



Available online at <http://scik.org>

J. Math. Comput. Sci. 11 (2021), No. 1, 839-855

<https://doi.org/10.28919/jmcs/5242>

ISSN: 1927-5307

ON COMPLEX FINSLER SPACE WITH INFINITE SERIES OF (α, β) -METRIC

K.S. VENKATESHA, S.K. NARASIMHAMURTHY*

Department of Mathematics, Kuvempu University, Shankaraghatta, Shivamoga, Karnataka, India-577451

Copyright © 2021 the author(s). This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. The purpose of the present paper is to investigate the complex Finsler space with infinite series of (α, β) -metric $F = \frac{|\beta|^2}{|\beta| - \alpha}$. Next we determine the fundamental metric tensor, angular metric tensor and Chern-Finsler connection coefficients. Further, we discussed the special approach to Kähler-Randers change of infinite series (α, β) -metric using some examples.

Keywords: complex Finsler space; Randers change; infinite series; Chern connection; Finsler connection.

2010 AMS Subject Classification: 53B40, 53C56.

1. INTRODUCTION

The real Randers metric were first introduced by G. Randers in the context of general relativity and they were applied to the theory of the electron microscope by R. S. Ingarden [16]. The importance of real Randers spaces is also pointed out in [7] and the obtained results are remarkable. Recently, it was shown that the real Randers metrics are solutions to Zermeto's navigation problem [9] and the classification of real Randers metrics of constant flag curvature was finally completed ([7],[9],[24]).

*Corresponding author

E-mail address: nmurthysk@gmail.com

Received November 19, 2020

The correct notion of complex Finsler metrics was probably proposed firstly by Rizza [14], and then developed by Rund [21] by defining the connection and geodesics. The famous complex Finsler metrics are the Kobayashi and Caratheodary metrics ([11],[17]), which play important roles in the theory of the moduli space of Riemann surface [13]. In recent years complex Finsler geometry has attracted renewed interest as examples of such metrics appear in a natural way in the geometric theory of several complex variables.

The current paper is organized in two sections. The first section is to introduce the complex Finsler spaces with infinite series of (α, β) -metrics, i.e., complex metrics constructed from just two pieces of familiar data: a purely Hermitian metric and a differential $(1, 0)$ -form, both globally defined on an underlying complex manifold. We determine the fundamental metric tensor of a complex Finsler spaces with infinite series of (α, β) -metric, its inverse and determinant. Moreover, a complex Randers change of infinite series also produces another invertible d -tensor (Theorem 5). By deformation of some purely Hermitian metrics we obtain some example of complex Randers change of infinite series.

In the second section, we find the conditions such that a complex Randers change of infinite series is weakly Kähler or strongly Kähler. A special attention is devoted to a class of complex Randers change of infinite series with some additional assumption, and finally our results are applied to some examples.

2. PRELIMINARIES

Let M be a complex manifold of $\dim_{\mathbb{C}} M = n$. The complexified of the real tangent bundle $T_{\mathbb{C}}M$ splits in to the sum of holomorphic tangent bundle $T'M$ and its conjugate $T''M$. The bundle $T'M$ is in its turn a complex manifold, The local coordinates in a chart will be denoted by $u = (z^k, \eta^k)$ and these are changed by the rules: $z'^k = z^k(z)$, $\eta^k = \frac{\partial z'^k}{\partial z^j} \eta^j$. The complexified tangent bundle of $T'M$ is decomposed as $T_{\mathbb{C}}(T'M) = T'(T'M) \oplus T''(T'M)$. A natural local frame for $T'_u M$ is $\{\frac{\partial}{\partial z^k}, \frac{\partial}{\partial \eta^k}\}$, which have changes by the rules obtained with Jacobi matrix of above transformations. Note that the change rules obtained with Jacobi matrix of above transformations. Note that the change rule of $\frac{\partial}{\partial z^k}$ contains the second order partial derivatives.

Let $V(T'M) = \ker \pi_* \subset T'(T'M)$ be the vertical bundle, spanned locally by $\{\frac{\partial}{\partial \eta^k}\}$. A complex nonlinear connection, briefly (c.n.c), determines a supplementary complex subbundle to

$V(T'M)$ in $T'(T'M)$, i.e., $T'(T'M) = H(T'M) \oplus V(T'M)$. It determines an adapted frame $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N_k^j \frac{\partial}{\partial \eta^j}\}$, where N_k^j are the coefficients of $(c.n.c)$, [2].

A continuous function $F : T'M \rightarrow \mathbb{R}$ is called complex Finsler metric on M if it satisfies the conditions

- (i) $L := F^2$ is smooth on $T\tilde{M} := T'M \setminus \{0\}$;
- (ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;
- (iii) $F(z, \lambda \eta) = |\lambda| F(z, \eta)$; for $\lambda \in \mathbb{C}$;
- (iv) the Hermitian matrix $(g_{i\bar{j}}(z, \eta))$, with $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$, called the fundamental metric tensor, is positive definite.

Let us write $L = F^2$, the pair (M, F) is called a complex Finsler space. The (iv)-th assumption involves the strongly pseudoconvexity of the Finsler metric F on complex indicatrix, $I_{F,z} = \{\eta \in T'_z M \mid F(z, \eta) < 1\}$.

Further, in a complex Finsler space a Hermitian connection of $(1, 0)$ -type has a special meaning, named in [1] the Chern-Finsler Connection. In notations from [19] it is $D\Gamma N = (L_{jk}^i, 0, C_{jk}^i, 0)$, where

$$(1) \quad N_j^i = g^{\bar{m}i} \frac{\partial g_{l\bar{m}}}{\partial z^j} \eta^l, \quad L_{jk}^i = g^{\bar{m}i} \frac{\partial g_{j\bar{m}}}{\partial z^j} = \frac{\partial N_k^i}{\partial \eta^j}, \quad C_{jk}^i = g^{\bar{m}i} \frac{\partial g_{j\bar{m}}}{\partial \eta^k}$$

Now, let us recall that in [1]’s terminology, the complex Finsler space (M, F) is strongly Kähler if and only if $L_{jk}^i - L_{kj}^i = 0$, Kähler if and only if $(L_{jk}^i - L_{kj}^i) \eta^j \bar{\eta}^l = 0$. In the particular case of purley Hermitian metrics, that is $g_{i\bar{j}} = g_{i\bar{j}}(z)$, those three nuances Kähler coincide [22]. For the vertical $\mathcal{L} = \eta^k \dot{\partial}_k$ with $\dot{\partial}_k := \frac{\partial}{\partial \eta^k}$, called the Liouville complex field (or the vertical radial vector field in [1]), we consider its horizontal lift $\chi := \eta^k \delta_k, (\delta_k := \frac{\delta}{\delta z^k})$.

According to the holomorphic curvature of the complex Finsler space (M, F) in direction η is

$$(2) \quad K_F(z, \eta) = \frac{2}{L^2} G(R(\chi, \bar{\chi})\chi, \bar{\chi}),$$

and locally it has the following expression ([3])

$$(3) \quad K_F(z, \eta) = \frac{2}{L^2} R_{\bar{j}k} \bar{\eta}^j \eta^k, \text{ where } R_{\bar{j}k} = -g_{l\bar{j}} \delta_{\bar{h}}^l (N_k^l) \bar{\eta}^h.$$

And more informations can be found in ([1],[2],[19]).

3. COMPLEX FINSLER SPACE WITH INFINITE SERIES OF (α, β) -METRIC

In the r -th series (α, β) -metric $F(\alpha, \beta) = |\beta| \sum_{k=0}^r \left(\frac{\alpha}{|\beta|}\right)^k$, where we assume $\alpha < |\beta|$. If $r = \infty$, then this metric is expressed in the form $F(\alpha, \beta) = \frac{|\beta|^2}{|\beta| - \alpha}$ and is called an infinite series (α, β) -metric. Interestingly this metric is the difference between a Randers metric and a Matsumoto metric. Following the ideas from real case ([7],[9],[24]), we shall introduce a new class of complex Finsler metrics. We consider $z \in M$ and $\eta \in T'_z M$, $\eta = \eta^i \frac{\partial}{\partial z^i}$. On M let

- $a := a_{i\bar{j}}(z) dz^i \otimes d\bar{z}^j$ be a purely Hermitian positive metric and
- $b = b_i(z) dz^i$ be a differential $(1, 0)$ -form.

By these objects we define the function F on $T'M$

$$(4) \quad F(z, \eta) := F(\alpha(z, \eta), |\beta(z, \eta)|),$$

where

$$(5) \quad \begin{aligned} \alpha(z, \eta) &:= \sqrt{a_{i\bar{j}}(z) \eta^i \bar{\eta}^j}, \\ |\beta(z, \eta)| &:= \sqrt{\beta(z, \eta) \overline{\beta(z, \eta)}} \text{ with } \beta(z, \eta) = b_i(z) \eta^i. \end{aligned}$$

By analogy with the real case the function from (4) the complex infinite series of (α, β) -metric and the pair $(M, \frac{|\beta|^2}{|\beta| - \alpha})$ a complex Finsler space with infinite series of (α, β) -metric such a metric is only quoted as an example of complex Finsler metric in [12].

Our goal in the sequel is to find the circumstances in which the function (4) is a complex Finsler metric. Some remarks are immediate due to the presence of $|\beta|$ be the complex Finsler space with infinite series of (α, β) -metric $F(\alpha, \beta) := \frac{|\beta|^2}{|\beta| - \alpha}$ is positive and smooth on $T'M \setminus \{0\}$. The complex Finsler space with infinite series of (α, β) -metric is purely Hermitian if and only if β vanishes identically.

Obviously, the function $L := F^2 = \left(\frac{|\beta|^2}{|\beta| - \alpha}\right)^2$ depends on z and η by means of the real valued functions $\alpha := (z, \eta)$ and $|\beta| := |\beta(z, \eta)|$. Moreover α and β are homogeneous with respect to η , i.e., $\alpha(z, \lambda \eta) = |\lambda| \alpha(z, \eta)$ and $\beta(z, \lambda \eta) = \lambda \beta(z, \eta)$ for any $\lambda \in \mathbb{C}$, thus $L(z, \lambda \eta) = \lambda \bar{\lambda} L(z, \eta)$ for any $\lambda \in \mathbb{C}$, and so the homogeneity property implies

$$(6) \quad \frac{\partial \alpha}{\partial \eta^i} \eta^i = \frac{1}{2} \alpha \quad \text{and} \quad \frac{\partial |\beta|}{\partial \eta^i} \eta^i = \frac{1}{2} |\beta|.$$

$$\begin{aligned}
 L_\alpha &= \frac{2F^2}{|\beta| - \alpha}, \\
 L_{|\beta|} &= 2F|\beta| \left(\frac{|\beta| - 2\alpha}{(|\beta| - \alpha)^2} \right), \\
 L_{\alpha|\beta|} &= 2F|\beta| \left(\frac{|\beta| - 4\alpha}{(|\beta| - \alpha)^3} \right), \\
 L_{\alpha\alpha} &= \frac{6F^2}{(|\beta| - \alpha)^2}, \\
 L_{|\beta||\beta|} &= 2F \left(\frac{|\beta|^2 - 4\alpha|\beta| + 6\alpha^2}{(|\beta| - \alpha)^3} \right).
 \end{aligned}$$

$$\alpha L_\alpha + |\beta| L_{|\beta|} = 2L, \quad \alpha L_{\alpha\alpha} + |\beta| L_{\alpha|\beta|} = L_\alpha,$$

$$\alpha L_{\alpha|\beta|} + |\beta| L_{|\beta||\beta|} = L_{|\beta|}, \quad \alpha^2 L_{\alpha\alpha} + 2\alpha|\beta| L_{\alpha|\beta|} + |\beta|^2 L_{|\beta||\beta|} = 2L,$$

where

$$L_\alpha := \frac{\partial L}{\partial \alpha}, \quad L_{|\beta|} = \frac{\partial L}{\partial |\beta|}, \quad L_{\alpha|\beta|} = \frac{\partial^2 L}{\partial \alpha \partial |\beta|}, \quad L_{\alpha\alpha} = \frac{\partial^2 L}{\partial \alpha^2}, \quad L_{|\beta||\beta|} = \frac{\partial^2 L}{\partial |\beta|^2}, \text{ etc.}$$

The main issue that needs to be checked is the strongly psuedoconvexity of the complex Finsler function. First we shall determine the fundamental tensor of the complex Finsler space with infinite series $(M, \frac{|\beta|^2}{|\beta| - \alpha})$ i.e., $g_{i\bar{j}} = \frac{\partial^2 (\frac{|\beta|^2}{|\beta| - \alpha})}{\partial \eta^i \partial \bar{\eta}^j}$. For this, let us consider the settings

$$\begin{aligned}
 \frac{\partial \alpha}{\partial \eta^i} &= \frac{1}{2\alpha} l_i; \quad \frac{\partial |\beta|}{\partial \eta^i} = \frac{\bar{\beta}}{2|\beta|} b_i \\
 b^i &:= a^{\bar{j}i} b_{\bar{j}}; \quad ||b||^2 := a^{\bar{j}i} b_i b_{\bar{j}}; \\
 \eta_i &:= \frac{\partial L}{\partial \eta^i} = L_\alpha \frac{\partial \alpha}{\partial \eta^i} + L_{|\beta|} \frac{\partial |\beta|}{\partial \eta^i} \\
 &= \frac{F^2}{(|\beta| - \alpha)} \frac{l_i}{\alpha} + \frac{|\beta| - 2\alpha}{(|\beta| - \alpha)^2} F \bar{\beta} b_i,
 \end{aligned}
 \tag{7}$$

where $l_i := a_{i\bar{j}} \bar{\eta}^j$ and $(a^{\bar{j}i})$ is the Hermitian inverse of $(a_{i\bar{j}})$ matrix.

Theorem 1. *The fundamental metric tensor of the complex Finsler space with infinite series of (α, β) -metric $F = \frac{|\beta|^2}{|\beta| - \alpha}$ is given by*

$$g_{i\bar{j}} = \frac{F^2}{\alpha V} h_{i\bar{j}} + \frac{F|\beta|}{2V^2} b_i b_{\bar{j}} + \frac{S}{2F^2} \eta_i \eta_{\bar{j}},$$

where $h_{i\bar{j}} := a_{i\bar{j}} + \frac{K}{2\alpha^2 V} l_i l_{\bar{j}}$.

Proof. Indeed, from

$$\begin{aligned}
 (9) \quad g_{i\bar{j}} &= \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j} \\
 &= L_{\alpha\alpha} \frac{\partial \alpha}{\partial \eta^i} \frac{\partial \alpha}{\partial \bar{\eta}^j} + L_{\alpha|\beta|} \left(\frac{\partial \alpha}{\partial \eta^i} \frac{\partial |\beta|}{\partial \bar{\eta}^j} + \frac{\partial |\beta|}{\partial \eta^i} \frac{\partial \alpha}{\partial \bar{\eta}^j} \right) \\
 &\quad + L_{|\beta||\beta|} \left(\frac{\partial |\beta|}{\partial \eta^i} \frac{\partial |\beta|}{\partial \bar{\eta}^j} \right) + L_{\alpha} \frac{\partial^2 \alpha}{\partial \eta^i \partial \bar{\eta}^j} + L_{|\beta|} \frac{\partial^2 |\beta|}{\partial \eta^i \partial \bar{\eta}^j}.
 \end{aligned}$$

Using (7) and (8) we have

$$\begin{aligned}
 (10) \quad g_{i\bar{j}} &= \frac{F^2}{\alpha(|\beta| - \alpha)} a_{i\bar{j}} + \frac{F^2}{2\alpha^3} \left(\frac{4\alpha - |\beta|}{(|\beta| - \alpha)^2} \right) l_i l_{\bar{j}} + \frac{F}{2} \left(\frac{8\alpha^2 + 2|\beta|^2 - 7\alpha|\beta|}{(|\beta| - \alpha)^3} \right) b_i b_{\bar{j}} \\
 &\quad + \frac{F}{2\alpha} \left(\frac{|\beta| - 4\alpha}{(|\beta| - \alpha)^3} \right) (l_i \beta b_{\bar{j}} + \bar{\beta} b_i l_{\bar{j}}),
 \end{aligned}$$

its equivalent form is given by[10]

$$(11) \quad g_{i\bar{j}} = \rho_0 a_{i\bar{j}} + \rho'_{-2} l_i l_{\bar{j}} + \mu''_0 b_i b_{\bar{j}} + \mu''_{-2} \eta_i \eta_{\bar{j}},$$

$$(12) \quad g_{i\bar{j}} = \frac{F^2}{\alpha(|\beta| - \alpha)} a_{i\bar{j}} + \frac{KF^2}{2\alpha^3(|\beta| - \alpha)^2} l_i l_{\bar{j}} + \frac{F|\beta|}{2(|\beta| - \alpha)^2} b_i b_{\bar{j}} + \frac{S}{2F^2} \eta_i \eta_{\bar{j}},$$

and also we simplify that

$$(13) \quad g_{i\bar{j}} = \frac{F^2}{\alpha V} a_{i\bar{j}} + \frac{KF^2}{2\alpha^3 V^2} l_i l_{\bar{j}} + \frac{F|\beta|}{2V^2} b_i b_{\bar{j}} + \frac{S}{2F^2} \eta_i \eta_{\bar{j}},$$

where

$$\begin{aligned}
 K &= (|\beta| - 4\alpha) \left[\frac{1}{|\beta| - \alpha} - \frac{\alpha}{|\beta| - 2\alpha} \right], \quad V = (|\beta| - \alpha) \\
 S &= \left(\frac{|\beta| - 4\alpha}{|\beta| - 2\alpha} \right)
 \end{aligned}$$

The next goal is to find the formulas for the inverse and the determinant of the fundamental metric tensor $g_{i\bar{j}}$. For this purpose we use the proposition proved in [5]. □

Proposition 2. *Suppose:*

- $(Q_{i\bar{j}})$ is a non-singular $n \times n$ complex matrix with inverse $(Q^{\bar{j}i})$.
- C_i and $C_{\bar{i}} = \bar{C}_i, i = 1, 2, 3, \dots, n$ are complex numbers.
- $C^i := Q^{\bar{j}i} C_{\bar{j}}$ and its conjugates; $C^2 := C^i C_i = \bar{C}^i C_{\bar{i}}; H_{i\bar{j}} := Q_{i\bar{j}} \pm C_i C_{\bar{j}}$.

Then

$$(1) \det(H_{i\bar{j}}) = (1 \pm C^2) \det(Q_{i\bar{j}}),$$

(2) whenever $(1 \pm C^2) \neq 0$, the matrix $(H_{i\bar{j}})$ is invertible and in this case its inverse is

$$H^{\bar{j}i} = Q^{\bar{j}i} \mp \frac{1}{1 \mp C^2} C^i \bar{C}^j.$$

Theorem 3. For the complex Finsler space with infinite series of (α, β) -metric $F = \frac{|\beta|^2}{|\beta| - \alpha}$ we have

$$(1) \quad g^{\bar{j}i} = \frac{\alpha V}{RF^2} \left[a^{\bar{j}i} + \frac{K}{\alpha^2 R} \left(1 + \frac{K|\beta|^3}{\alpha \bar{\gamma} R} \right) \eta^i \bar{\eta}^j + \frac{\alpha|\beta|}{\bar{\gamma}} b^i b^{\bar{j}} + \frac{K|\beta|}{R\alpha \bar{\gamma}} (b^i \bar{\eta}^j \beta + \bar{\beta} \eta^i b^{\bar{j}}) \right],$$

$$(2) \quad \det(g_{i\bar{j}}) = \left(\frac{F^2}{\alpha V} \right)^n \frac{MR\bar{\gamma}}{4FV^2} \det(a_{i\bar{j}}).$$

Proof. To prove the claims we apply the above proposition in a recursive algorithm in three steps. We write $g_{i\bar{j}}$ from (8) in the form

$$(14) \quad g_{i\bar{j}} = \frac{F^2}{\alpha V} \left(a_{i\bar{j}} + \frac{K}{2\alpha^2 V} l_i l_{\bar{j}} + \frac{\alpha|\beta|}{2FV} b_i b_{\bar{j}} + \frac{\alpha SV}{2F^4} \eta_i \eta_{\bar{j}} \right).$$

Step-1: We set $Q_{i\bar{j}} := a_{i\bar{j}}$ and $C_i := \frac{1}{\alpha} \sqrt{\frac{K}{2V}} l_i$. By applying the proposition (2) we obtain $Q^{\bar{j}i} = a^{\bar{j}i}$, $C^2 = \frac{K}{2V}$, $1 + C^2 = \frac{M}{2V}$ and $C^i = \frac{1}{\alpha} \sqrt{\frac{K}{2V}} \eta^i$. So the matrix $H_{i\bar{j}} = a_{i\bar{j}} + \frac{K}{2\alpha^2 V} l_i l_{\bar{j}}$ is invertible with $H^{\bar{j}i} = a^{\bar{j}i} + \frac{K}{\alpha^2 M} \eta^i \bar{\eta}^j$ and $\det(a_{i\bar{j}} + \frac{K}{2\alpha^2 V} l_i l_{\bar{j}}) = \frac{M}{2V} \det(a_{i\bar{j}})$.

Step-2: Now, we consider $Q_{i\bar{j}} := a_{i\bar{j}} + \frac{K}{2\alpha^2 V} l_i l_{\bar{j}}$ and $C_i = \sqrt{\frac{\alpha|\beta|}{2FV}} b_i$. By applying the proposition (2) we obtain this time:

$$\begin{aligned} Q^{\bar{j}i} &= a^{\bar{j}i} + \frac{K}{\alpha^2 M} \eta^i \bar{\eta}^j, \\ C^2 &= \frac{\alpha|\beta|}{2FV} \left(a^{\bar{j}i} + \frac{K}{\alpha^2 M} \eta^i \bar{\eta}^j \right) b_i b_{\bar{j}} = \frac{\alpha|\beta|}{2FV} \left(\|b\|^2 + \frac{K|\beta|^2}{\alpha^2 M} \right), \\ 1 + C^2 &= \frac{\bar{\gamma}}{2FV} \text{ where } \bar{\gamma} = 2FV + \alpha|\beta| \left(\|b\|^2 + \frac{K|\beta|^2}{\alpha^2 M} \right), \end{aligned}$$

and $C^i = \sqrt{\frac{\alpha|\beta|}{2FV}} (b^i + \frac{K\bar{\beta}}{\alpha^2 M} \eta^i)$. It results that the inverse of $H_{i\bar{j}} = a_{i\bar{j}} + \frac{K}{2\alpha^2 V} l_i l_{\bar{j}} + \frac{\alpha|\beta|}{2FV} b_i b_{\bar{j}}$ exists and it is

$$(15) \quad H^{\bar{j}i} = a^{\bar{j}i} + \frac{K}{\alpha^2 M} \eta^i \bar{\eta}^j + \frac{\alpha|\beta|}{\bar{\gamma}} \left(b^i + \frac{K}{\alpha^2 M} \bar{\beta} \eta^i \right) \left(b^{\bar{j}} + \frac{K}{\alpha^2 M} \beta \eta^{\bar{j}} \right),$$

and

$$(16) \quad \det(a_{i\bar{j}} + \frac{K}{2\alpha^2 V} l_i l_{\bar{j}} + \frac{\alpha|\beta|}{2FV} b_i b_{\bar{j}}) = \frac{\bar{\gamma} M}{4FV^2} \det(a_{i\bar{j}}).$$

Step-3: Here we put $Q_{i\bar{j}} := a_{i\bar{j}} + \frac{K}{2\alpha^2V} l_i l_{\bar{j}} + \frac{\alpha|\beta|}{2FV} b_i b_{\bar{j}}$ and $C_i := \frac{1}{F^2} \sqrt{\frac{\alpha SV}{2}} \eta_i$, we obtain

$$Q^{\bar{j}i} = \left[a^{\bar{j}i} + \frac{K}{\alpha^2 M} \left(1 + \frac{K|\beta|^3}{\alpha \bar{\gamma} M} \right) \eta^i \bar{\eta}^j + \frac{\alpha|\beta|}{\bar{\gamma}} b^i b^{\bar{j}} + \frac{K|\beta|}{\alpha \bar{\gamma} M} (\beta b^i \bar{\eta}^j + \bar{\beta} \eta^i b^{\bar{j}}) \right],$$

and

$$C^i = \frac{1}{F^2} \sqrt{\frac{\alpha SV}{2}} \left[a^{\bar{j}i} + \frac{K}{\alpha^2 M} \left(1 + \frac{K|\beta|^3}{\alpha \bar{\gamma} M} \right) \eta^i \bar{\eta}^j + \frac{\alpha|\beta|}{\bar{\gamma}} b^i b^{\bar{j}} + \frac{K|\beta|}{\alpha \bar{\gamma} M} (\beta b^i \bar{\eta}^j + \bar{\beta} \eta^i b^{\bar{j}}) \right] \eta_{\bar{j}},$$

therefore

$$C^2 = C^i C_i = \frac{\alpha SV}{2F^4} \left[a^{\bar{j}i} + \frac{K}{\alpha^2 M} \eta^i \bar{\eta}^j + \frac{\alpha|\beta|}{\bar{\gamma}} \left(b^i + \frac{K}{\alpha^2 M} \bar{\beta} \eta^i \right) \left(b^{\bar{j}} + \frac{K}{\alpha^2 M} \beta \eta^{\bar{j}} \right) \right] \eta_i \eta_{\bar{j}}.$$

Since $1 \pm C^2 \neq 0$, $H_{i\bar{j}} = Q_{i\bar{j}} \pm C_i C_{\bar{j}} = a_{i\bar{j}} + \frac{K}{2\alpha^2V} l_i l_{\bar{j}} + \frac{\alpha|\beta|}{2FV} b_i b_{\bar{j}} + \frac{\alpha SV}{2F^4} \eta_i \eta_{\bar{j}}$ is invertible with the inverse

$$(17) \quad H^{\bar{j}i} = \frac{a^{\bar{j}i} + \frac{K}{\alpha^2 M} \eta^i \bar{\eta}^j + \frac{\alpha|\beta|}{\bar{\gamma}} \left(b^i + \frac{K}{\alpha^2 M} \bar{\beta} \eta^i \right) \left(b^{\bar{j}} + \frac{K}{\alpha^2 M} \beta \eta^{\bar{j}} \right)}{1 + \frac{\alpha SV}{2F^4} \left[a^{\bar{j}i} + \frac{K}{\alpha^2 M} \eta^i \bar{\eta}^j + \frac{\alpha|\beta|}{\bar{\gamma}} \left(b^i + \frac{K}{\alpha^2 M} \bar{\beta} \eta^i \right) \left(b^{\bar{j}} + \frac{K}{\alpha^2 M} \beta \eta^{\bar{j}} \right) \right] \eta_i \eta_{\bar{j}}},$$

In view of (7), we have

$$(18) \quad a^{\bar{m}m} \eta_i \bar{\eta}_j = p_0^2 \alpha^2 + 2q_0 p_0 |\beta|^2 + q_0^2 \|b\|^2 |\beta|^2,$$

$$(19) \quad b^m \eta_m = (p_0 \bar{\beta} + q_0 \|b\|^2 \bar{\beta}),$$

$$(20) \quad b^{\bar{n}} \bar{\eta}_n = (p_0 \beta + q_0 \|b\|^2 \beta).$$

Putting (18), (19) and (20) in (17), we get

$$(21) \quad H^{\bar{j}i} = \frac{1}{R} \left[a^{\bar{j}i} + \frac{K}{\alpha^2 M} \eta^i \bar{\eta}^j + \frac{\alpha|\beta|}{\bar{\gamma}} \left(b^i + \frac{K}{\alpha^2 M} \bar{\beta} \eta^i \right) \left(b^{\bar{j}} + \frac{K}{\alpha^2 M} \beta \eta^{\bar{j}} \right) \right],$$

where

$$R = 1 + \frac{\alpha SV}{2F^4} \left[p_0^2 \alpha^2 + 2q_0 p_0 |\beta|^2 + q_0^2 \|b\|^2 |\beta|^2 + \frac{KF^4}{\alpha^2 M} + \frac{\alpha|\beta|}{\bar{\gamma}} \left(p_0^2 |\beta|^2 + 2p_0 q_0 |\beta|^2 \|b\|^4 + q_0^2 |\beta|^2 \|b\|^4 + 2p_0 \frac{KF^2}{\alpha^2 M} \bar{\beta} |\beta|^2 + 2q_0 \frac{K\beta}{\alpha^2 M} F^2 |\beta|^2 \|b\|^2 + \frac{K^2 F^4}{\alpha^4 M^2} |\beta|^4 \right) \right],$$

and

$$\det(H_{i\bar{j}}) = 1 + C^2 \det(Q_{i\bar{j}}) = R \det\left(a_{i\bar{j}} + \frac{K}{2\alpha^2V} l_i l_{\bar{j}} + \frac{\alpha|\beta|}{2FV} b_i b_{\bar{j}}\right),$$

on simplifying, we have

$$(22) \quad \det(H_{i\bar{j}}) = \frac{MR\bar{\gamma}}{4FV^2} \det(a_{i\bar{j}}).$$

Since $g_{i\bar{j}} = \frac{F^2}{\alpha V} H_{i\bar{j}}$, the inverse of the fundamental metric tensor is given by $g^{i\bar{j}} = \frac{\alpha V}{F^2} H^{i\bar{j}}$ and $\det(g_{i\bar{j}}) = (\frac{F^2}{\alpha V})^n \det(H_{i\bar{j}})$. we obtained the results (1) and (2). \square

Theorem 4. *A complex Finsler space with infinite series of (α, β) -metric with $\bar{\gamma} > 0$ is a complex Finsler metric.*

Proof. The formula for $\det(g_{i\bar{j}})$, we can say that $g_{i\bar{j}}(z, \eta)$ is positive definite if and only if $\bar{\gamma} > 0$ at each nonzero η in $T'_z M$. If the quadratic form $h(z, \eta) := (a_{i\bar{j}} - b_i b_{\bar{j}}) \eta^i \bar{\eta}^j$ is positive definite, then substituting $\eta^i \bar{\eta}^j$ with $b^i \bar{b}^j$ it follows that $\|b\|^2(1 - \|b\|^2) > 0$, which says that $\|b\|^2 \in (0, 1)$ and then $\bar{\gamma} > 0$, since $\bar{\gamma} = 2FV + \alpha|\beta| \left(\|b\|^2 + \frac{K|\beta|^2}{\alpha^2 M} \right) > 0$. Equivalently the positive definite of the quadratic form means that $\alpha^2 > |\beta|^2$, or in other words $\sup \frac{|\beta|}{\alpha} < 1$ for all $(z, \eta) \in T' M$. The last assumption is required in [21] and no other restrictive conditions are needful for a complex Finsler space with infinite series of (α, β) -metric. \square

Example 1. *We consider α is given by*

$$\alpha^2(z, \eta) = \frac{|\eta|^2 + \varepsilon(|z|^2|\eta|^2 - |\langle z, \eta \rangle|^2)}{(1 + \varepsilon|z|^2)^2},$$

where $|z|^2 := \sum_{k=1}^n z^k \bar{z}^k$, $\langle z, \eta \rangle := \sum_{k=1}^n z^k \bar{\eta}^k$, $|\langle z, \eta \rangle|^2 = \langle z, \eta \rangle \overline{\langle z, \eta \rangle}$. Defined over the disk $\Delta_r^n = \left\{ z \in \mathbb{C}^n, |z| < r, r := \sqrt{\frac{1}{|\varepsilon|}} \right\}$ if $\varepsilon < 0$, on \mathbb{C}^n if $\varepsilon = 0$ and on the complex projective space $\mathbb{P}^n(\mathbb{C})$ if $\varepsilon > 0$. Note that $\alpha^2(z, \eta) = a_{i\bar{j}}(z) \eta^i \bar{\eta}^j$ thus it determines a purely Hermitian metrics which have special properties. They are Kähler with constant holomorphic curvature $K_\alpha = 4\varepsilon$. Particularly, for $\varepsilon = -1$ we obtain the Bergman metric on the unit disk $\Delta^n := \Delta_1^n$; for $\varepsilon = 0$ the Euclidean metric on \mathbb{C}^n , and for $\varepsilon = 1$ the Fubini-Study metric on $\mathbb{P}^n(\mathbb{C})$. By setting $|\beta(z, \eta)| = \frac{\langle z, \eta \rangle}{1 + \varepsilon|z|^2}$ then we obtain the some examples of complex Finsler space with infinite series of (α, β) -metric F is of the form:

$$F_\varepsilon = \frac{|\langle z, \eta \rangle|^2}{(1 + \varepsilon|z|^2) \left(|\langle z, \eta \rangle| - \sqrt{|\eta|^2 + \varepsilon(|z|^2|\eta|^2 - |\langle z, \eta \rangle|^2)} \right)}.$$

For example, F_{-1} is of negative holomorphic curvature $K_{F_{-1}} = \frac{-2\alpha F_{-1}}{\bar{\gamma}}$, $\bar{\gamma} := L_{-1} - \alpha^2(1 - |z|^2)$.

Further we show that the complex Finsler space with infinite series of (α, β) -metric $F = \frac{|\beta|^2}{(|\beta|-\alpha)}$ offers a significant d -tensor with fairly many properties. Let us consider

$$(23) \quad t_{i\bar{r}} := \frac{\partial^2 F}{\partial \eta^i \partial \bar{\eta}^r} = \frac{1}{2F} \left(g_{i\bar{r}} - \frac{\partial F}{\partial \eta^i} \frac{\partial F}{\partial \bar{\eta}^r} \right) = \frac{1}{2F} \left(g_{i\bar{r}} - \frac{1}{2L} \eta^i \bar{\eta}^r \right).$$

We call (23) the complex angular metric tensor of the space. A direct computation yields $t_{i\bar{r}} \eta^i = \frac{1}{4F}$, $t_{i\bar{r}} \eta^i \bar{\eta}^r = \frac{F}{4}$, $\frac{\partial t_{i\bar{r}}}{\partial \eta^m} \eta^m = -\frac{1}{2} t_{i\bar{r}}$. Moreover we have:

Theorem 5. (1) $t_{i\bar{r}} = \frac{F^2}{2\alpha|\beta|} h_{i\bar{r}} + \frac{B\alpha^2 F}{|\beta|} b_i b_{\bar{r}} + \frac{C\alpha|\beta|}{F^4} \eta_i \bar{\eta}_r$,

where $h_{i\bar{r}} = a_{i\bar{r}} + \frac{AF}{|\beta|^3} l_i l_{\bar{r}}$,

$$(2) \quad t^{i\bar{r}} = \frac{2\alpha\beta}{F} \left[\frac{1}{FR_1} a^{i\bar{r}} - \frac{A}{R_1 M_1} \left(1 + \frac{A^2 B \alpha^2 F^3 |\beta|}{\bar{\gamma}_1} \right) \eta^i \bar{\eta}^r - \frac{B M_1 \alpha^2}{R_1 \bar{\gamma}_1} b^i b^{\bar{r}} + \frac{\alpha^2 F A B}{\bar{\gamma}_1} (\beta b^i \eta^{\bar{r}} + \bar{\beta} b^{\bar{r}} \eta^i) \right],$$

$$(3) \quad \det(t_{i\bar{r}}) = \left(\frac{F^2}{2\alpha} \right)^n \frac{R_1 \bar{\gamma}_1}{|\beta|^{(3+n)}} \det(a_{i\bar{r}}).$$

The proof is straight forward, by applying Proposition (2).

4. CHERN FINSLER CONNECTION COEFFICIENTS AND CARTAN TENSOR FOR INFINITE SERIES OF (α, β) -METRIC

The Chern-Finsler connection coefficients (c.n.c) of a complex Finsler space (M, F) with (α, β) -metric is defined by

$$(24) \quad N_j^i = g^{\bar{m}i} \frac{\partial g_{l\bar{m}}}{\partial z^j} \eta^l = g^{\bar{m}i} \frac{\partial \bar{\eta}_m}{\partial z^j}.$$

Once obtained the metric tensor of a complex Finsler space with infinite series, it is a technical computation to get the expression of Chern-Finsler connection by using (1). The First computation refers to the coefficients of the Chern-Finsler connection (c.n.c). A simplified writing for them is

$$(25) \quad N_j^i = N_j^i + \frac{\alpha V}{F^2} a^{\bar{m}i} \left(\frac{\partial \bar{\eta}_m}{\partial z^j} - \frac{F^2}{\alpha V} \frac{\partial a_{l\bar{m}}}{\partial z^j} \eta^l \right) + \frac{\alpha V}{F^2} \left\{ \frac{K}{\alpha^2 M} \eta^i \bar{\eta}^m + \frac{\alpha|\beta|}{\bar{\gamma}} \left(b^i + \frac{K}{\alpha^2 M} \bar{\beta} \eta^i \right) \left(b^{\bar{m}} + \frac{K}{\alpha^2 M} \beta \bar{\eta}^m \right) + \frac{\alpha S V}{2 R F^4} \left[a^{\bar{m}i} + \frac{K}{\alpha^2 M} \eta^i \bar{\eta}^m + \frac{\alpha|\beta|}{\bar{\gamma}} \left(b^i + \frac{K}{\alpha^2 M} \bar{\beta} \eta^i \right) \left(b^{\bar{m}} + \frac{K}{\alpha^2 M} \beta \bar{\eta}^m \right) \right]^2 \right\} \frac{\partial \bar{\eta}_m}{\partial z^j},$$

where

$$\begin{aligned} \frac{\partial \bar{\eta}_m}{\partial z^j} = & \left[\frac{F^2 l_{\bar{m}}}{2\alpha^2(|\beta| - \alpha)} \left(1 - \frac{1}{\alpha}\right) + \frac{F\beta b_{\bar{m}}}{(|\beta| - \alpha)^2} \left(\frac{(|\beta| - 2\alpha)}{(|\beta| - \alpha)^2} - 1\right) \right] \frac{\partial a_{i\bar{m}}}{\partial z^j} \eta^i \bar{\eta}^m \\ & + \frac{F}{\alpha(|\beta| - \alpha)} \left(F \frac{\partial a_{i\bar{m}}}{\partial z^j} \eta^i + \frac{\partial F}{\partial z^j} 2l_{\bar{m}} \right) + \frac{(|\beta| - 2\alpha)}{(|\beta| - \alpha)^2} \left[\frac{\partial F}{\partial z^j} \beta b_{\bar{m}} + F \left(\beta \frac{\partial b_{\bar{m}}}{\partial z^j} + b_{\bar{m}} \frac{\partial b_i}{\partial z^j} \eta^i \right) \right] \\ & + \frac{F}{(|\beta| - \alpha)^2} \left[\frac{\beta}{2} b_{\bar{m}} - \frac{F}{2\alpha} l_{\bar{m}} - \frac{(|\beta| - 2\alpha)}{(|\beta| - \alpha)^2} \beta b_{\bar{m}} \right] \frac{1}{|\beta|} \left(\beta \frac{\partial b_{\bar{m}}}{\partial z^j} \bar{\eta}^m + \bar{\beta} \frac{\partial b_i}{\partial z^j} \eta^i \right), \end{aligned}$$

and $N_j^i = a^{\bar{m}i} \frac{\partial a_{l\bar{m}}}{\partial z^j} \eta^l$.

Next, let us introduce the following complex Cartan tensors [3]

$$\begin{aligned} (26) \quad C_{j\bar{h}k} = & \frac{|\beta|^3}{\alpha} \left[\frac{|\beta| - 4\alpha}{(|\beta| - 4\alpha)^4} \left(\frac{\bar{\beta} b_k}{2|\beta|} - \frac{|\beta| l_k}{2\alpha^2} \right) \right] a_{j\bar{h}} + \frac{|\beta|^3}{2\alpha^3(|\beta| - 4\alpha)^4} \\ & \left\{ \left[\frac{3|\beta|}{(|\beta| - 4\alpha)^2} - \frac{(|\beta| - 8\alpha)}{(|\beta| - 2\alpha)} - \frac{2\alpha(|\beta| - 4\alpha)}{(|\beta| - 2\alpha)^2} \right. \right. \\ & \left. \left. + \frac{7\alpha - 3|\beta|}{\alpha(|\beta| - \alpha)} \left(\frac{|\beta| - 4\alpha}{|\beta| - \alpha} - \frac{\alpha(|\beta| - 4\alpha)}{|\beta| - 2\alpha} \right) \right] \frac{|\beta| l_k}{2\alpha} \right. \\ & \left. + \left[\left(\frac{3\alpha}{(|\beta| - \alpha)^2} - \frac{2\alpha^2}{(|\beta| - 2\alpha)^2} \right) |\beta| - \frac{|\beta| - 4\alpha}{|\beta| - \alpha} \right. \right. \\ & \left. \left. \left(\frac{4\alpha}{|\beta| - \alpha} - \frac{4\alpha^2}{|\beta| - 2\alpha} \right) \right] \frac{\bar{\beta} b_k}{2|\beta|} \right\} l_j l_{\bar{h}} + \frac{3|\beta|^2}{2(|\beta| - \alpha)^3} \\ & \left(\frac{l_k}{2\alpha} - \frac{3\alpha}{|\beta| - \alpha} \frac{\bar{\beta} b_k}{2|\beta|} \right) b_j b_{\bar{h}} + \left[\frac{1}{|\beta| - 2\alpha} \left(\frac{|\beta| - \alpha}{|\beta| - 2\alpha} - \frac{|\beta| - 4\alpha}{|\beta|} \right) \frac{l_k}{2\alpha} \right. \\ & \left. + \left(\frac{\alpha(|\beta| - \alpha)}{|\beta| - 2\alpha} + \frac{(2\alpha - |\beta|)(|\beta| - 4\alpha)}{|\beta|} \right) \frac{\bar{\beta} b_k}{2|\beta|^2} \right] \eta_j \eta_{\bar{h}}. \end{aligned}$$

Then the vertical coefficients of Chern-Finsler connections are

$$(27) \quad C_{jk}^i := g^{\bar{m}i} \frac{\partial g_{k\bar{m}}}{\partial \eta^j} = g^{\bar{m}i} \frac{\partial g_{j\bar{m}}}{\partial \eta^k} = g^{\bar{m}i} C_{j\bar{m}k}.$$

An expanded writing for these coefficients is

$$\begin{aligned}
 (28) \quad C_{jk}^i &= \frac{\alpha V}{RF^2} \left[a^{\bar{m}i} + \frac{K}{\alpha^2 R} \left(1 + \frac{K|\beta|^3}{\alpha \bar{\gamma} R} \right) \eta^i \bar{\eta}^m + \frac{\alpha |\beta|}{\bar{\gamma}} b^i b^{\bar{m}} \right. \\
 &+ \left. \frac{K|\beta|}{R\alpha \bar{\gamma}} (b^i \bar{\eta}^m \beta + \bar{\beta} \eta^i b^{\bar{m}}) \right] \left\{ \frac{|\beta|^3}{\alpha} \left[\frac{|\beta| - 4\alpha}{(|\beta| - 4\alpha)^4} \left(\frac{\bar{\beta} b_k}{2|\beta|} - \frac{|\beta| l_k}{2\alpha^2} \right) \right] a_{j\bar{m}} \right. \\
 &+ \frac{|\beta|^3}{2\alpha^3 (|\beta| - 4\alpha)^4} \left\{ \left[\frac{3|\beta|}{(|\beta| - 4\alpha)^2} - \frac{(|\beta| - 8\alpha)}{(|\beta| - 2\alpha)} - \frac{2\alpha (|\beta| - 4\alpha)}{(|\beta| - 2\alpha)^2} \right. \right. \\
 &+ \left. \left. \frac{7\alpha - 3|\beta|}{\alpha (|\beta| - \alpha)} \left(\frac{|\beta| - 4\alpha}{|\beta| - \alpha} - \frac{\alpha (|\beta| - 4\alpha)}{|\beta| - 2\alpha} \right) \right] \frac{|\beta| l_k}{2\alpha} \right. \\
 &+ \left. \left[\left(\frac{3\alpha}{(|\beta| - \alpha)^2} - \frac{2\alpha^2}{(|\beta| - 2\alpha)^2} \right) |\beta| - \frac{|\beta| - 4\alpha}{|\beta| - \alpha} \right. \right. \\
 &\left. \left. \left(\frac{4\alpha}{|\beta| - \alpha} - \frac{4\alpha^2}{|\beta| - 2\alpha} \right) \right] \frac{\bar{\beta} b_k}{2|\beta|} \right\} l_j l_{\bar{m}} + \frac{3|\beta|^2}{2(|\beta| - \alpha)^3} \\
 &\left(\frac{l_k}{2\alpha} - \frac{3\alpha}{|\beta| - \alpha} \frac{\bar{\beta} b_k}{2|\beta|} \right) b_j b_{\bar{m}} + \left[\frac{1}{|\beta| - 2\alpha} \left(\frac{|\beta| - \alpha}{|\beta| - 2\alpha} - \frac{|\beta| - 4\alpha}{|\beta|} \right) \frac{l_k}{2\alpha} \right. \\
 &\left. + \left(\frac{\alpha (|\beta| - \alpha)}{|\beta| - 2\alpha} + \frac{(2\alpha - |\beta|)(|\beta| - 4\alpha)}{|\beta|} \right) \frac{\bar{\beta} b_k}{2|\beta|^2} \right] \eta_j \eta_{\bar{m}} \left. \right\}.
 \end{aligned}$$

Also

$$(29) \quad C_k = C_{k\bar{h}j} g^{\bar{h}j}$$

plugging (27) in (29) gives us

$$\begin{aligned}
 (30) \quad C_k &= \left\{ \frac{|\beta|^3}{\alpha} \left[\frac{|\beta| - 4\alpha}{(|\beta| - 4\alpha)^4} \left(\frac{\bar{\beta} b_k}{2|\beta|} - \frac{|\beta| l_j}{2\alpha^2} \right) \right] a_{k\bar{h}} + \frac{|\beta|^3}{2\alpha^3 (|\beta| - 4\alpha)^4} \right. \\
 &\left\{ \left[\frac{3|\beta|}{(|\beta| - 4\alpha)^2} - \frac{(|\beta| - 8\alpha)}{(|\beta| - 2\alpha)} - \frac{2\alpha (|\beta| - 4\alpha)}{(|\beta| - 2\alpha)^2} + \frac{7\alpha - 3|\beta|}{\alpha (|\beta| - \alpha)} \right. \right. \\
 &\left. \left. \left(\frac{|\beta| - 4\alpha}{|\beta| - \alpha} - \frac{\alpha (|\beta| - 4\alpha)}{|\beta| - 2\alpha} \right) \right] \frac{|\beta| l_j}{2\alpha} + \left[\left(\frac{3\alpha}{(|\beta| - \alpha)^2} - \frac{2\alpha^2}{(|\beta| - 2\alpha)^2} \right) |\beta| \right. \right.
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{|\beta| - 4\alpha}{|\beta| - \alpha} \left(\frac{4\alpha}{|\beta| - \alpha} - \frac{4\alpha^2}{|\beta| - 2\alpha} \right) \left] \frac{\bar{\beta} b_k}{2|\beta|} \right\} l_k l_{\bar{h}} + \frac{3|\beta|^2}{2(|\beta| - \alpha)^3} \\
 & \left(\frac{l_j}{2\alpha} - \frac{3\alpha}{|\beta| - \alpha} \frac{\bar{\beta} b_k}{2|\beta|} \right) b_k b_{\bar{h}} + \left[\frac{1}{|\beta| - 2\alpha} \left(\frac{|\beta| - \alpha}{|\beta| - 2\alpha} - \frac{|\beta| - 4\alpha}{|\beta|} \right) \frac{l_j}{2\alpha} \right. \\
 & \left. + \left(\frac{\alpha(|\beta| - \alpha)}{|\beta| - 2\alpha} + \frac{(2\alpha - |\beta|)(|\beta| - 4\alpha)}{|\beta|} \right) \frac{\bar{\beta} b_k}{2|\beta|^2} \right] \eta_k \eta_{\bar{h}} \left. \right\} \frac{\alpha V}{RF^2} \\
 & \left[a^{\bar{h}j} + \frac{K}{\alpha^2 R} \left(1 + \frac{K|\beta|^3}{\alpha \bar{\gamma} R} \right) \eta^j \bar{\eta}^h + \frac{\alpha|\beta|}{\bar{\gamma}} b^j b^{\bar{h}} + \frac{K|\beta|}{R\alpha \bar{\gamma}} (b^j \bar{\eta}^h \beta + \bar{\beta} \eta^j b^{\bar{h}}) \right].
 \end{aligned}$$

Theorem 6. *The coefficients of Chern Finsler connection and Cartan tensor for an complex Finsler space with infinite series of (α, β) -metric is given in (25) and (30) respectively.*

We remark that a complex Finsler metric is purely Hermitian it leads to $F = \alpha(1 + ||b||)$ and $a_{i\bar{j}} ||b||^2 = b_i b_{\bar{j}}$. thus, we have proved.

5. KÄHLER-RANDERS CHANGE OF INFINITE SERIES OF (α, β) -METRICS

When trying to show more geometrical properties of complex Finsler space with infinite series of (α, β) -metrics, we face the fact that there are so many computations. Certainly, one should not infer that this class of complex Finsler metrics is less significant. On the contrary, beyond the computations in the sequel we show that there are interesting results.

Theorem 7. *Let $(M, \frac{|\beta|^2}{|\beta| - \alpha})$ be a complex Finsler space with infinite series with property $\frac{\partial |\beta|^2}{\partial z^i} = \varepsilon \frac{\partial \alpha^2}{\partial z^i}$, where $\varepsilon = \varepsilon(z)$ is a real valued function.*

- (1) *If $a_{i\bar{j}}$ is the Euclidian metric, then $F = \frac{|\beta|^2}{|\beta| - \alpha}$ is locally Minkowski.*
- (2) *If $\varepsilon = 0$ for any z , $C_k \neq 0$ for any k , and $a_{i\bar{j}}$ is Kähler, then $F = \frac{|\beta|^2}{|\beta| - \alpha}$ is weakly Kähler if and only if $b_i G^i := 0$, where $G^i := N_j^i \eta^j$.*
- (3) *If $\varepsilon = ||b||^2$, then $||b||^2$ is a constant.*

Proof. Indeed, if $a_{i\bar{j}}$ is the Euclidian metric, then $\frac{\partial |\beta|^2}{\partial z^i} = \frac{\partial \alpha^2}{\partial z^i} = 0$ and so $\frac{\partial F}{\partial z^i} = 0$. This means that there exists charts in any (z, η) such that the complex Finsler space with infinite series of (α, β) -metric $F = \frac{|\beta|^2}{|\beta| - \alpha}$ depends only on the η variable, i.e., it is locally Minkowski and (1) is true.

The assumption $\frac{\partial|\beta|^2}{\partial z^i} = \varepsilon \frac{\partial\alpha^2}{\partial z^i}$ can be written as

$$(31) \quad \bar{\beta} \frac{\partial b_r}{\partial z^i} \eta^r + \beta \frac{\partial b_s}{\partial z^i} \bar{\eta}^s = \varepsilon \frac{\partial a_{r\bar{s}}}{\partial z^i} \eta^r \bar{\eta}^s.$$

Deriving the relation (31) with respect to η and $\bar{\eta}$, we obtain

$$(32) \quad b_{\bar{s}} \frac{\partial b_r}{\partial z^i} + b_r \frac{\partial b_{\bar{s}}}{\partial z^i} = \varepsilon \frac{\partial a_{r\bar{s}}}{\partial z^i}.$$

Now, contracting in (32) by b^r and $b^{\bar{s}}$, we get

$$(33) \quad b^r \frac{\partial b_r}{\partial z^i} + b^{\bar{s}} \frac{\partial b_{\bar{s}}}{\partial z^i} = \frac{\varepsilon}{\|b\|^2} b^{\bar{s}} b^r \frac{\partial a_{r\bar{s}}}{\partial z^i}.$$

In [6], after the calculus of the L^i_{jk} coefficients we obtain the weakly Kähler condition for complex Finsler space with infinite series. By this, a straightforward computation using (33) with $\varepsilon = 0$ and $\Gamma^{\bar{s}}_{jk} = \frac{1}{2} \alpha^{\bar{s}k} \{ \frac{\partial a_{k\bar{j}}}{\partial z^i} - \frac{\partial a_{i\bar{j}}}{\partial z^k} \} = 0$ since $a_{i\bar{j}}$ is Kähler, show immediate that the assertion (2) is true.

Now, by (6) we obtain $\frac{\partial\|b\|^2}{\partial z^i} = -\left(1 - \frac{\varepsilon}{\|b\|^2}\right) b^{\bar{s}} b^r \frac{\partial a_{r\bar{s}}}{\partial z^i}$. Putting $\varepsilon = \|b\|^2$ in last relation we have $\frac{\partial\|b\|^2}{\partial z^i} = 0$, and so $\|b\|^2$ is a constant, i.e.,(3). □

Theorem 8. Let $(M, \frac{|\beta|^2}{|\beta|-\alpha})$ be a complex Finsler space with infinite series of $\frac{\partial|\beta|^2}{\partial z^i} = \|b\|^2 \frac{\partial\alpha^2}{\partial z^i}$. If one of equivalent conditions from lemma (3.1) in [5] holds, then $N_j^i = N_j^i$ $\overset{CF}{a}$

Proof. An elementary computation, taking into account lemma (3.1) in [5] in the formula (8) which give the coefficients of Chern-Finsler (c.n.c), prove that all terms are vanishes except the first term. Indeed we obtain $N_j^i = N_j^i$. □

Theorem 9. Let $(M, \frac{|\beta|^2}{|\beta|-\alpha})$ be a complex Finsler space with infinite series of $\frac{\partial|\beta|^2}{\partial z^i} = \|b\|^2 \frac{\partial\alpha^2}{\partial z^i}$. If one of equivalent conditions from lemma (3.1) in [5] holds, then $F = \frac{|\beta|^2}{|\beta|-\alpha}$ is strongly Kähler.

Proof. By Theorem (8), we have $N_j^i = N_j^i$ and $L^i_{jk} = \frac{\partial N_k^i}{\partial \eta^j} = \frac{\partial N_k^i}{\partial \eta^j}$, because $a_{i\bar{j}}$ is Kähler, it results $\frac{\partial N_k^i}{\partial \eta^j} = \frac{\partial N_j^i}{\partial \eta^k}$. Therefore $L^i_{jk} = L^i_{kj}$, i.e., $F = \frac{|\beta|^2}{|\beta|-\alpha}$ is strongly Kähler. □

Example 2. In order to reduce clutter, let us relabel the local coordinates z^1, z^2, η^1, η^2 as z, ω, η, θ , respectively. Let $\Delta = \{(z, \omega) \in C^2, |\omega| < |z| < 1\}$ be the Hartogs triangle with the

Kähler-purely Hermitian metric.

$$(34) \quad a_{i\bar{j}} = \frac{\partial^2}{\partial z^i \partial \bar{z}^j} \left(\log \frac{1}{(1 - |z|^2)(|z|^2 - |\omega|^2)} \right); \quad \alpha^2(z, \omega; \eta, \theta) = a_{i\bar{j}} \eta^i \bar{\eta}^j,$$

where $|z^i|^2 := z^i \bar{z}^i, z^i \in \{z, \omega\}, \eta^i \in \{\eta, \theta\}$. we choose

$$(35) \quad b_z = \frac{\omega}{|z|^2 - |\omega|^2}; \quad b_w = -\frac{z}{|z|^2 - |\omega|^2}.$$

With these tools we construct $\alpha(z, \omega, \eta, \theta) := \sqrt{a_{i\bar{j}}(z, \omega) \eta^i \bar{\eta}^j}$ and $\beta(z, \omega) = b_i(z, \omega) \eta^i$ and from the here complex Randers change infinite series $F = \frac{|\beta|^2}{|\beta| - \alpha}$. By a direct computation, we deduce

$$\begin{aligned} a_{z\bar{z}} &= \frac{1}{(1 - |z|^2)^2} + b_z b_{\bar{\omega}}; \quad a_{z\bar{\omega}} = b_z b_{\bar{\omega}}; \quad a_{\omega\bar{\omega}} = b_{\omega\bar{\omega}}; \\ a^{z\bar{z}} &= (1 - |z|^2)^2; \quad b^{\bar{\omega}z} = \frac{\bar{\omega}z(1 - |z|^2)^2}{|z|^2}; \\ a^{z\bar{z}} &= \frac{(|z|^2 - |\omega|^2)^2}{|z|^2} + \frac{|\omega|^2(1 - |z|^2)^2}{|z|^2}; \\ b^z &= 0; \quad b^\omega = -\frac{|z|^2 - |\omega|^2}{z}; \quad \|b\|^2 = 1; \quad \alpha^2 - |\beta|^2 = \frac{|\eta|^2}{(1 - |z|^2)^2} \end{aligned}$$

and the coefficients of the Chern-Finsler (c.n.c) are

$$\begin{aligned} N_z^{CF} &= N_z^a = \frac{2\bar{z}\eta}{1 - |z|^2}; \quad N_\omega^{CF} = N_\omega^a = 0; \\ N_z^{CF\omega} &= N_z^a = \frac{2\bar{z}\omega}{z} \left(\frac{1}{1 - |z|^2} + \frac{1}{|z|^2 - |\omega|^2} \right) \eta - \frac{|z|^2 + |\omega|^2}{z(|z|^2 - |\omega|^2)} \theta; \\ N_\omega^{CF} &= N_\omega^a = \frac{|z|^2 + |\omega|^2}{z(|z|^2 - |\omega|^2)} \eta + \frac{2\bar{\omega}\theta}{|z|^2 - |\omega|^2}. \end{aligned}$$

Thus we have another example of strongly Kähler-Randers changes of infinite series. It is a complex Finsler metric on Hartogs triangle and its holomorphic curvature is negative,

$$K_F = -\frac{4}{L_\alpha} [F(\alpha - |\beta|)^2 + |\beta|^3] < 0.$$

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

- [1] M. Abate, G. Patrizio, *Finsler Metrics—A Global Approach*, Springer, Berlin, Heidelberg, 1994.
- [2] T. Aikou, Projective flatness of complex Finsler metrics, *Publ. Math. Debrecen.* 63(3) (2003), 343-362.
- [3] N. Aldea, Complex Finsler spaces of constant holomorphic curvature, *Diff. Geom. and its Appl., Proc. Conf. Prague 2004, Charles Univ. Prague (Czech Republic) 2005*, 179–190.
- [4] N. Aldea, G. Munteanu, (α, β) -complex Finsler metrics, In: *Proceedings of the 4th international colloquium “Mathematics in Engineering and Numerical Physics”*, pp. 1–6, BGS Proc., 14, Geom. Balkan Press, Bucharest (2007).
- [5] N. Aldea, G. Munteanu, On complex Finsler spaces with Randers metric, *J. Korean Math. Soc.* 46(5) (2009), 949-966.
- [6] N. Aldea, G. Munteanu, On the geometry of Complex Randers space. In: *Proceedings of the 14th Nat. Sem. On Finsler, Lagrange and Hamilton spaces, Brasov*, pp 1–8.
- [7] D. Bao, S.S. Chern, Z. Shen, *An Introduction to Riemann–Finsler Geometry*. Springer, New York (2000).
- [8] D. Bao, C. Robles, On Randers space of constant flag curvature, *Rep. Math. Phys.* 51(1) (2003), 9-42.
- [9] D. Bao, C. Robles, Z. Shen, Zermelo navigation on Riemannian manifolds, *J. Differ. Geom.* 66(3) (2004), 377-435.
- [10] B. Chen, Y. Shen, Complex Randers metric, communicated to conference Zhejiang University, China, July, 2007.
- [11] C. Caratheodory, Über eine spezielle metric, die in der Theorie der analitischen Function antritt, *Atti Pontificia Acad. Sci. Nuovi Lincei* 80 (1927), 135-141.
- [12] M. Fukui, Complex Finsler manifolds, *J. Math. Kyoto Univ.* 29(4) (1989), 609-624.
- [13] K. Liu, X. Sun, S.T. Yau, Canonical metrics on the moduli space of Riemann surface, I, *J. Differ. Geom.* 68(3) (2004), 571-637.
- [14] G.B. Rizza, F-forme quadraticheed hermitiane, *Rend. Mat. Appl.* 23 (1965), 221-249.
- [15] G. Shankar, R. Kaushik Sharma, R -complex Finsler space with infinite series (α, β) -metric, *Int. J. Pure Appl. Math.* 120(3) (2018), 351-363.
- [16] R.S. Ingarden, On the geometrically absolute optical representation in the electron microscope, *Trav. Soc. Sci. Lettr. Wroclaw*, B45 (1957), 3–60.
- [17] S. Kobayashi, Invariant distances on complex manifolds and holomorphic mappings, *J. Math. Soc. Japan*, 19 (1976), 460-480.
- [18] R. Miron, The geometry of Ingarden spaces, *Rep. Math. Phys.* 54(2) (2004), 131-147.
- [19] G. Munteanu, *Complex spaces in Finsler, Lagrange, and Hamilton geometries*, Kluwer Academic Publishers, Dordrecht; Boston, 2004.
- [20] H.L. Royden, *Complex Finsler Metrics*, *Contemporary Math.* 49 (1984), 119-124.

- [21] H. Rund, Generalized metrics on complex Manifolds, *Math. Nachr.* 34 (1967), 55-77.
- [22] A. Spiro, The Structure Equations of a Complex Finsler Manifold, *Asian J. Math.* 5(2) (2001), 291-326.
- [23] P.M. Wong, A survey of complex Finsler geometry, *Adv. Stud. Pure Math.* 48 (2007), 375-433.
- [24] H. Yasuda, H. Shimada, On Randers spaces of scalar curvature, *Rep. Math. Phys.* 11(3) (1977), 347-360.