_{1,k}$ with $3 \leq k \leq 5$ and hence we conclude with the remark below:

Remark 3.17. *For any integer k, $3 \leq k \leq 5$, $l_{n^\star A^\star}(W_{1,k})$ does not exist.*

4. Neighborhood Alliance Sets of a Star Graph

Throughout this section, let v be the central vertex and $v_1, v_2, \dots, v_k$ be the k pendant vertices adjacent to v of a star S_k.

Theorem 4.1. *For any integer $k \geq 1$, $l_{na}(S_k) = 1 + \lfloor \frac{k}{2} \rfloor$.*

Proof. Let S be an na-set of S_k.

Case 1: $v \in S$. Since S is an a-set, by Remark 1.16, at least $\lfloor \frac{\deg(v)}{2} \rfloor$ neighbors of v should be in S and hence $|S| \geq 1 + \lfloor \frac{\deg(v)}{2} \rfloor \geq 1 + \lfloor \frac{k}{2} \rfloor$.

Case 2: $v \notin S$.

As S is an n-set, $S_k \cong \bigcup_{x \in S} \langle N[x] \rangle$ implies all the remaining k pendant vertices must be in S. Hence $|S| \geq k$

Therefore, by the above cases, $|S| \geq \min\{1 + \lfloor \frac{k}{2} \rfloor, k\} = 1 + \lfloor \frac{k}{2} \rfloor$. Hence $l_{na}(S_k) \geq 1 + \lfloor \frac{k}{2} \rfloor$.

Now to prove the reverse inequality, consider the set $S = \{v, v_1, v_2, v_3, \ldots, v_{\lfloor \frac{k}{2} \rfloor}\}$. The set S is an n-set as $v \in S$ (since $v \in S$ and by Remark 1.2). We have for k even, $|N[v] \cap S| = 1 + \lfloor \frac{k}{2} \rfloor > \frac{k}{2} = |N(v) \cap \overline{S}|$ and for k odd, $|N[v] \cap S| = 1 + \lfloor \frac{k}{2} \rfloor = \frac{k+1}{2} = |N(v) \cap \overline{S}|$. Also for every pendant vertex $u \in S$, $|N[u] \cap S| = 2 > 0 = |N(u) \cap \overline{S}|$ and hence S is an a-set. Thus, S is an na-set. Therefore, $l_{na}(S_k) \leq |S| = 1 + \lfloor \frac{k}{2} \rfloor$. Hence the theorem. □

Theorem 4.2. *For any integer $k \geq 1$, $l_{na^\star}(S_k) = 1 + \lfloor \frac{k}{2} \rfloor$.*

Proof. Let S be an $na^\star$-set. Then, by Theorem 4.1, $l_{na^\star}(S_k) \geq 1 + \lfloor \frac{k}{2} \rfloor$. On the other hand, for the na-set $S = \{v, v_1, v_2, \ldots, v_{\lfloor \frac{k}{2} \rfloor}\}$ considered in the proof of Theorem 4.1, its complement $\overline{S}$ consists of the remaining pendant vertices of S_k. For each vertex $x \in \overline{S}$, $|N[x] \cap \overline{S}| = 1 = |N(x) \cap S|$ implies that $\overline{S}$ is also an a-set. Hence S is an $na^\star$-set. Thus, $l_{na^\star}(S_k) \leq |S| = 1 + \lfloor \frac{k}{2} \rfloor$. Hence the theorem. □

Theorem 4.3. *For any integer $k \geq 1$,* $l_{nA}(S_k) = \begin{cases} 2 & \text{for} \quad k = 1. \\ k & \text{for} \quad k \geq 2. \end{cases}$

Proof. For $k = 1$, as $S_1 \cong K_2$, the result follows from Theorem 2.4.

For $k \geq 2$. Let S be an nA-set. Then, S is an na-set and $\overline{S}$ is not an a-set.

Claim 1: $v \notin S$

For, if $v \in S$, then from the proof of Theorem 4.1 we see that $|S| \geq 1 + \lfloor \frac{k}{2} \rfloor$ and $\overline{S}$ has remaining pendant vertices. And for each vertex $x \in \overline{S}$, $|N[x] \cap \overline{S}| = 1 = |N(x) \cap S|$ implies that $\overline{S}$ is also an a-set, a contradiction for S being an nA-set. Hence the Claim 1.

Now since $v \notin S$ and S is an na-set, S contains all k pendant vertices (as considered in the proof of Theorem 4.1). This indicates that $\overline{S} = \{v\}$ and $|N[v] \cap \overline{S}| = 1 < k = |N(v) \cap S|$ and hence

$\overline{S}$ is not an a-set. Therefore, S is an nA-set with $|S| = k$. This proves that $l_{nA}(S_k) = k$. Hence the theorem. □

Theorem 4.4. *For any integer $k \geq 1$, $l_{nA^\star}(S_k)$ does not exist.*

Proof. For $k = 1$, as $S_1 \cong K_2$, the result follows from Theorem 2.6.

For $k \geq 2$, let S be an $nA^\star$-set. Then, S is an n-set and both S and $\overline{S}$ is not an a-set.

Case 1: $v \in S$.

Then S is an n-set(since $v \in S$ and by Remark 1.2). As S is not an a-set and $v \in S$, $|S| < 1 + \lfloor \frac{k}{2} \rfloor$(by Theorem 4.1). But then $\overline{S}$ has only remaining pendant vertices each of which is defendable(since for each vertex $x \in \overline{S}$, $|N[x] \cap \overline{S}| = 1 = |N(x) \cap S|$) and hence $\overline{S}$ is an a-set, a contradiction for S being an $A^\star$-set.

Case 2: $v \notin S$.

Then, as considered in the proof of Theorem 4.1, S is an na-set (since $|S| = k$ with k pendant vertices and for every pendant vertex $u \in S$, $|N[u] \cap S| = 1 = |N(u) \cap \overline{S}|$), a contradiction for S being an $A^\star$-set.

Therefore, there is no $A^\star$-set and hence $l_{nA^\star}(S_k)$ does not exist. □

Theorem 4.5. *For any integer $k \geq 1$, $l_{n^\star a}(S_k) = k$.*

Proof. Let S be an $n^\star a$-set. Then, S is an na-set and $\overline{S}$ is an n-set.

Claim 1: $v \notin S$

For, if $v \in S$, then by Theorem 4.1, $|S| \geq 1 + \lfloor \frac{k}{2} \rfloor$. Let $S = \{v, v_1, v_2, v_3, \ldots, v_{\lfloor \frac{k}{2} \rfloor}\}$ then $\overline{S} = \{v_{\lfloor \frac{k}{2} \rfloor + 1}, v_{\lfloor \frac{k}{2} \rfloor + 2}, \ldots, v_k\}$ is not an n-set(since the edge $vv_1 \notin E(\bigcup_{x \in \overline{S}} \langle N[x] \rangle))$, a contradiction for S being an $n^\star a$-set. Hence the Claim 1.

Now $v \in \overline{S}$ and by Remark 1.2, $\overline{S}$ is an n-set. Then the set S containing all the k pendant vertices of S_k is an na-set (as considered in the proof of Theorem 4.1). Thus, S is an $n^\star a$-set with $|S| = k$. Hence $l_{n^\star a}(S_k) = k$. □

Theorem 4.6. *For any integer $k \geq 1$, $l_{n^\star a^\star}(S_1) = 1$ and $l_{n^\star a^\star}(S_k)$ does not exist for $k \geq 2$.*

Proof. For $k = 1$, as $S_1 \cong K_2$, the result follows from Theorem 2.8.

For $k \geq 2$, let S be an $n^\star a^\star$-set. Then, both S and $\overline{S}$ are na-sets.

Case 1: $v \in S$.

Then, from Theorem 4.1, $|S| = 1 + \lfloor \frac{k}{2} \rfloor$ say $S = \{v, v_1, v_2, v_3, \ldots, v_{\lfloor \frac{k}{2} \rfloor}\}$, then $\overline{S} = \{v_{\lfloor \frac{k}{2} \rfloor + 1}, v_{\lfloor \frac{k}{2} \rfloor + 2}, \ldots, v_k\}$ is not an n-set(since the edge $vv_1 \notin E(\bigcup_{x \in \overline{S}} \langle N[x] \rangle))$, a contradiction for S being an $n^\star$-set.

Case 2: $v \notin S$.

Then, from the proof of Theorem 4.1, $|S| = k$ with $S = \{v_1, v_2, v_3, \ldots, v_k\}$. In this case S is an na-set but $\overline{S} = \{v\}$ is not an a-set (since $|N[v] \cap \overline{S}| = 1 < k = |N(v) \cap S|$), a contradiction for S being an $a^\star$-set.

Thus, there is no set S such that both S and $\overline{S}$ are na-sets. Hence $l_{n^\star a^\star}(S_k)$ does not exist for $k \geq 2$. □

Theorem 4.7. *For any integer $k \geq 2, l_{n^\star A}(S_k) = k$ and for $k = 1, l_{n^\star A}(S_k)$ does not exist.*

Proof. As $S_1 \cong K_2$, the result follows from Theorem 2.4 for $k = 1$.

For $k \geq 2$, let S be an $n^\star A$-set. Then, S is an nA-set and $\overline{S}$ is an n-set. Now for the nA-set S containing all k pendant vertices as considered in the proof of Theorem 4.3, $\overline{S} = \{v\}$ and it is an n-set (since $v \in \overline{S}$ and by Remark 1.2). Thus, S is an $n^\star A$-set with $|S| = k$. Hence the theorem. □

Theorem 4.8. *For any integer $k \geq 1$, $l_{n^\star A^\star}(S_k)$ does not exist.*

Proof. As $S_1 \cong K_2$, the result follows from Theorem 2.6 for $k = 1$.

For $k \geq 2$, let S be an $n^\star A^\star$-set. Then, both S and $\overline{S}$ are n-sets and both are not a-sets. We have the following cases for S to be an $n^\star$-set.

Case 1: $v \in S$.

It is easy to see that $S = \{v\}$ is an n-set (since $v \in S$ and by Remark 1.2) and $\overline{S}$ contains all the remaining k pendant vertices(as S is an $n^\star$-set). Then, for every vertex $x \in \overline{S}$, $|N[x] \cap \overline{S}| = 1 = |N(x) \cap S|$ and hence $\overline{S}$ is an a-set, a contradiction for S being an $A^\star$-set.

Case 2: $v \notin S$.

Now $v \in \overline{S}$ implies S must contain all the remaining k pendant vertices(as S being an $n^\star$-set). Then, for every vertex $x \in S$, $|N[x] \cap S| = 1 = |N(x) \cap \overline{S}|$ and hence S is an a-set, a contradiction for S being an $A^\star$-set.

Therefore, there is no $n^\star A^\star$-set. Hence $l_{n^\star A^\star}(S_k)$ does not exist. $\square$

Theorem 4.9. *For any integer $k \geq 1$,* $l_{Na}(S_k) = \begin{cases} 2 & \textit{for} \quad k = 1. \\ 1 + \lfloor \frac{k}{2} \rfloor & \textit{for} \quad k \geq 2. \end{cases}$

Proof. As $S_1 \cong K_2$, the result follows from Theorem 2.9 for $k = 1$.

For $k \geq 2$, let S be an Na-set. Then, S is an na-set and $\overline{S}$ is not an n-set. As S is an na-set, by Theorem 4.1, $|S| \geq 1 + \lfloor \frac{k}{2} \rfloor$.

To prove the reverse inequality, consider the na-set $S = \{v, v_1, v_2, v_3, \ldots, v_{\lfloor \frac{k}{2} \rfloor}\}$ as considered in the proof of Theorem 4.1. Then, $\overline{S}$ is not an n-set as the edge $vv_1 \notin E(\bigcup_{x \in \overline{S}} \langle N[x] \rangle)$. Therefore, S is an Na-set. Thus $l_{Na}(S_k) \leq |S| = 1 + \lfloor \frac{k}{2} \rfloor$. Hence the theorem. $\square$

Theorem 4.10. *For any integer $k \geq 2$, $l_{Na^\star}(S_k) = 1 + \lfloor \frac{k}{2} \rfloor$.*

Proof. Let S be an $Na^\star$-set. Then, S is an na-set and $\overline{S}$ is an a-set but not an n-set. As S is an na-set, by Theorem 4.1, $|S| \geq 1 + \lfloor \frac{k}{2} \rfloor$.

To prove the reverse inequality, consider the Na-set $S = \{v, v_1, v_2, v_3, \ldots, v_{\lfloor \frac{k}{2} \rfloor}\}$ as considered in the proof of Theorem 4.9. Then $\overline{S} = \{v_{\lfloor \frac{k}{2} \rfloor + 1}, v_{\lfloor \frac{k}{2} \rfloor + 2}, \ldots, v_k\}$ is an a-set (as for every vertex $x \in \overline{S}$, $|N[x] \cap \overline{S}| = 1 = |N(x) \cap S|$). Therefore, S is an $Na^\star$-set. Thus, $l_{Na^\star}(S_k) \leq |S| = 1 + \lfloor \frac{k}{2} \rfloor$. Hence the theorem. $\square$

Remark 4.11. *For $k = 1, l_{Na^\star}(S_k)$ does not exist(since $S_1 \cong K_2$, the result follows from Theorem 2.9).*

Theorem 4.12. *For any integer $k \geq 1$, $l_{NA}(S_k) = k + 1$.*

Proof. Let S be an NA-set. Then, S is an na-set and $\overline{S}$ is neither an n-set nor an a-set. As S is an na-set, we consider the two cases as taken in the proof of Theorem 4.1.

Case 1: $v \in S$.

In this case, $|S| \geq 1 + \lfloor \frac{k}{2} \rfloor$ and $\overline{S}$ contains only remaining pendant vertices. That is, $\overline{S} = \{v_{\lfloor \frac{k}{2} \rfloor + 1}, v_{\lfloor \frac{k}{2} \rfloor + 2}, \ldots, v_k\}$ and for every vertex $x \in \overline{S}$, $|N[x] \cap \overline{S}| = 1 = |N(x) \cap S|$ and hence $\overline{S}$ is an a-set, a contradiction for S being an A-set.

Case 2: $v \notin S$.

In this case, $|S| \geq k$ and $v \in \overline{S}$ implies $\overline{S}$ is an n-set (since $v \in \overline{S}$ and by Remark 1.2), a contradiction for S being an N-set.

From the above cases we see that it is not possible to have an N-set with $v \in \overline{S}$. Further, if $\overline{S}$ contains any one pendant vertex v_i for $1 \leq i \leq k$ then $|N[v_i] \cap \overline{S}| = 1 = |N(v_i) \cap S|$ and hence $\overline{S}$ is an a-set. This proves that $\overline{S}$ cannot have any element and hence $\overline{S}$ must be an empty set. Therefore, a set S containing all the vertices of S_k is definitely an na-set and $\overline{S}$ being an empty set is neither an n-set nor an a-set. Thus, the set S with $|S| = k+1$ is an NA-set. Hence $l_{NA}(S_k) = k+1$. □

Theorem 4.13. *For any integer $k \geq 1$, $l_{NA^\star}(S_k)$ does not exist.*

Proof. For $k = 1$, the result follows from Theorem 2.9.

For $k \geq 2$, let S be an $NA^\star$-set. Then, S is an n-set and $\overline{S}$ is not an n-set. Also both S and $\overline{S}$ are not a-sets.

So, $v \in S$ (else if $v \in \overline{S}$, then $\overline{S}$ is an n-set by Remark 1.2). Also $\overline{S}$ cannot contain all the k pendant vertices, else $\overline{S}$ will be an n-set, a contradiction again to the fact that S is an N-set.

Further, if $\overline{S}$ contains any one pendant vertex v_i with $1 \leq i \leq k$, then for each vertex v_i, $|N[v_i] \cap \overline{S}| = 1 = |N(v_i) \cap S|$ and therefore $\overline{S}$ is an a-set, a contradiction for S being an $A^\star$-set. This indicates that $\overline{S}$ must be an empty set which implies $S = V(S_k)$. Hence S is an a-set in this case, again a contradiction to the fact that S is an $A^\star$-set. Thus, there is no $NA^\star$-set. Hence $l_{NA^\star}(S_k)$ does not exist. □

5. NEIGHBORHOOD ALLIANCE SETS OF A FAN GRAPH

Throughout this section, let v be the central vertex and $v_1, v_2, \ldots, v_k$ be the k vertices which are adjacent to v of a fan $F_{1,k}$ in order. The vertices $v_1, v_2, \ldots, v_k$ are called rim vertices of $F_{1,k}$.

Theorem 5.1. *For any integer $k \geq 1$,* $l_{na}(F_{1,k}) = \begin{cases} 2 & \textit{for} \quad k = 4. \\ 1 + \lfloor \frac{k}{2} \rfloor & \textit{for} \quad k \neq 4. \end{cases}$

Proof. For $k = 1, 2$, as $F_{1,1} \cong K_2$ and $F_{1,2} \cong K_3$, the result follows from Theorem 2.2.

For $k = 3$ and $k = 4$, the na-sets of minimum cardinality are shown in Figure 1 and Figure 2 respectively.

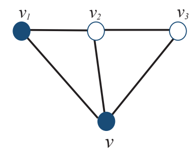

FIGURE 1. The Fan Graph $F_{1,3}$.

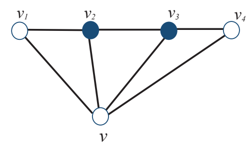

FIGURE 2. The Fan Graph $F_{1,4}$.

For $k \geq 5$, let S be an na-set. Now we consider the two subcases as below.

Case (i): $v \in S$.

Then, clearly S is an n-set(since $v \in S$ and by Remark 1.2). Also since S is an a-set, at least $\lfloor \frac{\deg(v)}{2} \rfloor$ neighbors of v should be in S (by Remark 1.16). Hence $|S| \geq 1 + \lfloor \frac{k}{2} \rfloor$.

Case(ii): $v \notin S$.

Define $S_i = \{v_{i-1}, v_i, v_{i+1}\}$ and Let $T_i = S \cap S_i$, for $i = 1, 2, \ldots, k-1$.

Claim 1: $|T_i| \geq 2$ for each $i, 1 \leq i \leq k-1$.

By Theorem 1.13, every edge of $\langle V(G) - S \rangle$ belongs to a triangle one of whose vertices belongs to S. Here as $v \notin S$, it is possible only if $T_i = S \cap S_i = \{v_i, v_{i+1}\}$ or $\{v_{i-1}, v_i\}$. Hence Claim 1 holds.

By Claim 1, it follows that for every three consecutive rim vertices of $F_{1,k}$ at least two of them have to be in S. Hence $|S| \geq \lfloor \frac{2k}{3} \rfloor$.

Therefore, $|S| \geq min\{1 + \lfloor \frac{k}{2} \rfloor, \lfloor \frac{2k}{3} \rfloor\} = 1 + \lfloor \frac{k}{2} \rfloor$. Hence $l_{na}(F_{1,k}) \geq 1 + \lfloor \frac{k}{2} \rfloor$.

Now to prove the reverse inequality, consider the set $S = \{v, v_1, v_2, v_3, \ldots, v_{\lfloor \frac{k}{2} \rfloor}\}$ then clearly S is an n-set(since $v \in S$ and by Remark 1.2). For each vertex $v_i \in S$, $|N[v_i] \cap S| \geq 3 > 1 \geq |N(v_i) \cap \overline{S}|$ for $i = 1 + \lfloor \frac{k}{2} \rfloor, \ldots, k$ and for $|N[v] \cap S| = 1 + \lfloor \frac{k}{2} \rfloor \geq |N(v) \cap \overline{S}|$, hence S is an a-set. This proves that $|S| \leq 1 + \lfloor \frac{k}{2} \rfloor$.

Hence $l_{na}(F_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$. $\square$

Theorem 5.2. *For any integer $k \geq 1$ and $k \neq 2$,* $l_{na^\star}(F_{1,k}) = \begin{cases} 2 & \text{for} \quad k = 4. \\ 1 + \lfloor \frac{k}{2} \rfloor & \text{for} \quad k \neq 4. \end{cases}$

Proof. Let S be an $na^\star$-set. Then, S is an na-set and $\overline{S}$ is also an a-set.

If $k \leq 4$, it is easy to observe $l_{na^\star}(F_{1,4}) = 2$. For $k > 4$. Now as S is an na-set, by Theorem 5.1, $|S| \geq 1 + \lfloor \frac{k}{2} \rfloor$. Consider the set na-set $S = \{v, v_1, v_2, \ldots, v_{\lfloor \frac{k}{2} \rfloor}\}$ as taken in the proof of Theorem 5.1, then for every vertex $w \in \overline{S}$, $|N[w] \cap \overline{S}| \geq 2 \geq 1 \geq |N(w) \cap S|$ and hence $\overline{S}$ is an a-set. Therefore, S is an $na^\star$-set. This proves that $|S| \leq 1 + \lfloor \frac{k}{2} \rfloor$. Thus, $l_{na^\star}(F_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$. $\square$

Remark 5.3. *As $F_{1,2} \cong K_3$, $l_{na^\star}(F_{1,2})$ does not exist (by Theorem 2.3).*

Theorem 5.4. *For any integer $k \geq 1$,* $l_{nA}(F_{1,k}) = \begin{cases} 2 & \text{for} \quad k = 1, 2. \\ 1 + \lfloor \frac{k}{2} \rfloor & \text{for} \quad k \geq 3. \end{cases}$

Proof. As $F_{1,1} \cong K_2$ and $F_{1,2} \cong K_3$, the result follows from Theorem 2.4 for $k = 1, 2$.

For $k \geq 3$, let S be an nA-set. Then, S is an na-set and hence by Theorem 5.1, $|S| \geq 1 + \lfloor \frac{k}{2} \rfloor$.

Now consider the set $S = \{v\} \cup \{v_2, v_3, \ldots, v_{1+\lfloor \frac{k}{2} \rfloor}\}$. Then S is an n-set (since $v \in S$ and by Remark 1.2). Also $|N[v] \cap S| = 1 + \lfloor \frac{k}{2} \rfloor \geq |N(v) \cap \overline{S}|$ and for other vertex $w \in S$, $|N[w] \cap S| \geq 3 > 1 \geq |N(w) \cap \overline{S}|$ and therefore every vertex in S is defendable and hence S is an na-set. Also $\overline{S}$ is not an a-set as for the vertex $v_1 \in \overline{S}$, $|N[v_1] \cap \overline{S}| = 1 < 2 = |N(v_1) \cap S|$ and hence not defendable in $\overline{S}$. Thus, S is an nA-set. Therefore, $|S| \leq 1 + \lfloor \frac{k}{2} \rfloor$. Hence $l_{nA}(F_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$. $\square$

Theorem 5.5. *For any integer $k \geq 4$, $l_{nA^\star}(F_{1,k}) = 2$.*

Proof. Let S be an $nA^\star$-set. If possible, let $|S| = 1$, then $S = \{v\}$ (since S is an n-set and by Remark 1.2). But then the remaining k vertices in $\overline{S}$ are defendable in $\overline{S}$ (since $|N[v_i] \cap \overline{S}| \geq 2 > 1 = |N(v_i) \cap S|$ for $i = 1, 2, \ldots k$) and hence forms an a-set, a contradiction to the fact that S is an $A^\star$-set. Therefore, $l_{nA^\star}(F_{1,k}) \geq 2$.

Now to prove the reverse inequality, consider the set $S = \{v, v_2\}$. Then S is an n-set (since $v \in S$ and by Remark 1.2). Further, for the vertex $v \in S$, $|N[v] \cap S| = 2 < k - 1 = |N(v) \cap \overline{S}|$ and for the vertex $v_1 \in \overline{S}$, $|N[v_1] \cap \overline{S}| = 1 < 2 = |N(v_1) \cap S|$ which proves that the sets S and $\overline{S}$ are not a-sets respectively. Therefore, S is an $nA^\star$-set. Hence $|S| \leq 2$. This proves $l_{nA^\star}(F_{1,k}) = 2$. $\square$

Remark 5.6. *A 2-element set S containing any two adjacent vertices of $F_{1,3}$ is always an a-set.*

Remark 5.7. *For any integer* $1 \leq k \leq 3$, $l_{nA^{\star}}(F_{1,k})$ *does not exist.*

Theorem 5.8. *For any integer* $k \geq 1$, $l_{Na}(F_{1,k}) = \begin{cases} k+1 & \textit{for} \quad k = 1,2. \\ k & \textit{for} \quad k = 3. \\ 1 + \lfloor \frac{k}{2} \rfloor & \textit{for} \quad k \geq 4. \end{cases}$

Proof. As $F_{1,1} \cong K_2$ and $F_{1,2} \cong K_3$, the result follows from Theorem 2.9 for $k = 1,2.$ and for $k = 3$ one can easily observe that $l_{Na}(F_{1,3}) = 3$.

Let S be an Na-set. Then, S is an na-set and $\overline{S}$ is not an n-set. For $k \geq 4$, as S is an na-set, by Theorem 5.1, $|S| \geq 1 + \lfloor \frac{k}{2} \rfloor$.

Now consider the na-set $S = \{v, v_1, v_2, v_3, \ldots, v_{\lfloor \frac{k}{2} \rfloor}\}$ as taken in the proof of the Theorem 5.1. Then the set $\overline{S}$ is not an n-set (since the edge $vv_1 \notin E(\bigcup_{x \in \overline{S}} \langle N[x] \rangle))$. This proves that S is an Na-set and thus $|S| \leq 1 + \lfloor \frac{k}{2} \rfloor$. Hence $l_{na}(F_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$. □

Theorem 5.9. *For any integer* $k \geq 4$, $l_{Na^{\star}}(F_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$.

Proof. Let S be an $Na^{\star}$-set. Then, S is an Na-set and $\overline{S}$ is also an a-set. Since S is an Na-set, by Theorem 5.8, $|S| \geq 1 + \lfloor \frac{k}{2} \rfloor$.

Now consider the Na-set $S = \{v, v_1, v_2, v_3, \ldots, v_{\lfloor \frac{k}{2} \rfloor}\}$ as considered in the proof of Theorem 5.8. Then $\overline{S} = \{v_{1+\lfloor \frac{k}{2} \rfloor}, \ldots, v_k\}$ and for each vertex $v_i \in \overline{S}$, $|N[v_i] \cap \overline{S}| \geq 2 \geq 2 \geq |N(v_i) \cap S|$ for $i = 1 + \lfloor \frac{k}{2} \rfloor, \ldots, k$, which proves that $\overline{S}$ is also an a-set. Hence S is an $Na^{\star}$-set. This proves that $|S| \leq 1 + \lfloor \frac{k}{2} \rfloor$. Hence $l_{Na^{\star}}(F_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$. □

Remark 5.10. *For any integer* $1 \leq k \leq 3$, $l_{Na^{\star}}(F_{1,k})$ *does not exist.*

Proof. For $k = 1,2$, since $F_{1,1} \cong K_2$ and $F_{1,2} \cong K_3$, the result follows from Theorem 2.9.

For $k = 3$, the Na-set S as considered in the proof of Theorem 5.8 are either $S = \{v, v_2, v_3\}$ or $S = \{v, v_2, v_1\}$. And $\overline{S} = \{v_1\}$ or $\overline{S} = \{v_3\}$ respectively. In both the cases the $\overline{S}$ is not an a-set(since $|N[v_j] \cap \overline{S}| = 1 < 2 = |N(v_j) \cap S|$ for $j = 1,3$). Thus, there is no $Na^{\star}$-set S in $F_{1,3}$. Hence $l_{Na^{\star}}(F_{1,3})$ does not exist. □

Theorem 5.11. *For any integer* $k \geq 1$, $l_{NA}(F_{1,k}) = \begin{cases} k+1 & \textit{for} \quad k = 1,2. \\ k & \textit{for} \quad k = 3. \\ 1 + \lfloor \frac{k}{2} \rfloor & \textit{for} \quad k \geq 4. \end{cases}$

Proof. If $k \leq 3$ the result is trivial. Let$k \geq 4$ and S be an NA-set. Since S is an na-set, by Theorem 5.1, $|S| \geq 1 + \lfloor \frac{k}{2} \rfloor$. Now to prove the reverse inequality, consider the nA-set $S = \{v\} \cup \{v_2, v_3, \ldots, v_{1+\lfloor \frac{k}{2} \rfloor}\}$ as considered in the proof of Theorem 5.4. Further, $\overline{S}$ is not an n-set since the edge $v_2v_3 \notin E(\bigcup_{x \in \overline{S}} \langle N[x] \rangle)$. Hence S is an NA-set. This proves that $|S| \leq 1 + \lfloor \frac{k}{2} \rfloor$. Hence $l_{NA}(F_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$. □

Theorem 5.12. *For any integer $k \geq 6$, $l_{NA^\star}(F_{1,k}) = 3$.*

Proof. Let S be an $NA^\star$-set. Then, S is an n-set and $\overline{S}$ is not an n-set. Then $v \in S$, for, if $v \in \overline{S}$ then $\overline{S}$ is an n-set(by Remark 1.2), a contradiction for S being an N-set.

Moreover, as $\overline{S}$ is not an n-set, there must be two adjacent vertices, say v_i, v_{i+1} for $1 \leq i \leq k-1$ of $F_{1,k}$ in S such that the edge v_iv_{i+1} is not in $E(\bigcup_{x \in \overline{S}} \langle N[x] \rangle)$. Hence S has at least three elements. This proves that $|S| \geq 3$.

Now to prove the reverse inequality, consider the set $S = \{v, v_2, v_3\}$. Then S is an n-set (since $v \in S$ and by Remark 1.2). And $\overline{S}$ is not an n-set since edge $v_2v_3 \notin E(\bigcup_{x \in \overline{S}} \langle N[x] \rangle)$. The vertex v has more attackers than defenders in S (as $|N[v] \cap S| = 3 < k-2 = |N(v) \cap \overline{S}|$) and hence not defendable in S. Thus S is not an a-set. Further, $\overline{S}$ is not an a-set (as $|N[v_1] \cap \overline{S}| = 1 < 2 = |N(v_1) \cap S|$). Thus, S is an $NA^\star$-set. This proves that $|S| \leq 3$. Hence $l_{NA^\star}(F_{1,k}) = 3$. □

Remark 5.13. *For any integer $1 \leq k \leq 5$, $l_{NA^\star}(F_{1,k})$ does not exist.*

Theorem 5.14. *For any integer $k \geq 1$,* $l_{n^\star a}(F_{1,k}) = \begin{cases} 2 & \textit{for} \quad k = 4. \\ 1 + \lfloor \frac{k}{2} \rfloor & \textit{for} \quad k \neq 4. \end{cases}$

Proof. For $k = 1, 2$, the result follows from Theorem 2.7 because $F_{1,1} \cong K_2$ and $F_{1,2} \cong K_3$. Let S be an $n^\star a$-set. Then, S is an na-set and $\overline{S}$ is also an n-set.

Case 1: $k = 3, 4$.

For the na-set $S = \{v, v_1\}$ in $F_{1,3}$ and $S = \{v_2, v_3\}$ in $F_{1,4}$ as considered in the proof of Theorem 5.1, we have $\overline{S} = \{v_2, v_3\}$ in $F_{1,3}$ and $\overline{S} = \{v, v_1, v_4\}$ in $F_{1,4}$ respectively. From the Figure 1 and Figure 2 we see that the $\overline{S} = \{v_2, v_3\}$ in $F_{1,3}$ and $\overline{S} = \{v, v_1, v_4\}$ in $F_{1,4}$ are n-sets. Hence $S = \{v, v_1\}$ in $F_{1,3}$ and $S = \{v_2, v_3\}$ in $F_{1,4}$ is an $n^\star a$-set of the graphs $F_{1,3}$ and $F_{1,4}$ respectively. Hence the result holds.

Case 2: $k \geq 5$.

As S is an na-set, by Theorem 5.1, $|S| \geq 1 + \lfloor \frac{k}{2} \rfloor$. Now to prove the reverse inequality, consider the set $S = \{v\} \cup \{v_2, v_4, v_6, \ldots, v_{2\lfloor \frac{k}{2} \rfloor}\}$. Then clearly S is an n-set(since $v \in S$ and by Remark 1.2). And by Theorem 1.13, $\overline{S}$ is also an n-set. Hence S is an $n^\star$-set. Further, every vertex in S is defendable (since $|N[v] \cap S| = 1 + \lfloor \frac{k}{2} \rfloor \geq |N(v) \cap \overline{S}|$ and for every vertex w in S, $|N[w] \cap S| = 2 = |N(w) \cap \overline{S}|$). Hence S is an $n^\star a$-set. This proves that $|S| \leq 1 + \lfloor \frac{k}{2} \rfloor$.

Hence $l_{n^\star a}(F_{1,k}) = 1 + \lfloor \frac{k}{2} \rfloor$. □

Theorem 5.15. *For an integer k,* $l_{n^\star a^\star}(F_{1,k}) = \begin{cases} 1 & \text{for } \ k = 1. \\ 2 & \text{for } \ k = 3, 4. \end{cases}$

Proof. For $k = 1$, as $F_{1,1} \cong K_2$, by Theorem 2.8, it follows that $l_{n^\star a^\star}(F_{1,1}) = 1$.

For $k = 3, 4$, let S be an $n^\star a^\star$-set. Then, S is an $n^\star a$-set and hence by Theorem 5.14, $|S| \geq 2$.

Now to prove the reverse inequality, consider the $n^\star a$-set $S = \{v, v_1\}$ in $F_{1,3}$ and $S = \{v_2, v_3\}$ in $F_{1,4}$ respectively as taken in the proof of the Theorem 5.14. Then $\overline{S} = \{v_2, v_3\}$ in $F_{1,3}$ is an a-set (since $|N[v_j] \cap \overline{S}| = 2 \geq |N[v_j] \cap S|$ for $j = 2, 3$). And $\overline{S} = \{v, v_1, v_4\}$ in $F_{1,4}$ is an a-set (since $|N[v] \cap \overline{S}| = 3 > 2 = |N(v) \cap S|$ and $|N[v_j] \cap \overline{S}| = 2 > 1 = |N(v) \cap S|$ for $j = 1, 4$). Thus S is an an $n^\star a^\star$-set of the graphs $F_{1,3}$ and $F_{1,4}$ respectively. Therefore, $|S| \leq 2$. Hence the $l_{n^\star a^\star}(F_{1,k}) = 2$. □

Remark 5.16. *For $k = 2$ and $k \geq 5$, $l_{n^\star a^\star}(F_{1,k})$ does not exist.*

Theorem 5.17. *For any integer $k \geq 2$,* $l_{n^\star A}(F_{1,k}) = \begin{cases} 2 & \text{for } \ k = 2, 3. \\ 3 & \text{for } \ k = 4. \\ \lceil \frac{2k}{3} \rceil & \text{for } \ 5 \leq k \leq 7. \\ \lfloor \frac{2k}{3} \rfloor & \text{for } \ k \geq 8. \end{cases}$

Proof. For $k \leq 4$, the result is trivial. Let $k \geq 5$ and S be an $n^\star A$-set. Then, S is an na-set and $\overline{S}$ is an n-set but not an a-set. As S is an na-set, we consider the cases below for na-set as considered in the proof of Theorem 5.1.

Case(i): $v \in S$.

Here in this case, we have an na-set $S = \{v, v_1, v_2, v_3, \ldots, v_{\lfloor \frac{k}{2} \rfloor}\}$ such that $|S| = 1 + \lfloor \frac{k}{2} \rfloor$. Then $\overline{S}$ is not an n-set (since the edge $vv_1 \notin E(\bigcup_{x \in \overline{S}} \langle N[x] \rangle)$),a contradiction for S being an $n^\star$-set. Hence this case is not possible.

Case(ii):$v \notin S$.

Here in this case, we have an na-set S such that $|S| \geq \lfloor \frac{2k}{3} \rfloor$ (as taken in the proof of Theorem 5.1). Since $v \in \overline{S}$, by Remark 1.2, $\overline{S}$ is an n-set. Thus, we have an $n^\star$-set in this case.

Now in the $n^\star$-set as considered above, for the vertex $v \in \overline{S}$, $|N[v] \cap \overline{S}| = (k+1) - \lfloor \frac{2k}{3} \rfloor < \lfloor \frac{2k}{3} \rfloor = |N(v) \cap S|$ except for $k = 5$ and $k = 7$. Therefore, $|S| \geq \lceil \frac{2k}{3} \rceil$ for $k = 5$ and $k = 7$ (or else v and the other rim vertices in $\overline{S}$ are defendable in $\overline{S}$, a contradiction for S being an A-set). Since for $k = 6$, $\lfloor \frac{2k}{3} \rfloor = \lceil \frac{2k}{3} \rceil$, we have $|S| \geq \lceil \frac{2k}{3} \rceil$ for $5 \leq k \leq 7$ and $|S| \geq \lfloor \frac{2k}{3} \rfloor$ for $k \geq 8$.

We now prove the reverse inequality in the following cases.

For $k = 5$, consider the set $S = \{v_2, v_3, v_4, v_5\}$ then $\overline{S} = \{v, v_1\}$. Clearly, S is an $n^\star a$-set and $\overline{S}$ is not an a-set(since $v \in \overline{S}$, $|N[v] \cap \overline{S}| = 2 < 4 = |N(v) \cap S|$. Thus, S is an $n^\star A$-set. Therefore, $|S| \leq 4 = \lceil \frac{2k}{3} \rceil$. Hence the result.

For $k = 6$, consider the set $S = \{v_2, v_3, v_5, v_6\}$ then $\overline{S} = \{v, v_1, v_4\}$. Clearly, S is an $n^\star a$-set and $\overline{S}$ is not an a-set(since $v \in \overline{S}$, $|N[v] \cap \overline{S}| = 3 < 4 = |N(v) \cap S|$. Thus, S is an $n^\star A$-set. Therefore, $|S| \leq 4 = \lceil \frac{2k}{3} \rceil$. Hence the result.

For $k = 7$, consider the set $S = \{v_2, v_3, v_5, v_6, v_7\}$ then $\overline{S} = \{v, v_1, v_4\}$. Clearly, S is an $n^\star a$-set and $\overline{S}$ is not an a-set(since $v \in \overline{S}$, $|N[v] \cap \overline{S}| = 3 < 5 = |N(v) \cap S|$. Thus, S is an $n^\star A$-set. Therefore, $|S| \leq 5 = \lceil \frac{2k}{3} \rceil$. Hence the result.

For $k \geq 8$, consider the set $S = \{v_2, v_3, v_5, v_6, \ldots, v_{k-3}, v_{k-1}, v_k\}$ for $k \equiv 0(mod 3)$, $S = \{v_2, v_3, v_5, v_6, \ldots, v_{k-4}, v_{k-2}, v_{k-1}\}$ for $k \equiv 1(mod 3)$ and $S = \{v_2, v_3, v_5, v_6, \ldots, v_{k-3}, v_{k-2}, v_{k-1}\}$ for $k \equiv 2(mod 3)$. Then S is an na-set and since $v \in \overline{S}$, $\overline{S}$ is an n-set(by Remark 1.2). Further, $\overline{S}$ is not an a-set(since $v \in \overline{S}$, $|N[v] \cap \overline{S}| = (k+1) - \lfloor \frac{2k}{3} \rfloor < \lfloor \frac{2k}{3} \rfloor = |N(v) \cap S|$). Therefore, S is an $n^\star A$-set. Thus, $|S| \leq \lfloor \frac{2k}{3} \rfloor$.

Hence the theorem. □

Remark 5.18. *For $k = 1$, $l_{n^\star A}(F_{1,1})$ does not exist, which follows from Theorem 2.5 as $F_{1,1} \cong K_2$.*

Theorem 5.19. *For any integer $k \geq 4$, $l_{n^\star A^\star}(F_{1,k}) = 2$.*

Proof. Let S be an $n^\star A^\star$-set. Then, S is an $nA^\star$-set. Thus, by Theorem 5.5, $|S| \geq 2$. Now consider the set $S = \{v, v_2\}$. Then as $v \in S$, S is an n-set (by Remark 1.2). Also the remaining $k-1$ rim vertices in $\overline{S}$ forms an n-set(since $\bigcup_{x \in \overline{S}} < N[x] > \cong F_{1,k}$). Further, for the vertex $v \in S$,

$|N[v] \cap S| = 2 < k-1 = |N(v) \cap \overline{S}|$ and for the vertex $v_1 \in \overline{S}$, $|N[v_1] \cap \overline{S}| = 1 < 2 = |N(v_1) \cap S|$ which proves that the sets S and $\overline{S}$ are not a-sets respectively. Therefore, S is an $n^\star A^\star$-set. Hence $|S| \leq 2$. Hence $l_{n^\star A^\star}(F_{1,k}) = 2$. □

Remark 5.20. *For any integer* $1 \leq k \leq 3$, $l_{n^\star A^\star}(F_{1,k})$ *does not exist.*

Proof. For $k = 1, 2$, as $F_{1,1} \cong K_2$ and $F_{1,2} \cong K_3$, the result follows from Theorem 2.6.

For $k = 3$, let S be an $n^\star A^\star$-set. Then, S is an $nA^\star$-set. Now the result follows from Remark 5.7 (as there is no $nA^\star$-set in $F_{1,3}$). □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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