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A GENERAL CLASS OF NEW CONTINUOUS MIXTURE DISTRIBUTION AND APPLICATION

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Abstract. A generalization of a distribution increases the flexibility particularly in studying of a phenomenon and its properties. Many generalizations of continuous univariate distributions are available in literature. In this study, an investigation is conducted on a distribution and its generalization. Several available generalizations of the distribution are reviewed and recent trends in the construction of generalized classes with a generalized mixing parameter are discussed. To check the suitability and comparability, real data set have been used.

Keywords: Bonferroni and Lorenz; MRLF; Renyi entropy; MGF; K-S.

2010 AMS Subject Classification: 62E10.

1. Introduction

Modelling and analysis helps in explaining the lifetime events in various aspects of applied sciences. These phenomenon can be studied using various popular statistical distributions such as exponential, beta, gamma, pareto, weibull, lognormal etc. but each of these lifetime distributions has own advantages and disadvantages over one another due to the number of parameters involved, its shape, nature of hazard function and mean residual life function. Lindley

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[12, 13] introduced one parameter continuous distribution as an alternative and competent distribution to the existing distributions, with the help of convex combination of the particular case of gamma distribution. Lindley distribution is useful for analysing lifetime data, especially in stress-strength reliability modeling.

Ghitany et al. [31] studied the properties of the one parameter Lindley distribution under a careful mathematical treatment. Shanker et al. [40] made a comparison study of the goodness of fit of exponential and Lindley distributions on modeling of lifetime data. A generalized Lindley distribution, which includes as special cases the Exponential and Gamma distributions, was proposed by Zakerzadeh and Dolati [18], and Nadarajah and Haghighi [48] introduced the exponentiated Lindley distribution. Ghitany and Al-Mutari [32] considered a size-biased Poisson-Lindley distribution and Sankaran [30] proposed the Poisson-Lindley distribution to model count data. Some properties of Poisson-Lindley distribution and its derived distributions were considered in Borah and Begum [25] while Borah and Deka [27], Singh et al. [3] considered the Poisson-Lindley and some of its mixture distributions. The zero-truncated Poisson-Lindley distribution and the generalized Poisson-Lindley distribution were considered in Ghitany et al. [35] and Mahmoudi and Zakerzadeh [16], respectively.

A study on the inflated Poisson-Lindley distribution was presented in Borah and Deka [26], Singh et al. [4] and Zamani and Ismail [19] considering the Negative Binomial Lindley distribution. The weighted and extended Lindley distribution was considered by Ghitany et al. [33], Elbatal and Elgarhy [21] and Bakouch et al. [20], respectively. The one parameter Lindley distribution in the competing risks scenario was considered in Mazucheli and Achcar [22]. The exponential Poisson Lindley distribution was presented in Barreto-Souza and Bakouch [52]. Ghitany et al. [34] introduced the power Lindley distribution. Ali [45] investigated various properties of the weighted Lindley distribution which main focus was the Bayesian analysis. Gomez-Deniz et al. [15] studied Log Lindley distribution, another extended form of generalized Lindley distribution with applications to lifetime data is proposed by Torabi et al. [17], Ashour and Eltehiwy [47] considered exponentiated power Lindley distribution. A new four-parameter class of generalized Lindley distribution called the beta-generalized Lindley distribution is proposed by Oluyede and Yang [7].

Nedjar and Zeghdoudi [50] suggested Gamma Lindley distribution and a generalized weighted Lindley distribution is discussed by Ramos and Louzada [36]. Kumar and Jose [8] introduced a two-tailed version of the Lindley distribution through the name double Lindley distribution (DLD). Nadarajah et al. [49], Roozegar and Nadarajah [44], Abouammoh et al. [1] proposed a generalized Lindley distribution and Tomy [24] discussed several extension of Lindley distribution published at a glance. Ibrahim et al. [28] proposed a Topp Leone mixture of Generated Lindley (TLGLi), which is constructed based on the Topp Leone Generated (TLG) family introduced by Rezaei et al. [51]. Sah [6] discussed two-parameter quasi-Lindley mixture of generalised Poisson distribution, Shanker and Amanuel [38] studied quasi Lindley distribution and its properties. Shanker and Mishra [41, 42] introduced two parameter Lindley and its mixture with poisson distribution. Also three parameter Lindley distribution is considered by Shanker et al. [43]. Alkarni [46] and Pararai et al. [29] discussed about power Lindley distribution and its extension. Mahmoudi and Zakerzadeh [16] proposed an extended version of the compound Poisson-generalized Lindley distribution. Deniz and Ojeda [14] proposed discretized version of Lindley distribution.

For exploration of flexibility of existing statistical distribution, generalization is also an interesting tool as well as important research area. Shanker and Shukla [39] has proposed a two parameter life time distribution named Rama-Kamlesh distribution which is a generalised form of Lindley distribution containing various one parameter life time distributions (such as Lindley, Akash, Prakaamy distribution etc.). Again, Shukla et al. [23] has proposed another generalisation named Shukla distribution which contains many other one parameter life time distribution like exponential, Shanker, Ishita, Pranav, Rani and Ram-Awadh distributions as particular cases of Shukla distribution. The aim of this study is to propose another generalized distribution which provides a more flexible distribution for modeling lifetime data. We propose a generalised distribution mixing gamma (2, θ) with gamma ($\alpha+2$, θ) with different mixing parameters.

Lindley distribution is a mixture of gamma (1, θ) i.e. exponential distribution (θ) and gamma (2, θ) as given

$$f(x;\theta) = pf_1(x) + (1-p)f_2(x)$$

where

$$p = \frac{\theta}{\theta+1}; \quad f_1(x) = \theta e^{-\theta x}, \quad f_2(x) = \frac{\theta^2}{\Gamma(2)} x e^{-\theta x}$$

Thus the probability density function (pdf) of one parameter Lindley distribution is given by

$$f_1(x,\theta) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \quad x > 0, \quad \theta > 0$$

Correspondingly to above pdf, the cumulative distribution function (cdf) is given as

$$F_1(x,\theta) = 1 - \left(1 + \frac{\theta x}{\theta+1}\right) e^{-\theta x} \tag{1}$$

For a random variable with the one parameter Lindley distribution, the probability distribution function is unimodal for $0 < \theta < 1$ and decreasing when $\theta > 1$. The hazard rate function is an increasing function in t and θ and given by

$$h_1(t) = \frac{\theta^2(t+1)}{\theta t + \theta + 1}; \quad \theta > 0, t > 0$$

The generalised Lindley with two parameters named Rama-Kamlesh distribution (RKD) is mixture of exponential (θ) and gamma (α, θ)

$$f(x;\alpha,\theta) = \frac{\theta^{\alpha+1}}{\theta^{\alpha} + \Gamma(\alpha+1)}(1+x^{\alpha})e^{-\theta x}; \quad x > 0, \theta > 0, \alpha \geq 0$$

and Shukla distribution (SD) is a mixture of exponential (θ) and gamma ($\alpha+1$, θ).

$$f(x;\alpha,\theta) = \frac{\theta^{\alpha+1}}{\theta^{\alpha+1} + \Gamma(\alpha+1)}(\theta+x^{\alpha})e^{-\theta x}; \quad x > 0, \theta > 0, \alpha \geq 0$$

In this study a new more flexible generalized distribution is proposed that is the mixture of gamma (2, θ) i.e. length biased exponential (θ) and gamma ($\alpha+2$, θ). The pdf of two parameter proposed distribution named SSD distribution with parameters (α, θ) is written as

$$f(x;\alpha,\theta) = \frac{\theta^{\alpha+2}}{\theta^{\alpha} + \Gamma(\alpha+2)} e^{-\theta x}(x + x^{\alpha+1}); \quad x > 0, \theta > 0, \alpha \geq -1 \tag{2}$$

we have

$$f(x;\alpha,\theta) = p.f_1(x;\alpha,\theta) + (1-p)f_2(x;\alpha,\theta)$$

where

$$p=\frac{\theta^{\alpha}}{\theta^{\alpha}+\Gamma(\alpha+2)}, \quad f_1(x;\alpha,\theta)=\frac{\theta^2}{\Gamma 2}e^{-\theta x}x^{2-1}=\theta^2 x.e^{-\theta x}$$

$$f_2(x;\alpha,\theta)=\frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)}e^{-\theta x}x^{\alpha+2-1}=\frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)}x^{\alpha+1}e^{-\theta x}$$

Length biased exponential distribution is a particular case of the proposed distribution when $\alpha=0$ and in cae of $\alpha=-1$ the above distribution converted into Lindley distribution.

The plot of probability density function of SSD distribution is given in figure (1)

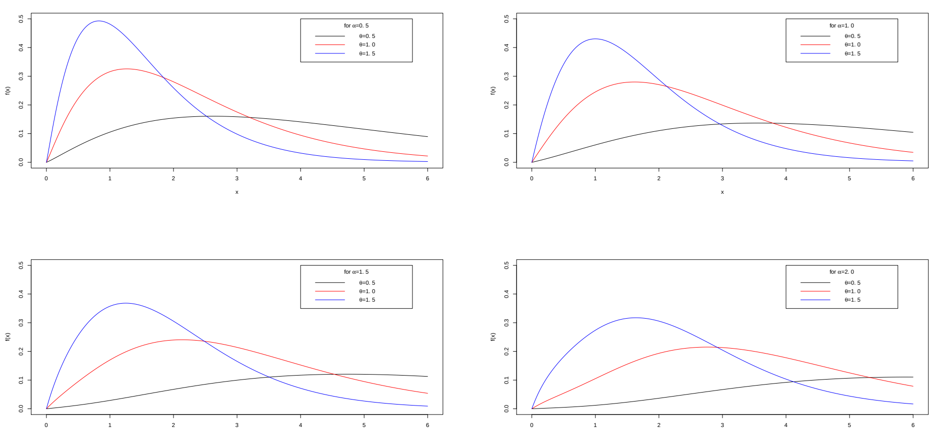


FIGURE 1. pdf plot of SSD distribution for different values of α and θ

In the above pdf putting n=1,2,3,... we can get series of various distributions. The cumulative distribution function for the proposed distribution can be written as

$$F(x)=\int_0^x f(t)dt=\frac{\theta^{\alpha}\left[1-(1+\theta x)e^{-\theta x}\right]+\gamma(\alpha+2,\theta x)}{\Gamma(\alpha+2)+\theta^{\alpha}} \tag{3}$$

Where, γ is Lower incomplete gamma function, defined as $\gamma(s,x)=\int_0^x t^{s-1}e^{-t}dt$

The plot of cumulative distribution function of SSD distribution is given in figure (2).

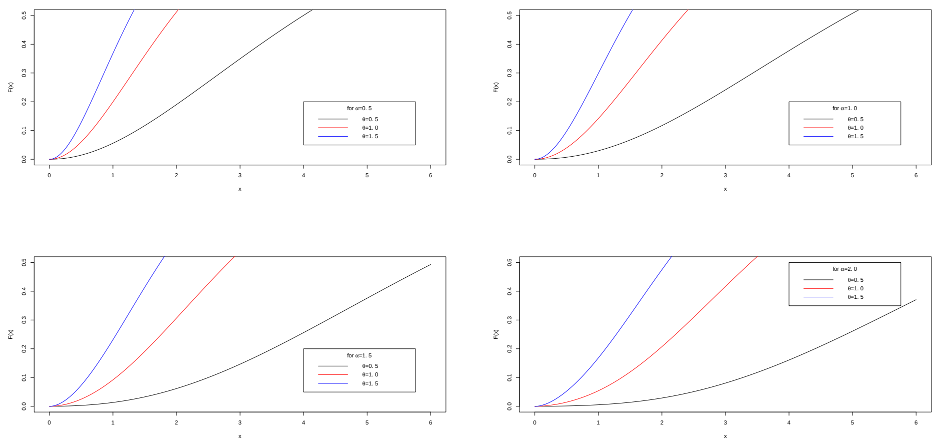


FIGURE 2. cdf plot of SSD distribution for different values of α and θ

2. Statistical Properties and Related Measures of SSD

2.1. Moments, Moment generating function and Characteristics function. The moment and moment generating functions play important roles for analysing any distributions functions. The moment generating function characterizes the distribution function. Although, we could not obtain the moments in explicit forms, they can be obtained as infinite summation of beta functions.

Moments are important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g. location, dispersion, skewness, and kurtosis).

The r-th moment about origin of the generalised Lindley distribution is given by:

$$\mu_r' = E(X^r) = \int_0^\infty x^r f(x) dx$$

so

$$\mu_r' = \frac{\theta^{\alpha-r}}{\Gamma(\alpha+2)+\theta^\alpha}\left[\Gamma(r+2)+\frac{\Gamma(\alpha+r+2)}{\theta^\alpha}\right] \tag{4}$$

Now putting $r = 1,2,3\ldots$ we get the first four moments aspects

$$\mu_1' = E(X) = \frac{2\theta^\alpha+\Gamma(\alpha+3)}{\theta(\theta^\alpha+\Gamma(\alpha+2))}$$

$$\mu_2' = E(X^2) = \frac{6\theta^\alpha + \Gamma(\alpha+4)}{\theta^2(\theta^\alpha + \Gamma(\alpha+2))}$$

$$\mu_3' = E(X^3) = \frac{24\theta^\alpha + \Gamma(\alpha+5)}{\theta^3(\theta^\alpha + \Gamma(\alpha+2))}$$

$$\mu_4' = E(X^4) = \frac{120\theta^\alpha + \Gamma(\alpha+6)}{\theta^4(\theta^\alpha + \Gamma(\alpha+2))}$$

So,

$$V(x) = \mu_2' - (\mu_1')^2 = \frac{1}{\theta^2\left[\theta^\alpha + \Gamma(\alpha+2)\right]}\left[6\theta^\alpha + \Gamma(\alpha+4) - \frac{\left[2\theta^\alpha + \Gamma(\alpha+3)\right]^2}{\theta^\alpha + \Gamma(\alpha+2)}\right] \tag{5}$$

Now, we derive the moment generating function and characteristic function. The moment generating function is given by the relation

$$M_x(t) = \int_0^\infty e^{tx} f(x)dx$$

then, we have,

$$M_x(t) = \int_0^\infty e^{tx}\frac{\theta^{\alpha+2}}{\theta^\alpha + \Gamma(\alpha+2)}e^{-\theta x}(x + x^{\alpha+1})dx$$

After solving the above equation, we get,

$$M_x(t) = \frac{\theta^{\alpha+2}}{(\theta-t)^{\alpha+2}}\frac{(\theta-t)^{\alpha+2} + \Gamma(\alpha+2)}{\theta^\alpha + \Gamma(\alpha+2)} \tag{6}$$

Now we replace *it* for *t* in equation number (6) we get the corresponding Characteristics function for the SSD distribution and it can be written as in equation number (7).

$$\phi_x(t) = \frac{\theta^{\alpha+2}}{(\theta-it)^{\alpha+2}}\frac{(\theta-it)^{\alpha+2} + \Gamma(\alpha+2)}{\theta^\alpha + \Gamma(\alpha+2)} \tag{7}$$

2.2. Hazard function and Mean Residual Life function. Let $f\ (x)$ and $F\ (x)$ be the p.d.f. and c.d.f. of a continuous random variable. The hazard rate function (alsoknown as the failure rate function), survival function and the mean residual life function of X are respectively given as

$$h(x) = \frac{f(x)}{R(x)} = \frac{f(x)}{1-F(x)} = \frac{\left(\frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)+\theta^\alpha}\right)e^{-\theta x}\left(x + x^{\alpha+1}\right)}{1 - \left(\frac{\theta^\alpha\left\{1-(1+\theta x)e^{-\theta x}\right\} + \gamma(\alpha+2,\theta x)}{\Gamma(\alpha+2)+\theta^\alpha}\right)}$$

$$h(x) = \frac{\theta^{\alpha+2}e^{-\theta x}\left(x + x^{\alpha+1}\right)}{\Gamma(\alpha+2) + \theta^\alpha(1+\theta x)e^{-\theta x} - \gamma(\alpha+2,\theta x)} \tag{8}$$

Now, differentiating (8) with respect to x, we get,

$$h'(x) = \frac{\theta^{\alpha+2} e^{-\theta x} \left[\begin{array}{c} (1+(\alpha+1)x^{\alpha} - \theta x(1+x^{\alpha}))\left(\Gamma(\alpha+2)+\theta^{\alpha}(1+\theta x)e^{-\theta x} - \gamma(\alpha+2,\theta x)\right) \\ +(x+x^{\alpha+1})(\theta^{\alpha+2} x e^{-\theta x} + \gamma'(\alpha+2,\theta x)) \end{array} \right]}{\left[\Gamma(\alpha+2)+\theta^{\alpha}(1+\theta x)e^{-\theta x} - \gamma(\alpha+2,\theta x)\right]^2} \tag{9}$$

Now, as $x \to 0$, we get,

$$h'(0) = \frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)+\theta^{\alpha}} > 0 \forall \quad \theta > 0, \alpha > 0 \tag{10}$$

Hence, from (10) we can say that hazard is increasing and the plot of hazard function of SSD distribution is given in figure (3).

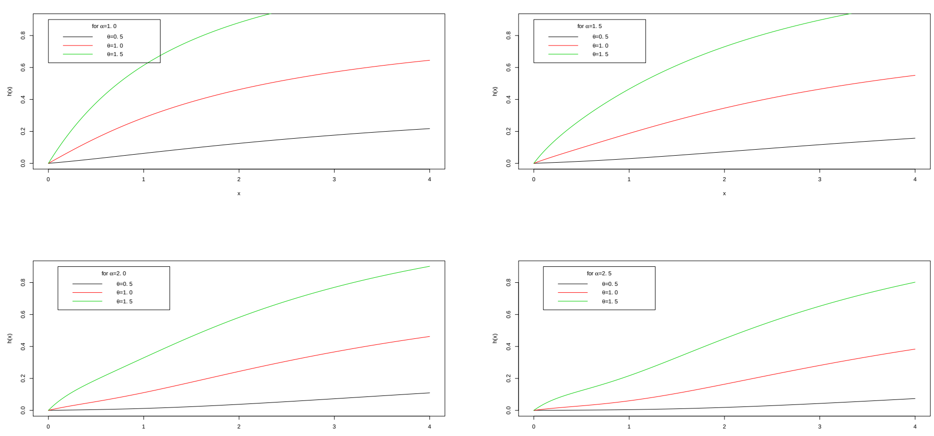


FIGURE 3. Hazard plot for SSD distribution for different values of α and θ

The survival function of SSD distribution is given as

$$S(x) = \int_x^{\infty} f(t)dt = 1 - F(x) = \frac{\Gamma(\alpha+2)+\theta^{\alpha}(1+\theta x)e^{-\theta x} - \gamma(\alpha+2,\theta x)}{\Gamma(\alpha+2)+\theta^{\alpha}} \tag{11}$$

The plot of survival function of SSD distribution is given in figure (4)

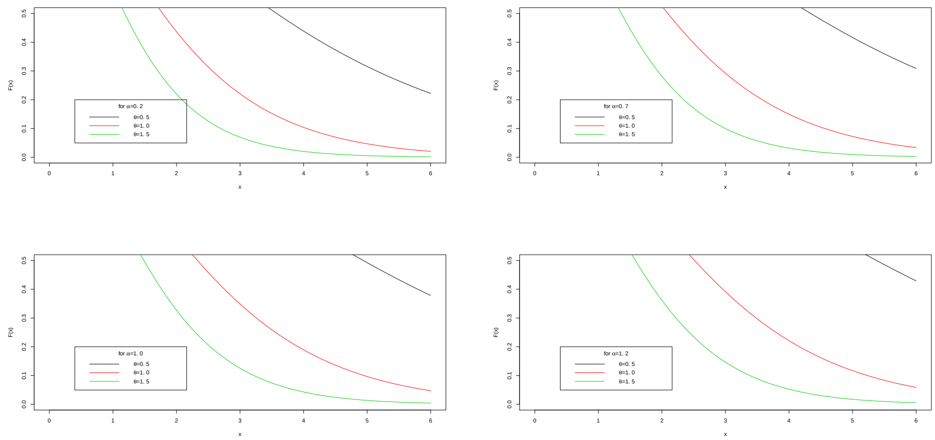


FIGURE 4. Survival plot for SSD distribution for different values of α and θ

Mean residual life function for the SSD distribution can be given as

$$m(x) = E(X - x|X > x) = \frac{1}{S(x)} \int_x^\infty t f(t) dt - x$$

$$= \frac{1}{S(x)} \int_x^\infty t \left(\frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)+\theta^\alpha} \right) e^{-\theta t} \left(t + t^{\alpha+1}\right) - x$$

$$= \frac{1}{S(x)} \left(\frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)+\theta^\alpha} \right) \int_x^\infty t e^{-\theta t} \left(t + t^{\alpha+1}\right) - x$$

$$= \frac{1}{S(x)} \left(\frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)+\theta^\alpha} \right) \left[\frac{\left(\theta^2 x^2 + 2\theta x + 2\right) e^{-\theta x}}{\theta^3} + \frac{\Gamma(\alpha+3, \theta x)}{\theta^{\alpha+3}} \right] - x$$

After solving the above integral, we get,

$$m(x) = \frac{\left[\theta^\alpha \left(\theta^2 x^2 + 2\theta x + 2\right) e^{-\theta x} + \Gamma(\alpha+3, \theta x)\right]}{\theta \left[\Gamma(\alpha+2) + \theta^\alpha (1+\theta x) e^{-\theta x} - \gamma(\alpha+2, \theta x)\right]} - x \tag{12}$$

Where $\Gamma(*,*)$ is upper incomplete gamma function , $\Gamma(s,x) = \int_x^\infty t^{s-1} e^{-t} dt$ and $\lim_{x \to 0} \Gamma(s,x) = (s-1)!$. and $\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt$ is the lower incomplete gamma function.

2.3. Rényi entropy. Entropy is a measure of the uncertainty associated with a random variable. The Shannon entropy is a measure of the average information contentone is missing when one doesn't know the value of the random variable. A useful generalization of Shannon entropy is the Rényi entropy. It is also important in quantum information, where it can beused as a measure of entanglement; e.g. see Shannon [10]; Renyi [2].

The Rényi entropy of the proposed distribution is given as

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\int_0^\infty f^\gamma(x)dx$$

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\int_0^\infty \left[\frac{\theta^{\alpha+2}}{\theta^\alpha+\Gamma(\alpha+2)}e^{-\theta x}(x+x^{\alpha+1})\right]^\gamma dx$$

Hence, after simplification above integral we get,

$$T_R(\gamma) = \frac{1}{1-\gamma}\log\left[\frac{\theta^{\gamma(\alpha+2)}}{\{\theta^\alpha+\Gamma(\alpha+2)\}^\gamma}\sum_{k=0}^{\gamma}\binom{\gamma}{k}\frac{(\alpha k+\gamma)!}{(\theta\gamma)^{\alpha k+\gamma+1}}\right] \tag{13}$$

2.4. Bonferroni and Lorenz curves. The Bonferroni and Lorenz curves (Bonferroni, [9]) and Bonferroni and Gini indices have applications not only in economics to study income and poverty, but also in other fields like reliability, demography, insurance and medicine. The Bonferroni curve $B(p)$ and Lorenz curves $L(p)$ are defined as

$$B(p) = \frac{1}{p\mu}\int_0^q xf(x)dx = \frac{1}{p\mu}\left[\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right] = \frac{1}{p\mu}\left[\mu - \int_q^\infty xf(x)dx\right]$$

and

$$L(p) = \frac{1}{p}\int_0^q xf(x)dx = \frac{1}{p}\left[\int_0^\infty xf(x)dx - \int_q^\infty xf(x)dx\right] = \frac{1}{p\mu}\left[\mu - \int_q^\infty xf(x)dx\right]$$

Using the proposed pdf, we get,

$$\int_q^\infty xf(x)dx = \left(\frac{\theta^{\alpha+2}}{\Gamma(\alpha+2)+\theta^\alpha}\right)\left[\frac{\left(\theta^2q^2+2\theta q+2\right)e^{-\theta q}}{\theta^3} + \frac{\Gamma(\alpha+3,\theta q)}{\theta^{\alpha+3}}\right] \tag{14}$$

So, from the above equations we get,

$$B(p) = \frac{1}{p}\left[1 - \frac{\left\{\theta^\alpha\left(\theta^2q^2+2\theta q+2\right)e^{-\theta q}+\Gamma(\alpha+3,\theta q)\right\}}{2\theta^\alpha+\Gamma(\alpha+3)}\right] \tag{15}$$

and

$$L(p) = 1 - \frac{\left\{\theta^\alpha\left(\theta^2q^2+2\theta q+2\right)e^{-\theta q}+\Gamma(\alpha+3,\theta q)\right\}}{2\theta^\alpha+\Gamma(\alpha+3)} \tag{16}$$

2.5. Order Statistics. Let $x_1, x_2, \ldots x_n$ be a random sample of size n from the SSD distribution. Let $X_{(1)} < X_{(2)} < \cdots < X_{(n)}$ denote the corresponding order statistics. The p.d.f. and the c.d.f. of the k th order statistic, say $Y = X_{(k)}$ are given by

$$f_Y(y) = \frac{n!}{(k-1)!(n-k)!} F^{k-1}(y)\{1-F(y)\}^{n-k} f(y)$$

or

$$f_Y(y) = \frac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^l F^{k+l-1}(y) f(y) \tag{17}$$

and

$$F_Y(y) = \sum_{j=k}^{n} \binom{n}{j} F^j(y)\{1-F(y)\}^{n-j}$$

or

$$F_Y(y) = \sum_{i=k}^{n} \sum_{l=0}^{n-j} \binom{n}{j} \binom{n-i}{l} (-1)^l F^{i+l}(y) \tag{18}$$

Now, using equation number (2) and (3) in equation (17) and (18) we get the corresponding pdf and the cdf of $k-th$ order statistics of the SSD distribution are obtained as

$$f_Y(y) = \tfrac{n!}{(k-1)!(n-k)!} \sum_{l=0}^{n-k} \left(\begin{array}{c} n-k \\ l \end{array} \right) (-1)^l \left(\tfrac{\theta^\alpha}{\Gamma(\alpha+2)+\theta^\alpha} \right)^{k+l}$$
$$\left[\left\{ 1-(1+\theta x) e^{-\theta x} \right\} + \tfrac{\gamma(\alpha+2,\theta x)}{\theta^\alpha} \right]^{k+l-1} \left(x + x^{\alpha+1} \right) e^{-\theta x}$$

And

$$F_Y(y) = \sum_{i=k}^{n} \sum_{l=0}^{n-i} \binom{n}{i} \binom{n-i}{l} (-1)^l \left[\frac{\theta^\alpha}{\Gamma(\alpha+2)+\theta^\alpha} \right]^{i+l}$$
$$\left[\left\{ 1-(1+\theta x) e^{-\theta x} \right\} + \frac{\gamma(\alpha+2,\theta x)}{\theta^\alpha} \right]^{i+l} \tag{19}$$

3. ESTIMATION OF THE PARAMETER

The parameters of above discussed SSD distribution are estimated by method of maximum likelihood. The likelihood function for the proposed distribution can be written as

$$L(\theta) = \prod_{i=1}^{n} \left[\frac{\theta^{\alpha+2}}{\theta^\alpha + \Gamma(\alpha+2)} \right] e^{-\theta x_i} (x_i + x_i^{\alpha+1})$$

or

$$L(\theta)=\left[\frac{\theta^{n(\alpha+2)}}{\{\theta^{\alpha}+\Gamma(\alpha+2)\}^{n}}\right]e^{-\theta\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}(x_i+x_i^{\alpha+1})$$

Now, log-likelihood can be given as

$$\log L(\theta)=n(\alpha+2)\log\theta-n\log\{\theta^{\alpha}+\Gamma(\alpha+2)\}-\theta\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}\log(x_i+x_i^{\alpha+1})$$

Differentiating the above equation with respect to θ and α partially, we get,

$$\frac{\partial\log L}{\partial\theta}=\frac{n(\alpha+2)}{\theta}-\frac{n\alpha\theta^{\alpha-1}}{\theta^{\alpha}+\Gamma(\alpha+2)}-\sum_{i=1}^{n}x_i \tag{20}$$

and

$$\frac{\partial\log L}{\partial\alpha}=n\log\theta-\frac{n\left[\frac{\partial\Gamma(\alpha+2)}{\partial\alpha}+\theta^{\alpha}\log\theta\right]}{\theta^{\alpha}+\Gamma(\alpha+2)}+\frac{\partial}{\partial\alpha}\left[\sum_{i=1}^{n}\log(x_i+x_i^{\alpha+1})\right] \tag{21}$$

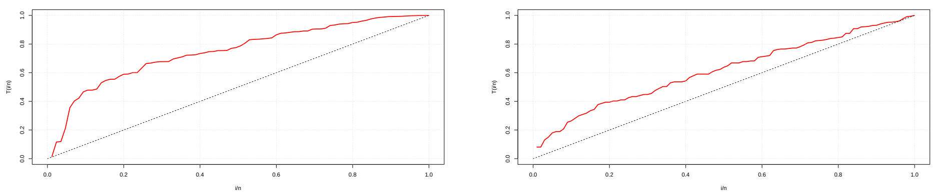
Which are non-linear equations and cannot be solved analytically so applying numerical method, we use Newton-Raphson algorithm to evaluate the equations (20) and (21).

4. TOTAL TIME ON TEST (TTT)

We know that distribution function $F(t)$ and and μ the mean time to failure (MTTF). The $F(t)$ is continuous and strictly increasing which indicates that $F^{-1}(t)$ exists. thus the Total time on test (TTT) of $F(t)$ is defined as

$$\phi(x)=\frac{1}{\mu}\int_{0}^{F^{-1}(x)}[1-F(t)]\,dt\quad\text{for}\quad 0\leq\mu\leq 1 \tag{22}$$

The value of $\phi(x)$ is interpreted as the area below $\frac{1-F(t)}{\mu}$ between 0 and $F^{-1}(\mu)$. The TTT plot, an empirical and scale independent plot based on failure data and corresponding to asymptotic curve, The scaled TTT-transformation see Barlow and Campo [37] is used to illustrate some test statistics for testing exponentiality.



(A) TTT plot for 1st data set of SSD distribution

(B) TTT plot for 2nd data set of SSD distribution

FIGURE 5. TTT plot for fitted data sets

5. Application on Real Data

The applications of SSD distribution have been discussed with the following two data sets. First data set is relating to failure times of mechanical components reported in the book "Weibull Models" by Murthy et al. [11] page number-297 and the second data set is waiting time (measured in min) bank customers before service is being rendered, this data set previously used by Ghitany et al. [35]. The graph of total test time (TTT) is shown in the figure (5).

For the above two data set, SSD has been fitted along with two-parameter distributions including Shukla distribution proposed by Shukla et al. [23], gamma distribution, RKD [39] and one parameter lifetime distributions including exponential, Lindley, length biased exponential distribution (LBED) used in Singh and Das [5]. The ML estimates, value of $-2\log L$, Akaike Information criteria (AIC), Corrected Akaike Information criteria (AICc), K-S statistics and p-value of the fitted distributions are presented in tables 2 and 3.The AIC, BIC, AICc and K-S Statistics are computed using the following formulae:

$$AIC = -2loglik + 2k, \qquad BIC = -2loglik + k\log n$$

$$AICc = AIC + \frac{2k^2+2k}{n-k-1}, \qquad D = \sup_x |F_n(x) - F_0(x)|$$

where k= the number of parameters, n= the sample size, and the $F_n(x)$=empirical distribution function and $F_0(x) =$ is the theoretical cumulative distribution function. The best distribution is the distribution corresponding to lower values of$-2\log L$, AIC, BIC, AICc and K-S statistics and associated higher p-value. Therefore SSD distribution is found best among the considered other distribution here in this study.

TABLE 1. MLE's, - 2ln L, AIC, K-S and p-values of the fitted distributions for the first dataset.

Distribution	Estimate		-2LL	AIC	BIC	AICc	K-S	p-value
	α	θ						
SSD	5.7104	2.7713	258.73	262.73	267.62	262.88	0.0808	0.6354
SD	5.1116	1.8652	285.35	289.35	294.23	289.49	0.1451	0.0305
RKD	6.6402	2.5467	270.66	274.66	279.55	274.81	0.0982	0.2557
Gamma	1.3769	3.5285	280.79	285.67	280.94	276.79	0.0919	0.3201
LBED		0.7805	289.52	291.52	293.97	291.57	0.1686	0.0079
Lindley		0.6297	307.93	309.93	312.38	309.98	0.2304	0.0000
Exponential		0.3902	329.98	331.98	334.42	332.02	0.2914	0.0000

TABLE 2. MLE's, - 2ln L, AIC, K-S and p-values of the fitted distributions for the second dataset.

Distribution	Estimate		-2LL	AIC	BIC	AICc	K-S	p-value
	α	θ						
SSD	0.0143	0.2032	634.60	638.60	643.81	638.72	0.0425	0.9937
SD	1.0848	0.2070	635.17	639.17	644.38	639.29	0.0450	0.9874
RKD	6.6402	2.5467	636.73	640.73	645.94	640.85	0.0504	0.9616
Gamma	0.2034	2.0088	634.60	638.60	643.81	638.72	0.0425	0.9936
LBED		0.2025	634.60	636.60	639.21	636.64	0.0427	0.9922
Lindley		0.1866	638.07	640.07	642.68	640.11	0.0577	0.7495
Exponential		0.1012	658.04	660.04	662.65	660.08	0.1630	0.0050

6. Conclusion

This paper studied the well established and widely used exponential, Lindley distribution and its generalizations. The proposed SSD distribution is discussed with various statistical properties and also with application on real data set. It performs better than various other distributions. Generalization proposed here with its parameters can be used to handle various real data sets with complex structure. It can be also used to develop various new probability models to explain real phenomena.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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