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A FITTED DEVIATING ARGUMENT AND INTERPOLATION SCHEME FOR THE SOLUTION OF SINGULARLY PERTURBED DIFFERENTIAL-DIFFERENCE EQUATION HAVING LAYERS AT BOTH ENDS

RAGHVENDRA PRATAP SINGH, Y. N. REDDY*

Department of Mathematics, National Institute of Technology Warangal-506004, India

Abstract: In this paper**,** problem of singularly perturbed differential-difference equation having boundary layers at both ends is solved and analyzed numerically by fitted method. To do this, original problem is transformed into an asymptotically equivalent singularly perturbed differential equation by Taylor's series expansion. By introducing deviating argument concept, SPDE is replaced by first order differential equation. Resulting equation having deviating argument is solved with proper choice of fitting factor and interpolation. To demonstrate the applicability of this numerical method, three test examples are solved and numerical results are compared with the available/exact results.

Keywords: differential-difference equation**;** dual layer; deviating argument; interpolation.

Mathematics Subject Classification: 65L11, 65Q10.

1. INTRODUCTION

A singularly perturbed delay differential equation is an ordinary differential equation in which the highest derivative is multiplied by a small parameter and containing delay term. In recent years, there has been a growing interest in the numerical treatment of such differential equations. This is due to the versatility of such type of differential equations in the mathematical modeling

*Corresponding author

E-mail address: ynreddy@nitw.ac.in

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of processes in various application fields, for e.g., the first exit time problem in the modeling of the activation of neuronal variability, in the study of bistable devices, and variational problems in control theory where they provide the best and in many cases the only realistic simulation of the observed. Stein [12] gave a differential-difference equation model incorporating stochastic effects due to neuron excitation. Lange and Miura [4-5] gave an asymptotic approach for a class of boundary-value problems for linear second-order differential-difference equations. Kadalbajoo and Sharma [9-10], presented a numerical approaches to solve singularly perturbed differential-difference equation, which contains negative shift in the either in the derivative term or the function but not in the derivative term. Asymptotic-numerical method for singularly perturbed differential difference equations of mixed-type is discussed by Salama and Al-Amery [1]. Erdogan [6], has presented an Exponentially fitted method for singularly perturbed delay differential equations. Venkat and Palli [15] presented a numerical approach for solving singularly perturbed convection delay problems via exponentially fitted spline method. Rao and Chakrravarthy [16-17] have described a finite difference method for singularly perturbed differential-difference equations arising from a model of neuronal variability. Reddy and Chakravarthy [20] presented an initial-value approach for solving singularly perturbed two-point boundary value problems. Reddy et al [19] described a numerical integration method for singularly perturbed delay differential equations. Reddy and Awoke [18] presented a method for solving singularly perturbed differential difference equations via fitted method, In this paper**,** problem of singularly perturbed differential-difference equation having boundary layers at both ends is solved and analyzed numerically by fitted method. To do this, original problem is transformed into an asymptotically equivalent singularly perturbed differential equation by Taylor's series expansion. By introducing deviating argument concept, SPDE is replaced by first order differential equation. Resulting equation having deviating argument is solved with proper choice of fitting factor and interpolation. To demonstrate the applicability of this numerical method, three test examples are solved and numerical results are compared with the available/exact results. For detailed theory is available in books [2,7,11,13,14].

2. DESCRIPTION OF THE FITTED METHOD

Consider a class of differential-difference equation with small shifts of mixed type

$$\varepsilon y''(x) + a(x)y(x-\delta) + c(x)y(x) + b(x)y(x+\eta) = f(x), \quad 0 \le x \le 1 \tag{1}$$

under the boundary conditions

$$y(x) = \alpha(x), \quad -\delta \le x \le 0 \tag{2}$$

$$y(x) = \beta(x), 1 \le x \le 1+\eta \tag{3}$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter, $0 < \delta = O(\varepsilon)$is the small delay parameter,$0 < \eta = O(\varepsilon)$ is the small advanced parameter, $a(x)$,$b(x)$, $c(x)$, $f(x)$, $\alpha(x)$and $\beta(x)$are sufficiently differentiable in $(0,1)$.If$a(x) + b(x) + c(x) \le 0$on the interval [0, 1], then the solution of (1) exhibits boundary layers at both ends of the interval [0, 1], whereas it exhibits oscillatory behaviour $a(x) + b(x) + c(x) > 0$.

Using Taylor series expansion, in the neighbourhood of x.

$$y(x-\delta) \approx y(x) - \delta y'(x) + \frac{\delta^2}{2} y''(x) \tag{4}$$

$$y(x+\eta) \approx y(x) + \eta y'(x) + \frac{\delta^2}{2} y''(x) \tag{5}$$

From Equations (4), (5) and (1), we obtain singularly perturbed differential equation

$$\varepsilon' y''(x) + A(x)y'(x) + B(x)y(x) = f(x) \tag{6}$$

with the boundary conditions

$$y(0) = \alpha(0) = \varphi_0 \tag{7}$$

$$y(1) = \beta(1) = \gamma_1 \tag{8}$$

where

$$A(x) = b(x)\eta - a(x)\delta,$$

$$B(x) = a(x) + b(x) + c(x),$$

$$\varepsilon' = \varepsilon + a(x)\frac{\delta^2}{2} + b(x)\frac{\eta^2}{2}$$

,φ_0 and γ_1 are constants. Since $0 < \delta \ll 1$ and $0 < \eta \ll 1$, the transformation from Eqn. (1) to Eqn. (6) is admitted. For more details on the validity of this transformation one can refer El'sgolt's and Norkin [8].

Since problem exhibits two boundary layers across the interval, we divide the interval [0, 1] into two sub intervals $\left[0,\frac{1}{2}\right]$ and $\left[\frac{1}{2},1\right]$. Clearly in the interval $\left[0,\frac{1}{2}\right]$ the boundary layer will be at the left end i.e. at $x = 0$, and in the interval $\left[\frac{1}{2},1\right]$ the boundary layer will be at right end i.e. at $x = 1$.

2.1 Problem with left end boundary layer in $\left[\mathbf{0},\frac{\mathbf{1}}{\mathbf{2}}\right]$

From Taylor's series expansion about the deviating argument $\sqrt{\varepsilon'}$ in the neighbourhood of the point x, we have

$$y\left(x-\sqrt{\varepsilon'}\right) \approx y(x) - \sqrt{\varepsilon'}y'(x) + \frac{\varepsilon'}{2}y''(x) \tag{9}$$

From Equations (9) and (6), we get

$$y'(x) = p(x)y\left(x-\sqrt{\varepsilon'}\right) + q(x)y(x) + r(x) \tag{10}$$

where

$$p(x) = \frac{-2}{2\sqrt{\varepsilon'}+A(x)} \tag{11}$$

$$q(x) = \frac{2-B(x)}{2\sqrt{\varepsilon'}+A(x)} \tag{12}$$

$$r(x) = \frac{f(x)}{2\sqrt{\varepsilon'}+A(x)} \tag{13}$$

The transition from equation (6) to (10) is valid, because of the condition that $\sqrt{\varepsilon'}$ is small. For more details on the validity of the transition, one can refer El'sgolt's and Norkin [8].

Now, we divide the interval [0, 1] into n equal parts with constant mesh length h.

Let $0 = x_0,\ x_1, \dots, x_n = 1$ be the mesh points, then we have $x_i = ih$, $i = 0, 1, 2, \dots, n$.

We choose N such that $x_N = \frac{1}{2}$.

Equation (10) can be written as

$$y'(x) - qy(x) = py\left(x-\sqrt{\varepsilon'}\right) + r(x) \tag{14}$$

By taking an integrating factor e^{-qx} for equation (14) and producing (as in McCartin [3])

$$\frac{d}{dx}\left[e^{-qx}y(x)\right] = e^{-qx}\left[py\left(x-\sqrt{\varepsilon'}\right)+r(x)\right] \tag{15}$$

On integrating equation (15) from x_i to x_{i+1}, we get

$$e^{-qx_{i+1}}y_{i+1} - e^{-qx_i}y_i = \int_{x_i}^{x_{i+1}} e^{-qx}\, py\left(x-\sqrt{\varepsilon'}\right)dx + \int_{x_i}^{x_{i+1}} e^{-qx}\, r(x)dx \tag{16}$$

Using the Hermite interpolation on $[x_i x_{i+1}]$ for $y\left(x-\sqrt{\varepsilon'}\right)$ and $r(x)$ into the above equation, we get

$$\begin{aligned} y_{i+1} = e^{qh}y_i + p\int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\Big\{& h_i\left(x-\sqrt{\varepsilon'}\right) * y\left(x_i-\sqrt{\varepsilon'}\right) + h_{i+1}\left(x-\sqrt{\varepsilon'}\right) * y\left(x_{i+1}-\sqrt{\varepsilon'}\right) \\ &+ \overline{h_i}\left(x-\sqrt{\varepsilon'}\right) * y'\left(x_i-\sqrt{\varepsilon'}\right) + \overline{h_{i+1}}\left(x-\sqrt{\varepsilon'}\right) * y'\left(x_{i+1}-\sqrt{\varepsilon'}\right)\Big\}dx \\ &+ \int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\left\{h_i(x) * r(x_i) + h_{i+1}(x) * r(x_{i+1}) + \overline{h_i}(x) * r'(x_i) + \overline{h_{i+1}}(x) \right. \\ &\left. * r'(x_{i+1})\right\}dx \end{aligned} \tag{17}$$

where h_i, h_{i+1}, $\overline{h_i}$ and $\overline{h_{i+1}}$ are given by Hermite interpolation

$$h_i = [(-2)x^3 + (3x_i + 3x_{i+1})x^2 + (-6x_i x_{i+1})x + 3x_i x_{i+1}^2 - x_{i+1}^3]/(-h^3)$$

$$h_{i+1} = [(-2)x^3 + (3x_i + 3x_{i+1})x^2 + (-6x_i x_{i+1})x + 3x_{i+1} x_i^2 - x_i^3]/(h^3)$$

$$\overline{h_i} = [x^3 + (-x_i - 2x_{i+1})x^2 + (2x_i x_{i+1} + x_{i+1}^2)x - x_i x_{i+1}^2]/(h^2)$$

$$\overline{h_{i+1}} = [x^3 + (-2x_i - x_{i+1})x^2 + (2x_i x_{i+1} + x_i^2)x - x_{i+1} x_i^2]/(h^2)$$

To solve equation (17), we first solve integrals

$$\begin{aligned} \int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\, h_i dx = \frac{1}{-h^3}\Bigg[&(3x_i + 3x_{i+1})\left\{x_{i+1}^2\left(-\frac{1}{q}\right) - (2x_{i+1})\left(\frac{1}{q^2}\right) + 2\left(-\frac{1}{q^3}\right) - \right. \\ &\left. x_i^2\left(\frac{e^{qh}}{-q}\right) + 2x_i\left(\frac{e^{qh}}{q^2}\right) - 2\left(\frac{e^{qh}}{-q^3}\right)\right\} - 6x_i x_{i+1}\left\{x_{i+1}\left(-\frac{1}{q}\right) - \left(\frac{1}{q^2}\right) - \right. \\ &\left. x_i\left(\frac{e^{qh}}{-q}\right) + \left(\frac{e^{qh}}{q^2}\right)\right\} - 2\left\{x_{i+1}^3\left(-\frac{1}{q}\right) - 3x_{i+1}^2\left(\frac{1}{q^2}\right) + 6x_{i+1}\left(-\frac{1}{q^3}\right) - \right. \\ &\left. 6\left(\frac{1}{q^4}\right) - x_i^3\left(\frac{e^{qh}}{-q}\right) + 3x_i^2\left(\frac{e^{qh}}{q^2}\right) - 6x_i\left(\frac{e^{qh}}{-q^3}\right) + 6\left(\frac{e^{qh}}{q^4}\right)\right\} + \\ &(3x_i x_{i+1}^2 - x_{i+1}^3)\left\{-\frac{1}{q} + \frac{e^{qh}}{q}\right\}\Bigg] = X(i) \end{aligned} \tag{18}$$

$$\int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\, h_{i+1} dx$$

$$= \frac{1}{h^3}\Bigg[(3x_i + 3x_{i+1})\left\{x_{i+1}^2\left(-\frac{1}{q}\right) - (2x_{i+1})\left(\frac{1}{q^2}\right) + 2\left(-\frac{1}{q^3}\right) - x_i^2\left(\frac{e^{qh}}{-q}\right)\right.$$

$$\left. + 2x_i\left(\frac{e^{qh}}{q^2}\right) - 2\left(\frac{e^{qh}}{-q^3}\right)\right\} - 6x_i x_{i+1}\left\{x_{i+1}\left(-\frac{1}{q}\right) - \left(\frac{1}{q^2}\right) - x_i\left(\frac{e^{qh}}{-q}\right) + \left(\frac{e^{qh}}{q^2}\right)\right\}$$

$$- 2\left\{x_{i+1}^3\left(-\frac{1}{q}\right) - 3x_{i+1}^2\left(\frac{1}{q^2}\right) + 6x_{i+1}\left(-\frac{1}{q^3}\right) - 6\left(\frac{1}{q^4}\right) - x_i^3\left(\frac{e^{qh}}{-q}\right)\right.$$

$$\left. + 3x_i^2\left(\frac{e^{qh}}{q^2}\right) - 6x_i\left(\frac{e^{qh}}{-q^3}\right) + 6\left(\frac{e^{qh}}{q^4}\right)\right\} + (3x_{i+1}x_i^2 - x_i^3)\left\{-\frac{1}{q} + \frac{e^{qh}}{q}\right\}\Bigg]$$

$$= Y(i) \tag{19}$$

$$\int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\overline{h_i}\, dx = \frac{1}{h^2}\Bigg[(-x_i - 2x_{i+1})\left\{x_{i+1}^2\left(-\frac{1}{q}\right) - (2x_{i+1})\left(\frac{1}{q^2}\right) + 2\left(-\frac{1}{q^3}\right) - \right.$$

$$\left. x_i^2\left(\frac{e^{qh}}{-q}\right) + 2x_i\left(\frac{e^{qh}}{q^2}\right) - 2\left(\frac{e^{qh}}{-q^3}\right)\right\} + (2x_i x_{i+1} + x_{i+1}^2)\left\{x_{i+1}\left(-\frac{1}{q}\right) - \right.$$

$$\left. \left(\frac{1}{q^2}\right) - x_i\left(\frac{e^{qh}}{-q}\right) + \left(\frac{e^{qh}}{q^2}\right)\right\} + \left\{x_{i+1}^3\left(-\frac{1}{q}\right) - 3x_{i+1}^2\left(\frac{1}{q^2}\right) + \right.$$

$$6x_{i+1}\left(-\frac{1}{q^3}\right) - 6\left(\frac{1}{q^4}\right) - x_i^3\left(\frac{e^{qh}}{-q}\right) + 3x_i^2\left(\frac{e^{qh}}{q^2}\right) - 6x_i\left(\frac{e^{qh}}{-q^3}\right) +$$

$$\left. 6\left(\frac{e^{qh}}{q^4}\right)\right\} - x_i x_{i+1}^2\left\{-\frac{1}{q} + \frac{e^{qh}}{q}\right\}\Bigg] = Z(i) \tag{20}$$

$$\int_{x_i}^{x_{i+1}} e^{q(x_{i+1}-x)}\overline{h_{i+1}}\, dx = \frac{1}{h^2}\Bigg[(-x_{i+1} - 2x_i)\left\{x_{i+1}^2\left(-\frac{1}{q}\right) - (2x_{i+1})\left(\frac{1}{q^2}\right) + 2\left(-\frac{1}{q^3}\right) - \right.$$

$$\left. x_i^2\left(\frac{e^{qh}}{-q}\right) + 2x_i\left(\frac{e^{qh}}{q^2}\right) - 2\left(\frac{e^{qh}}{-q^3}\right)\right\} + (2x_i x_{i+1} + x_i^2)\left\{x_{i+1}\left(-\frac{1}{q}\right) - \right.$$

$$\left. \left(\frac{1}{q^2}\right) - x_i\left(\frac{e^{qh}}{-q}\right) + \left(\frac{e^{qh}}{q^2}\right)\right\} + \left\{x_{i+1}^3\left(-\frac{1}{q}\right) - 3x_{i+1}^2\left(\frac{1}{q^2}\right) + \right.$$

$$6x_{i+1}\left(-\frac{1}{q^3}\right) - 6\left(\frac{1}{q^4}\right) - x_i^3\left(\frac{e^{qh}}{-q}\right) + 3x_i^2\left(\frac{e^{qh}}{q^2}\right) - 6x_i\left(\frac{e^{qh}}{-q^3}\right) +$$

$$\left. 6\left(\frac{e^{qh}}{q^4}\right)\right\} - x_{i+1}x_i^2\left\{-\frac{1}{q} + \frac{e^{qh}}{q}\right\}\Bigg] = W(i) \tag{21}$$

After Substituting equations (18), (19), (20) and (21) in equation (17), we obtain

$$y_{i+1} = e^{qh}y_i + \left\{py\left(x_i - \sqrt{\varepsilon'}\right) + r(x_i)\right\}X(i) + \left\{py\left(x_{i+1} - \sqrt{\varepsilon'}\right) + r(x_{i+1})\right\}Y(i) +$$

$$\left\{py'\left(x_i - \sqrt{\varepsilon'}\right) + r'(x_i)\right\}Z(i) + \left\{py'\left(x_{i+1} - \sqrt{\varepsilon'}\right) + r'(x_{i+1})\right\}W(i) \tag{22}$$

From finite difference approximation, we have

$$y\left(x_{i+1}-\sqrt{\varepsilon'}\right) \approx \left(1-\frac{\sqrt{\varepsilon'}}{h}\right)y_{i+1}+\frac{\sqrt{\varepsilon'}}{h}y_i$$

$$y\left(x_{i}-\sqrt{\varepsilon'}\right) \approx \left(1-\frac{\sqrt{\varepsilon'}}{h}\right)y_{i}+\frac{\sqrt{\varepsilon'}}{h}y_{i-1}$$

$$y'\left(x_i-\sqrt{\varepsilon'}\right) \approx y_i'-\sqrt{\varepsilon'}y_i'' \approx (y_i-y_{i-1})/h$$

$$y'\left(x_{i+1}-\sqrt{\varepsilon'}\right) \approx y_{i+1}'-\sqrt{\varepsilon'}y_{i+1}'' \approx (y_{i+1}-y_i)/h$$

Therefore equation (22) becomes

$$y_{i-1}-F_iy_i+G_iy_{i+1}=H_i, \quad i=1,2,\dots,N-1 \tag{23}$$

where,

$$E_i=-\frac{p\sqrt{\varepsilon'}}{h}X(i)+\frac{p}{h}Z(i)$$

$$F_i=e^{qh}+p\left(1-\frac{\sqrt{\varepsilon'}}{h}\right)X(i)+\frac{p\sqrt{\varepsilon'}}{h}Y(i)+\frac{p}{h}Z(i)-\frac{p}{h}W(i)$$

$$G_i=1-P\left(1-\frac{\sqrt{\varepsilon'}}{h}\right)Y(i)-\frac{p}{h}W(i)$$

$$H_i=r_iX(i)+r_{i+1}Y(i)+\ r'_i\,Z(i)+r'_{i+1}\,W(i)$$

2.2 Problem with right end boundary layer in $\left[\frac{1}{2},1\right]$

From Taylor's series expansion about the deviating argument $\sqrt{\varepsilon'}$ in the neighbourhood of the point x, we have

$$y\left(x+\sqrt{\varepsilon'}\right) \approx y(x)+\sqrt{\varepsilon'}y'(x)+\frac{\varepsilon'}{2}y''(x) \tag{24}$$

From Equations (24) and (6), we get

$$y'(x)=p(x)y\left(x+\sqrt{\varepsilon'}\right)+q(x)y(x)+r(x) \tag{25}$$

where

$$p(x)=\frac{-2}{-2\sqrt{\varepsilon'}+A(x)} \tag{26}$$

$$q(x)=\frac{2-B(x)}{-2\sqrt{\varepsilon'}+A(x)} \tag{27}$$

$$r(x) = \frac{f(x)}{-2\sqrt{\varepsilon'} + A(x)} \tag{28}$$

Equation (25) can be written as

$$y'(x) - qy(x) = py\left(x + \sqrt{\varepsilon'}\right) + r(x) \tag{29}$$

By taking an integrating factor e^{-qx} for equation (29) and producing (as in McCartin [3])

$$\frac{d}{dx}\left[e^{-qx} y(x)\right] = e^{-qx}\left[py\left(x + \sqrt{\varepsilon'}\right) + r(x)\right] \tag{30}$$

On integrating equation (30) from x_{i-1} to x_i, we get

$$e^{-qx_i} y_i - e^{-qx_{i-1}} y_{i-1} = \int_{x_{i-1}}^{x_i} e^{-qx}\, py\left(x + \sqrt{\varepsilon'}\right) dx + \int_{x_{i-1}}^{x_i} e^{-qx}\, r(x) dx \tag{31}$$

Using the Hermite interpolation on $[x_{i-1} x_i]$ for $y\left(x + \sqrt{\varepsilon'}\right)$ and $r(x)$ into the above equation, we get

$$y_i = e^{qh} y_{i-1} + p \int_{x_{i-1}}^{x_i} e^{q(x_i - x)} \left\{h_{i-1}\left(x + \sqrt{\varepsilon'}\right) * y\left(x_{i-1} + \sqrt{\varepsilon'}\right) + h_i\left(x + \sqrt{\varepsilon'}\right) * y\left(x_i + \sqrt{\varepsilon'}\right) + \overline{h_{i-1}}\left(x + \sqrt{\varepsilon'}\right) * y'\left(x_{i-1} + \sqrt{\varepsilon'}\right) + \overline{h_i}\left(x + \sqrt{\varepsilon'}\right) * y'\left(x_i + \sqrt{\varepsilon'}\right)\right\} dx + \int_{x_{i-1}}^{x_i} e^{q(x_i - x)} \left\{h_{i-1}(x) * r(x_{i-1}) + h_i(x) * r(x_i) + \overline{h_{i-1}}(x) * r'(x_{i-1}) + \overline{h_i}(x) * r'(x_i)\right\} dx \tag{32}$$

where h_{i-1}, h_i, $\overline{h_{i-1}}$ and $\overline{h_i}$ are given by Hermite interpolation as in case of $\left[0, \frac{1}{2}\right]$. In the similar way, we get

$$\int_{x_{i-1}}^{x_i} e^{q(x_i - x)}\, h_{i-1} dx = X(i) \tag{33}$$

$$\int_{x_{i-1}}^{x_i} e^{q(x_i - x)}\, h_i dx = Y(i) \tag{34}$$

$$\int_{x_{i-1}}^{x_i} e^{q(x_i - x)}\, \overline{h_{i-1}} dx = Z(i) \tag{35}$$

$$\int_{x_{i-1}}^{x_i} e^{q(x_i - x)}\, \overline{h_i} dx = W(i) \tag{36}$$

After Substituting equations (33), (34), (35) and (36) in equation (32) and using finite difference approximation, we obtain

$$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i, \quad i = N+1, N+2, \ldots, n-1 \tag{37}$$

where,

$$E_i = -e^{qh} - p\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right)X(i) + \frac{p}{h}Z(i)$$

$$F_i = -1 + p\left(1 - \frac{\sqrt{\varepsilon'}}{h}\right)Y(i) + \frac{p\sqrt{\varepsilon'}}{h}X(i) + \frac{p}{h}Z(i) - \frac{p}{h}W(i)$$

$$G_i = -\frac{p\sqrt{\varepsilon'}}{h}Y(i) - \frac{p}{h}W(i)$$

$$H_i = r_{i-1}X(i) + r_iY(i) + r'_{i-1}Z(i) + r'_i W(i)$$

We have a system of $n-2$ equations from both left and right end boundary layer problem with $n+1$ unknowns. From the given boundary conditions, Eqn. (7) and Eqn. (8), we get two equations i.e.

$$y(0) = \alpha(0) = \varphi_0$$

$$y(1) = \beta(1) = \gamma_1$$

We need one more equation to solve for the unknowns $(y_0, y_1, \ldots, y_n)$. For this, we consider the Eqn. (6) at $\varepsilon = 0$ and the point $x = x_N$, we get

$$A(x_N)y'(x_N) + B(x_N)y(x_N) = f(x_N)$$

Using second order central finite difference formula, we get

$$\frac{A_N}{2h}y_{N-1} - B_N y_N + \left(-\frac{A_N}{2h}\right)y_{N+1} = -f_N \tag{38}$$

With this equation (38), we now have $n+1$ equations to solve for the unknowns $(y_0, y_1, \ldots, y_n)$. Using invariant imbedding algorithm also knowns as Thomas algorithm, we get the solution.

3. NUMERICAL EXPERIMENTS

In this section to demonstrate the applicability of the method , we tested it on three standard model examples and solutions obtained from this scheme are compared with the available/exact solutions.

The exact solution of the differential -difference equation

$$\varepsilon y''(x) + a(x)y(x-\delta) + c(x)y(x) + b(x)y(x+\eta) = f(x), \quad 0 < x < 1$$

with the boundary conditions $y(x) = \alpha(x), \quad -\delta \le x \le 0$and $\quad y(x) = \beta(x), 1 \le x \le 1 + \eta$ with constant coefficients (i.e. $a(x) = a, b(x) = b, c(x) = c, f(x) = f, \alpha(x) = \alpha, \beta(x) = \beta$ are constants)is given by

$$y(x) = \frac{[\{(1-a-b-c)\exp(m_2)-1\}\exp(m_1 x)-\{(1-a-b-c)\exp(m_1)-1\}\exp(m_2 x)]}{[(a+b+c)(\exp(m_1)-\exp(m_2))]} + \frac{1}{(a+b+c)} \qquad (39)$$

where

$$m_1 = \frac{\left[(a\delta - b\eta) + \sqrt{(b\eta - a\delta)^2 - 4\varepsilon(a+b+c)}\right]}{2\varepsilon}$$

$$m_2 = \frac{\left[(a\delta - b\eta) - \sqrt{(b\eta - a\delta)^2 - 4\varepsilon(a+b+c)}\right]}{2\varepsilon}$$

Example 1.

Consider the differential-difference equation having dual boundary layer

$$\varepsilon y''(x) - 2y(x-\delta) - y(x) - 2y(x+\eta) = 1, \quad 0 < x < 1$$

with the boundary conditions$y(0) = 1$ and $y(1) = 0$.

The exact solution is given by Eqn. (39). Results are shown in Table-1 & 2 and the layer behaviour in fig.1 & 2 for different values of parameters.

Example 2.

Consider the differential-difference equation having dual boundary layer

$$\varepsilon y''(x) + 0.25y(x-\delta) - y(x) + 0.25y(x+\eta) = 1, \quad 0 < x < 1$$

with the boundary conditions $y(0) = 1$ and $y(1) = 0$.

The exact solution is given by Eqn. (39). Results are shown in Table-3 & 4 and the layer behaviour in fig.3 & 4 for different values of parameters.

Example 3.

Consider the differential-difference equation having dual boundary layer

$$\varepsilon y''(x) - y(x-\delta) - y(x) - 3y(x+\eta) = 1, \quad 0 < x < 1$$

with the boundary conditions$y(0) = 1$ and $y(1) = 0$.

The exact solution is given by Eqn. (39). Results are shown in Table-5 & 6 and the layer behaviour in fig.5 & 6 for different values of Parameters.

3. DISCUSSION AND CONCLUSIONS

A fitted numerical scheme is presented for solving singularly perturbed differential-difference equations having layers at both ends. In this scheme deviating argument and Hermite interpolation concepts are used. This scheme is implemented on three standard examples for those numerical solutions are found to be in agreement with available or exact solution. Numerical, exact results and layer behaviour are presented in their respective figures and tables for different values of the parameters. This scheme is very simple and easy to implement on the class of singularly perturbed differential-difference equations having layers at both ends.

Fig-1 Example **1**: $h = 0.01, \varepsilon = 0.0001, \delta = 0.007$ and $\eta = 0.003$

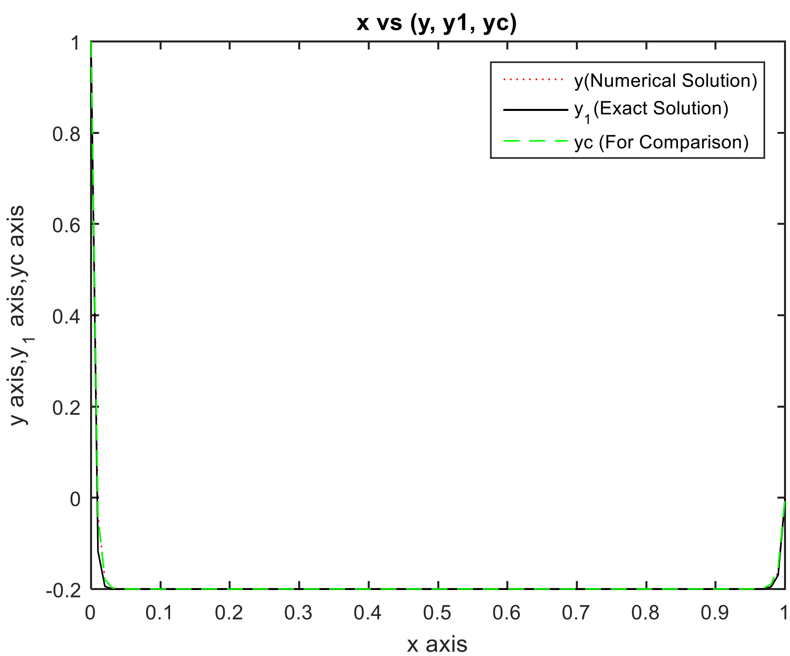

Table-1.

Example **1**: $h = 0.01, \varepsilon = 0.0001, \delta = 0.007$ and $\eta = 0.003$

x	Numerical Solution	Exact Solution	Solution by[9]
0.00	1	1	1
0.02	-0.17794562	-0.19426292	-0.17901486
0.04	-0.19959467	-0.19997257	-0.19963302
0.06	-0.19999255	-0.19999986	-0.19999358
0.08	-0.19999986	-0.19999999	-0.19999988
0.1	-0.19999999	-0.19999999	-0.19999999
0.2	-0.20000000	-0.20000000	-0.20000000
0.3	-0.20000000	-0.20000000	-0.20000000
0.4	-0.20000000	-0.20000000	-0.20000000
0.5	-0.20000000	-0.20000000	-0.20000000
0.6	-0.20000000	-0.20000000	-0.20000000
0.7	-0.20000000	-0.20000000	-0.20000000
0.8	-0.19999999	-0.20000000	-0.19999999
0.9	-0.19999998	-0.19999999	-0.19999988
1.0	0	0	0

Fig2 Example **1**: $h = 0.01, \varepsilon = 0.0001, \delta = 0.005$ and $\eta = 0.007$

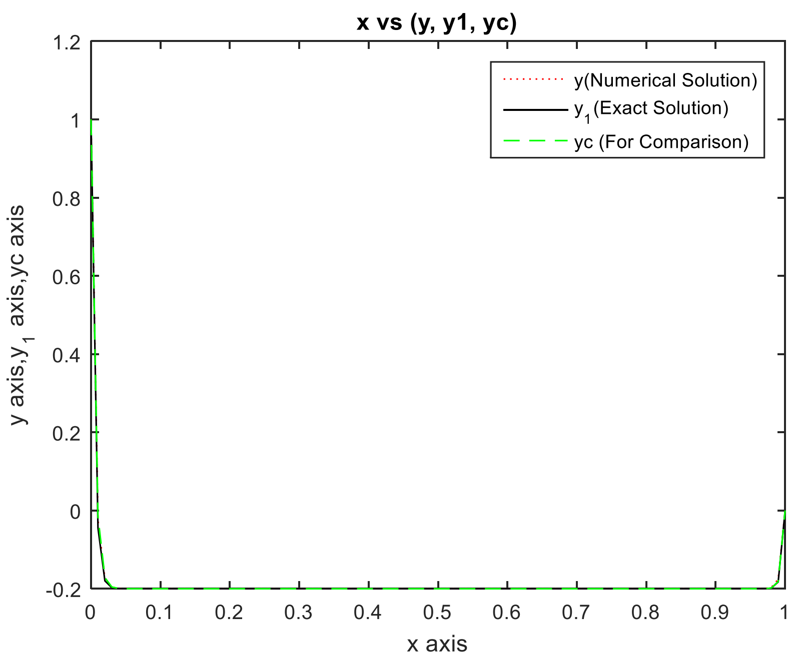

Table-2.

Example **1**: $h = 0.01, \varepsilon = 0.0001, \delta = 0.005$ and $\eta = 0.007$

x	Numerical Solution	Exact Solution	Solution by [9]
0.00	1	1	1
0.02	-0.17438754	-0.17991269	-0.17166570
0.04	-0.19945333	-0.19966375	-0.19933097
0.06	-0.19998833	-0.19999437	-0.19998420
0.08	-0.19999975	-0.19999990	-0.19999962
0.1	-0.19999999	-0.19999999	-0.19999999
0.2	-0.20000000	-0.20000000	-0.20000000
0.3	-0.20000000	-0.20000000	-0.20000000
0.4	-0.20000000	-0.20000000	-0.20000000
0.5	-0.20000000	-0.20000000	-0.20000000
0.6	-0.20000000	-0.20000000	-0.20000000
0.7	-0.20000000	-0.20000000	-0.20000000
0.8	-0.20000000	-0.20000000	-0.20000000
0.9	-0.19999999	-0.19999999	-0.19999999
1.0	0	0	0

Fig-3 **Example 2**: $h = 0.01, \varepsilon = 0.0001, \delta = 0.007$ and $\eta = 0.003$

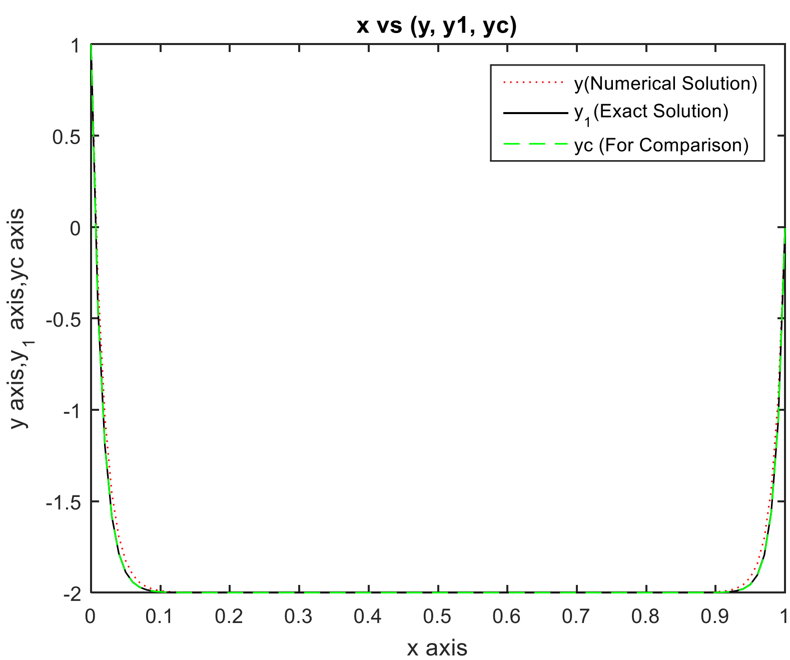

Table-3.

Example 2: $h = 0.01, \varepsilon = 0.0001, \delta = 0.007$ and $\eta = 0.003$

x	Numerical Solution	Exact Solution	Solution by [9]
0.00	1	1	1
0.02	-1.04099515	-1.19678463	-1.19863931
0.04	-1.69343656	-1.78494835	-1.78594035
0.06	-1.90200139	-1.94242240	-1.94282033
0.08	-1.96867295	-1.98458426	-1.98472615
0.1	-1.98998573	-1.99587261	-1.99592004
0.2	-1.99996657	-1.99999432	-1.99999445
0.3	-1.99999988	-1.99999999	-1.99999999
0.4	-1.99999999	-1.99999999	-1.99999999
0.5	-1.99999999	-1.99999999	-1.99999999
0.6	-1.99999999	-1.99999999	-1.99999999
0.7	-1.99999998	-1.99999999	-1.99999999
0.8	-1.99999376	-1.99999948	-1.99999955
0.9	-1.99646871	-1.99898774	-1.99905160
1.0	0	0	0

Fig-4 **Example 2**: $h = 0.01, \varepsilon = 0.0001, \delta = 0.005$ and $\eta = 0.007$

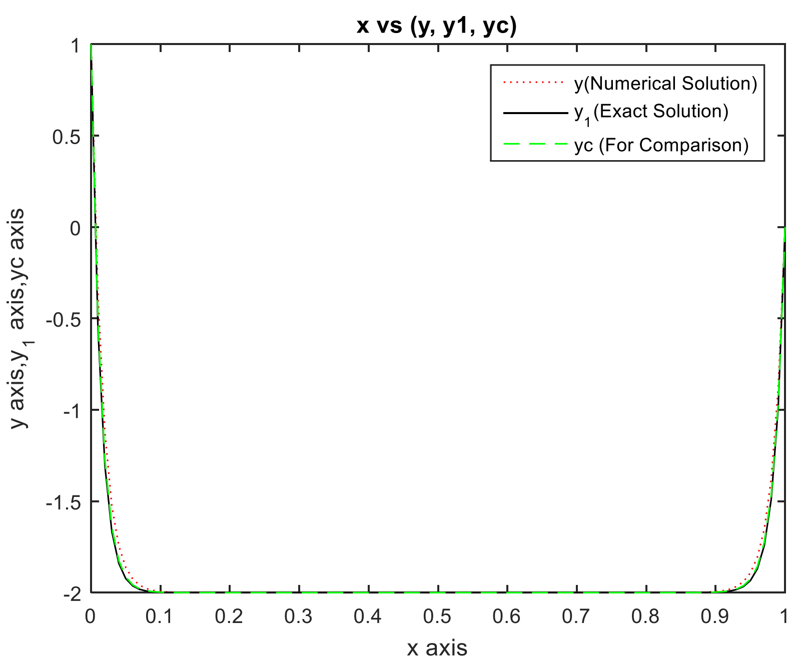

Table-4.

Example 2: $h = 0.01, \varepsilon = 0.0001, \delta = 0.005$ and $\eta = 0.007$

x	Numerical Solution	Exact Solution	Solution by [9]
0.00	1	1	1
0.02	-1.12308773	-1.30683338	-1.27464990
0.04	-1.74367496	-1.83984001	-1.82462241
0.06	-1.92507514	-1.96299414	-1.95759661
0.08	-1.97809915	-1.99144959	-1.98974756
0.1	-1.99359829	-1.99802438	-1.99752113
0.2	-1.99998633	-1.99999869	-1.99999795
0.3	-1.99999997	-1.99999999	-1.99999999
0.4	-1.99999999	-1.99999999	-1.99999999
0.5	-1.99999999	-1.99999999	-1.99999999
0.6	-1.99999999	-1.99999999	-1.99999999
0.7	-1.99999994	-1.99999999	-1.99999999
0.8	-1.99998286	-1.99999764	-1.99999637
0.9	-1.99414563	-1.99782850	-1.99730812
1.0	0	0	0

Fig-5 **Example 3:** $h = 0.01, \varepsilon = 0.0001, \delta = 0.007$ and $\eta = 0.003$

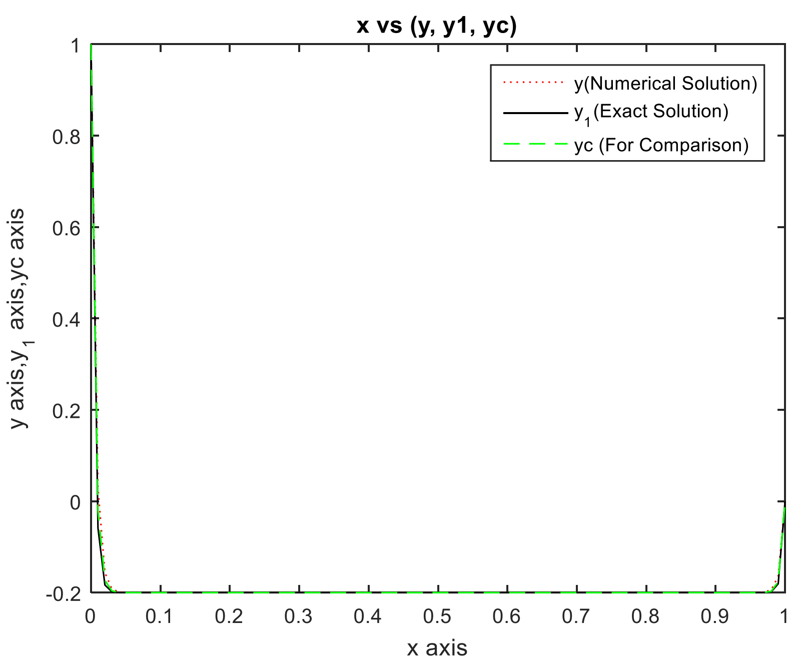

Table-5.

Example 3: $h = 0.01, \varepsilon = 0.0001, \delta = 0.007$ and $\eta = 0.003$

x	Numerical Solution	Exact Solution	Solution by [9]
0.00	1	1	1
0.02	-0.15638578	-0.18333232	-0.17310964
0.04	-0.19841483	-0.19976849	-0.19939742
0.06	-0.19994238	-0.19999678	-0.19998649
0.08	-0.19999790	-0.19999995	-0.19999969
0.1	-0.19999992	-0.19999999	-0.19999999
0.2	-0.19999999	-0.20000000	-0.20000000
0.3	-0.20000000	-0.20000000	-0.20000000
0.4	-0.20000000	-0.20000000	-0.20000000
0.5	-0.20000000	-0.20000000	-0.20000000
0.6	-0.20000000	-0.20000000	-0.20000000
0.7	-0.20000000	-0.20000000	-0.20000000
0.8	-0.20000000	-0.20000000	-0.20000000
0.9	-0.19999999	-0.19999999	-0.19999999
1.0	0	0	0

Fig 6 **Example 3**: $h = 0.01, \varepsilon = 0.0001, \delta = 0.007$ and $\eta = 0.005$

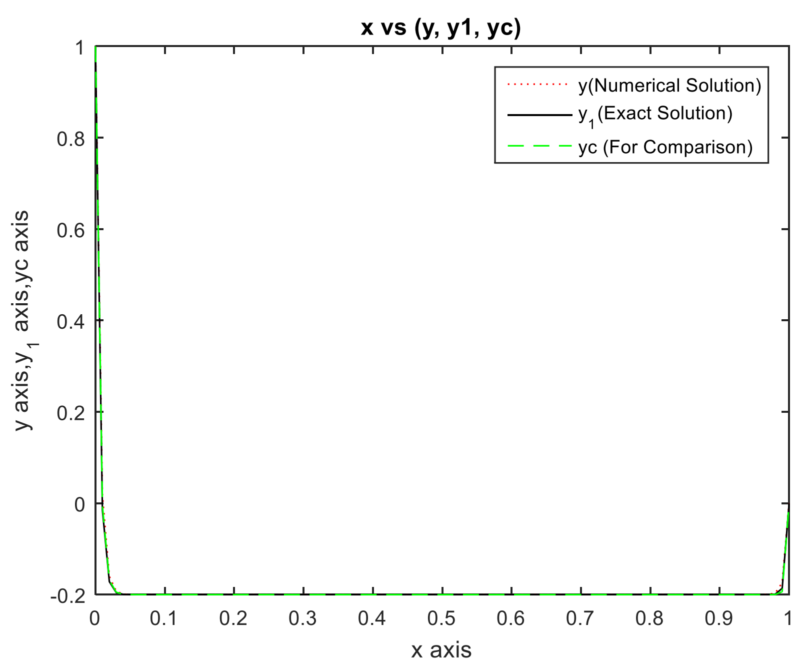

Table-6.

Example 3: $h = 0.01, \varepsilon = 0.0001, \delta = 0.007\ and\ \eta = 0.005$

x	Numerical Solution	Exact Solution	Solution by [9]
0.00	1	1	1
0.02	-0.16030035	-0.16030035	-0.16845338
0.04	-0.19868661	-0.19868661	-0.19917067
0.06	-0.19995654	-0.19995654	-0.19997819
0.08	-0.19999856	-0.19999856	-0.19999942
0.1	-0.19999995	-0.19999995	-0.19999998
0.2	-0.19999999	-0.19999999	-0.20000000
0.3	-0.20000000	-0.20000000	-0.20000000
0.4	-0.20000000	-0.20000000	-0.20000000
0.5	-0.20000000	-0.20000000	-0.20000000
0.6	-0.20000000	-0.20000000	-0.20000000
0.7	-0.20000000	-0.20000000	-0.20000000
0.8	-0.20000000	-0.20000000	-0.20000000
0.9	-0.19999999	-0.19999999	-0.20000000
1.0	0	0	0

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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