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# THE (NORMALIZED) LAPLACIAN SPECTRUM AND RELATED INDEXES OF GENERALIZED QUADRILATERAL GRAPHS

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**Abstract.** In this paper, we introduce the generalized quadrilateral graph $Q^{(n)}(G)$, which can be got by replacing each edge of the given graph $G$ with a complete bipartite graph $K_{n,n}$. We characterize all the spectrum of the graph $Q^{(n)}(G)$ in terms of the given graph. Then we derive the formula for the multiplicative degree-Kirchhoff index, the Kemeny's constant and the number of spanning trees of $Q^{(n)}(G)$. Finally, we can obtain more about the iterative graph $Q_r^{(n)}(G)$.

**Keywords:** normalized Laplacian; multiplicative degree-Kirchhoff index; Kemeny's constant; spanning tree.

**2010 AMS Subject Classification:** 05C50.

## 1. INTRODUCTION

### 1.1. Notions and definitions.

Throughout all the paper, we consider a simple and connected graph $G = (V(G), E(G))$ with $N_0$ vertices and denote the vertex set of $G$ by $V(G) = \{1, 2, \cdots, N_0\}$. For any two adjacent vertices $s$ and $t$, we denote it by $s \sim t$. Denote the degree of a vertex $s$ by $d_s$ in $G$.

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Let $A_G$ be the *adjacency matrix* of $G$, where the $(s,t)$-entry equals to 1 if $s \sim t$ and 0 otherwise. Let $d_s$ be the degree of the vertex $s$ and $D_G = diag(d_1, d_2, \cdots, d_{N_0})$ be the *diagonal matrix* of $G$. We call $L_G = D_G - A_G$ the *Laplacian matrix.*

**Definition 1.** *Given a matrix $M$, let $M(s,t)$ denote the $(s,t)$-entry of $M$. For the eigenvalue $\lambda$ of the matrix $M$, denote by $m_M(\lambda)$ the multiplicity of $\lambda$ in $M$.*

*For the $N_0$ eigenvalues of $\mathscr{L}_G$, we label them by $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N_0}$.*

**Definition 2.** *Define the normalized Laplacian spectrum on $\mathscr{L}_G$ as $\sigma = \{\lambda_1, \lambda_2, \ldots, \lambda_{N_0}\}$.*

**Definition 3.** *The probability of jumping from the current vertex $s$ to another vertex $t$ is $p_{st}$,*

$$p_{st} = \begin{cases} \frac{1}{d_s}, & if\ s \sim t, \\ 0, & otherwise. \end{cases}$$

*We call $P_G = (p_{st})_{N_0 \times N_0} = D_G^{-1} A_G$ the transition probability matrix.*

**Definition 4.** *The normalized Laplacian matrix can be expressed by*

$$\mathscr{L}_G = I - D_G^{\frac{1}{2}} P_G D_G^{-\frac{1}{2}},$$

*where $I$ is an $N_0 \times N_0$ identity marix. According to the definition of $\mathscr{L}_G$, we have that:*

$$\mathscr{L}_G(s,t) = \delta_{st} - \frac{A_G(s,t)}{\sqrt{d_s d_t}}.$$

*Where $\delta_{st}$ is the Kronecker delta.*

We often use the normalized Laplacian to characterize parameters of graphs, see [4].

**Definition 5.** [3] *The multiplicative degree-Kirchhoff index of $G$ is expressed by $Kf^*(G) = \sum\limits_{s<t} d_s d_t r_{st}$.*

**Definition 6.** *For a stationary distribution of unbiased random walks on $G$, let the transition from an initial vertex $s$ to a target vertex $t$ be selected randomly, we define the expected number of steps we need by $K_e(G)$, called the Kemeny's constant.*

**Definition 7.** *Define the number of spanning trees of $G$ by $\tau(G)$.*

**1.2. Backgrounds.** Many graph invariants, including $Kf^*(G)$ and $K_e(G)$, $\tau(G)$, can be calculated according to the spectrum of the graph. In recent years, some researchers focused on expanding a given graph by replacing each edge with another graph and characterize its spectrum in terms of the given graph.

Wang et al. [7] generalized the result of [8] by replacing each edge with $k$ triangles, i.e., they added $k$ edge-disjoint paths of length two between each two adjacent two vertices.

Huang and Li [6] further added $k$ paths of length three between each two adjacent vertices to get the so-called $k$-quadrilateral graph $Q^k(G)$. Luckily, the normalized Laplacian spectra of these resulting graphs can be characterized completely in terms of the given graph $G$. As applications, one can calculate $Kf^*(G)$, $K_e(G)$ and $\tau(G)$ of these graphs again in terms of the host graph $G$.

## 2. PRELIMINARIES

Let $n \geq 2$. For each edge $e = st$, add $2n-2$ vertices to form a complete bipartite graph, where $s$ and $t$ belong to part $X$ and part $Y$ respectively, and name the vertices in $X$ with $p_m^e(m = 1,2,\cdots,n-1)$, the vertices in $Y$ with $q_y^e(y = 1,2,\cdots,n-1)$. We denote by $Q^{(n)}(G)$ the new graph. The Figure 1 gives an example of the $Q^{(n)}(G)$ for $G = K_3$ and $n = 3$.

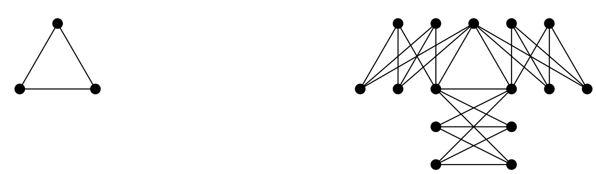

FIGURE 1. The graph $G = K_3$ and $Q^{(n)}(G)$ for $n = 3$.

Let $E_1 = |E(Q^{(n)}(G)_|$ and $N_1 = |V(Q^{(n)}(G)_|$. Obviously,

$$E_1 = n^2 E_0, \quad N_1 = N_0 + (2n-2)E_0.$$

**Lemma 2.1.** [4] *For the graph G with* $\sigma = \{0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_{N_0}\}$. *We have*

(i) $\frac{N_0}{N_0-1} \leq \lambda_{N_0} \leq 2$. *Besides,* $\lambda_{N_0} = 2$ *if and only if G is a bipartite graph;*

(ii) *For any eigenvalue* $\lambda_s$ *of* $\mathscr{L}_G$, $2-\lambda_s$ *is also an eigenvalue of* $\mathscr{L}_G$ *and* $m_{\mathscr{L}_G}(\lambda_s) = m_{\mathscr{L}_G}(2-\lambda_s)$ *otherwise.*

**Lemma 2.2.** [5] For the given connected graph $G$, the rank of the incidence matrix $B$ is

$$r(B)=\begin{cases} N_0-1, & \text{G is bipartite,}\\ N_0, & \text{otherwise.}\end{cases}$$

**Lemma 2.3.** [1] For the simple connected graph $G$, $r(L_G)=N_0-1$.

**Lemma 2.4.** *For the given graph $G$ with $\sigma=\{0=\lambda_1<\lambda_2\le\cdots\le\lambda_{N_0}\}$, we have*

(i)[3] $Kf^*(G)=2E_0\sum\limits_{s=2}^{N_0}\frac{1}{\lambda_s}$.

(ii)[2] $K_e(G)=\sum\limits_{s=2}^{N_0}\frac{1}{\lambda_s}$.

(iii)[4] $\tau(G)=\frac{1}{2E_0}\prod\limits_{s=1}^{N_0}d_s\cdot\prod\limits_{k=2}^{N_0}\lambda_k$.

(iv) $Kf^*(G)=2E_0K_e(G)$.

## 3. The Normalized Laplacian Spectrum of $Q^{(n)}(G)$

For $Q^{(n)}(G)$, denote the normalized Laplacian by $\mathscr{L}_Q$. Let $d_s^{'}$ be the degree of the vertex $s\in V(Q^{(n)}(G))$. Denote the *adjacency matrix* by $A_Q$ and the degree matrix $D_Q$. Let $N_G=D_G^{-\frac{1}{2}}A_GD_G^{-\frac{1}{2}}$ and $N_Q=D_Q^{-\frac{1}{2}}A_QD_Q^{-\frac{1}{2}}$.

At first, we consider the eigenvalue and its eigenvector in the graph $Q^{(n)}(G)$. Take a eigenvector $v=(v_1,v_2,\dots v_{N_1})^T$ for the eigenvalue $\lambda$ of $\mathscr{L}_Q$, so we have,

$$\mathscr{L}_Qv=(I-N_Q)v=\lambda v. \tag{1}$$

For $u\in V(Q^{(n)}(G))$, from the Eqn. (1), we have

$$(1-\lambda)v_u=\sum_{k=1}^{N_1}N_Q(u,k)v_k=\sum_{k=1}^{N_1}\frac{A_Q(u,k)}{\sqrt{d_u^{'}d_k^{'}}}v_k. \tag{2}$$

For simplicity, let $V_O=V(G)$. And for $s\in V_O$, let $N_s=N_G(s)$. Let $e=st\in E(G)$. By Eqn. (2), we have that

$$\begin{aligned}(1-\lambda)v_s&=\sum_{t\in N_s}\frac{v_t}{\sqrt{d_s^{'}d_t^{'}}}+\sum_{e\in E(G)\text{ is incident with } s}\sum_{l=1}^{n-1}\frac{v_{q_l^e}}{\sqrt{d_s^{'}d_{q_l^e}^{'}}}\\ &=\sum_{t\in N_s}\frac{v_t}{n\sqrt{d_sd_t}}+\sum_{e\in E(G)\text{ is incident with } s}\sum_{l=1}^{n-1}\frac{v_{q_l^e}}{n\sqrt{d_s}}\end{aligned} \tag{3}$$

$$(1-\lambda)v_t = \sum_{s\in N_t} \frac{v_s}{\sqrt{d'_t d'_s}} + \sum_{e\in E(G) \text{ is incident with } t} \sum_{l=1}^{n-1} \frac{v_{p_l^e}}{\sqrt{d'_t d'_{p_l^e}}} \tag{4}$$
$$= \sum_{s\in N_t} \frac{v_s}{n\sqrt{d_t d_s}} + \sum_{e\in E(G) \text{ is incident with } t} \sum_{l=1}^{n-1} \frac{v_{p_l^e}}{n\sqrt{d_s}}$$

Similarly, for any $t \in N_s$, we have

$$(1-\lambda)v_{p_1^e} = \frac{v_t}{\sqrt{d'_t d'_{p_1^e}}} + \frac{v_{q_1^e}}{\sqrt{d'_{p_1^e} d'_{q_1^e}}} + \frac{v_{q_2^e}}{\sqrt{d'_{p_1^e} d'_{q_2^e}}} + \cdots + \frac{v_{q_{n-1}^e}}{\sqrt{d'_{p_1^e} d'_{q_{n-1}^e}}} \tag{5}$$
$$= \frac{v_t}{n\sqrt{d_t}} + \frac{v_{q_1^e} + v_{q_2^e} + \cdots + v_{q_{n-1}^e}}{n}$$

and

$$(1-\lambda)v_{p_2^e} = \frac{v_t}{\sqrt{d'_t d'_{p_2^e}}} + \frac{v_{q_1^e}}{\sqrt{d'_{p_2^e} d'_{q_1^e}}} + \frac{v_{q_2^e}}{\sqrt{d'_{p_2^e} d'_{q_2^e}}} + \cdots + \frac{v_{q_{n-1}^e}}{\sqrt{d'_{p_2^e} d'_{q_{n-1}^e}}} \tag{6}$$
$$= \frac{v_t}{n\sqrt{d_t}} + \frac{v_{q_1^e} + v_{q_2^e} + \cdots + v_{q_{n-1}^e}}{n}.$$

And for any $s \in N_t$, we have

$$(1-\lambda)v_{q_1^e} = \frac{v_s}{\sqrt{d'_s d'_{q_1^e}}} + \frac{v_{p_1^e}}{\sqrt{d'_{q_1^e} d'_{p_1^e}}} + \frac{v_{p_2^e}}{\sqrt{d'_{q_1^e} d'_{p_2^e}}} + \cdots + \frac{v_{p_{n-1}^e}}{\sqrt{d'_{q_1^e} d'_{p_{n-1}^e}}} \tag{7}$$
$$= \frac{v_s}{n\sqrt{d_s}} + \frac{v_{p_1^e} + v_{p_2^e} + \cdots + v_{p_{n-1}^e}}{n}$$

and

$$(1-\lambda)v_{q_2^e} = \frac{v_s}{\sqrt{d'_s d'_{q_2^e}}} + \frac{v_{p_1^e}}{\sqrt{d'_{q_2^e} d'_{p_1^e}}} + \frac{v_{p_2^e}}{\sqrt{d'_{q_2^e} d'_{p_2^e}}} + \cdots + \frac{v_{p_{n-1}^e}}{\sqrt{d'_{q_2^e} d'_{p_{n-1}^e}}} \tag{8}$$
$$= \frac{v_s}{n\sqrt{d_s}} + \frac{v_{p_1^e} + v_{p_2^e} + \cdots + v_{p_{n-1}^e}}{n}.$$

**Lemma 3.1.** *Let $\lambda \neq \frac{1}{n}$, 1 and $\frac{2n-1}{n}$. If $\lambda$ is an eigenvalue of $\mathscr{L}_Q$, then $\frac{\lambda(2n-n\lambda-1)}{1-\lambda}$ is also an eigenvalue of $\mathscr{L}_G$ with $m_{\mathscr{L}_G}(\frac{\lambda(2n-n\lambda-1)}{1-\lambda}) = m_{\mathscr{L}_Q}(\lambda)$.*

**Proof:** Take an eigenvector $v = (v_1, v_2, \cdots, v_{N_1})^T$ for the the eigenvalue $\lambda$ of $\mathscr{L}_Q$. Let $e = st \in E(G)$. Since $\lambda \neq 1$, from Eqns. (5) and (6), we have $v_{p_1^e} = v_{p_2^e}$. By Eqns. (7) and (8), we can get $v_{q_1^e} = v_{q_2^e}$. Easily, we have

$$v_{p_1^e} = v_{p_2^e} = \cdots = v_{p_{n-1}^e}$$

and

$$v_{q_1^e} = v_{q_2^e} = \cdots = v_{q_{n-1}^e}.$$

For convenience, let $v_{p_1^e} = x_p, v_{q_1^e} = x_q$. According to (3)(4)(5) and (7) , we have

$$(1-\lambda)v_s = \sum_{t\in N_s} \frac{v_t}{n\sqrt{d_s d_t}} + (n-1) \sum_{e\in E(G) \text{ is incident with } s} \frac{x_q}{n\sqrt{d_s}}, \tag{9}$$

$$(1-\lambda)v_t = \sum_{s\in N_t} \frac{v_s}{n\sqrt{d_t d_s}} + (n-1) \sum_{e\in E(G) \text{ is incident with } t} \frac{x_p}{n\sqrt{d_t}},$$

$$(1-\lambda)x_p = \frac{v_t}{n\sqrt{d_t}} + \frac{n-1}{n} x_q \tag{10}$$

and

$$(1-\lambda)x_q = \frac{v_s}{n\sqrt{d_s}} + \frac{n-1}{n} x_p. \tag{11}$$

Combining Eqns. (10) and (11) , we have

$$(2n-n\lambda-1)(1-n\lambda)x_p = \frac{n(1-\lambda)}{\sqrt{d_t}} v_t + \frac{(n-1)}{\sqrt{d_s}} v_s. \tag{12}$$

Similarly, we can have

$$(2n-n\lambda-1)(1-n\lambda)x_q = \frac{n(1-\lambda)}{\sqrt{d_s}} v_s + \frac{(n-1)}{\sqrt{d_t}} v_t. \tag{13}$$

Combining Eqns. (9) and (13), for $\lambda \neq \frac{1}{n}$, 1 and $\frac{2n-1}{n}$, it follows

$$\begin{aligned} n(1-\lambda)v_s &= \frac{n-1}{(2n-n\lambda-1)(1-n\lambda)} \sum_{t\in N_s} \left(\frac{n(1-\lambda)}{d_s} v_s + \frac{n-1}{\sqrt{d_s d_t}} v_t\right) + \sum_{t\in N_s} \frac{v_t}{\sqrt{d_s d_t}} \\ &= \frac{n(n-1)(1-\lambda)}{(2n-n\lambda-1)(1-n\lambda)} v_s + \sum_{t\in N_s} \left(\frac{(n-1)^2}{(2n-n\lambda-1)(1-n\lambda)} + 1\right) \frac{v_t}{\sqrt{d_s d_t}} \end{aligned}$$

Therefore, the equation

$$\frac{n\lambda^2 - 2n\lambda + 1}{1-\lambda} v_s = \sum_{t\in N_s} \frac{v_t}{\sqrt{d_s d_t}} \tag{14}$$

holds for $\lambda \neq \frac{1}{n}$, 1 and $\frac{2n-1}{n}$.

From Eqn. (14), it is obvious that $\frac{n\lambda^2-2n\lambda+1}{1-\lambda}$ is the eigenvalue of $N_G$ when $\lambda \neq \frac{1}{n}$, 1 and $\frac{2n-1}{n}$. So for any eigenvalue $\lambda$ ($\lambda \neq \frac{1}{n}$, 1 and $\frac{2n-1}{n}$) and a corresponding eigenvector $v$ of $\mathscr{L}_Q$, $\frac{\lambda(2n-n\lambda-1)}{1-\lambda}$ and $(v_s)^T_{s\in V_O}$ are an eigenvalue and a corresponding eigenvector of $\mathscr{L}_G$, respectively. This implies that $m_{\mathscr{L}_G}(\frac{\lambda(2n-n\lambda-1)}{1-\lambda}) \geq m_{\mathscr{L}_Q}(\lambda)$.

On the other hand, for any eigenvalue $\frac{\lambda(2n-n\lambda-1)}{1-\lambda}$ $(\neq 0,2)$ and a corresponding eigenvector $(v_s)^T_{s\in V_O}$ of $\mathscr{L}_G$, $\lambda$ is a eigenvalue of $\mathscr{L}_Q$ and the vector determined by $(v_s)^T_{s\in V_O}$ and Eqn. (12) and Eqn. (13) together is a corresponding eigenvector. Hence $m_{\mathscr{L}_G}(\frac{\lambda(2n-n\lambda-1)}{1-\lambda}) \leq m_{\mathscr{L}_Q}(\lambda)$. So we have that $m_{\mathscr{L}_G}(\frac{\lambda(2n-n\lambda-1)}{1-\lambda}) = m_{\mathscr{L}_Q}(\lambda)$. □

**Theorem 3.2.** *For the given simple connected graph G, we have the followings*

(i) $m_{\mathscr{L}_Q}(0) = 1$. *And* $m_{\mathscr{L}_Q}(2) = 1$ *if G is bipartite;*

(ii) *For* $\lambda \neq 0$ *and* 2, *both* $\frac{\lambda+2n-1+\sqrt{\lambda^2-2\lambda+4n^2-4n+1}}{2n}$ *and* $\frac{\lambda+2n-1-\sqrt{\lambda^2-2\lambda+4n^2-4n+1}}{2n}$ *are the eigenvalues of* $\mathscr{L}_Q$ *with*
$m_{\mathscr{L}_Q}(\frac{\lambda+2n-1+\sqrt{\lambda^2-2\lambda+4n^2-4n+1}}{2n}) = m_{\mathscr{L}_Q}(\frac{\lambda+2n-1-\sqrt{\lambda^2-2\lambda+4n^2-4n+1}}{2n}) = m_{\mathscr{L}_G}(\lambda)$;

(iii) *If G is non-bipartite,* $m_{\mathscr{L}_Q}(\frac{1}{n}) = E_0 - N_0$;

(iv) *If G is bipartite,* $m_{\mathscr{L}_Q}(\frac{1}{n}) = E_0 - N_0 + 1$;

(v) $m_{\mathscr{L}_Q}(\frac{2n-1}{n}) = E_0 - N_0 + 1$;

(vi) $m_{\mathscr{L}_Q}(1) = (2n-4)E_0 + N_0$.

**Proof:** (i) It is obvious from Lemma 2.1.

(ii) Assume $x$ is the eigenvalue of $\mathscr{L}_Q$ and $x \neq \frac{1}{n}$, 1 and $\frac{2n-1}{n}$. By Lemma 3.1, we have that $\lambda = \frac{x(2n-nx-1)}{1-x}$, for $\lambda \neq 0$ and 2. Thus $x = \frac{\lambda+2n-1\pm\sqrt{\lambda^2-2\lambda+4n^2-4n+1}}{2n}$.

Since each of the eigenvalues $\lambda$ ( $\lambda \neq \frac{1}{n}$, 1 and $\frac{2n-1}{n}$) and its multiplicity in $Q^{(n)}(G)$ have been determined in the statement above, here we only need to consider the eigenvalues $\lambda \in \{\frac{1}{n}, 1, \frac{2n-1}{n}\}$.

Let $v = (v_1, v_2, \cdots, v_{N_1})^T$ be the eigenvector corresponding to the eigenvalue $\lambda$ of $\mathscr{L}_Q$. Let $e \in E(G)$ with end vertices $s$ and $t$. For $n \geq 2$, substituting $\lambda = \frac{1}{n}$ into Eqns. (5) and (6), we have $v_{p_1^e} = v_{p_2^e}$. By Eqns. (7) and (8), we can get $v_{q_1^e} = v_{q_2^e}$. Easily, we have

$$v_{p_1^e} = v_{p_2^e} = \cdots = v_{p_{n-1}^e}$$

and

$$v_{q_1^e} = v_{q_2^e} = \cdots = v_{q_{n-1}^e}.$$

For convenience, let $v_{p_1^e} = x_p, v_{q_1^e} = x_q$. When $\lambda = \frac{1}{n}$, according to (3)(4)(5) and (7), we have

$$(n-1)v_s = \sum_{t\in N_s} \frac{v_t}{\sqrt{d_s d_t}} + (n-1) \sum_{e\in E(G) \text{ is incident with } s} \frac{x_q}{\sqrt{d_s}}, \tag{15}$$

$$(n-1)v_t = \sum_{s\in N_t} \frac{v_s}{\sqrt{d_t d_s}} + (n-1) \sum_{e\in E(G) \text{ is incident with } t} \frac{x_p}{\sqrt{d_t}},$$

$$(n-1)x_p = \frac{v_t}{\sqrt{d_t}} + (n-1)x_q \tag{16}$$

and

$$(n-1)x_q = \frac{v_s}{\sqrt{d_s}} + (n-1)x_p. \tag{17}$$

Combining Eqns. (16) and (17), we get

$$\frac{v_s}{\sqrt{d_s}} = -\frac{v_t}{\sqrt{d_t}},\ s \in V_O,\ t \in N_s. \tag{18}$$

(iii) Let $G$ be non-bipartite. Take an odd cycle $C$ of length $h$ with vertices $s_1, s_2, ..., s_h$ in turn. By Eqn. (18), we have

$$\frac{v_{i_1}}{\sqrt{d_{s_1}}} = -\frac{v_{s_2}}{\sqrt{d_{s_2}}} = \cdots = \frac{v_{i_h}}{\sqrt{d_{s_h}}} = -\frac{v_{s_1}}{\sqrt{d_{s_1}}},$$

which implies that $v_{s_k} = 0$ for any $s_k$, and hence we have

$$v_s = 0 \text{ for all } s \in V_O. \tag{19}$$

Together with Eqns. (15) and (16), we have that

$$\sum_{e\in E(G) \text{ is incident with } s} x_q = 0, \text{ for all } s \in V_O \tag{20}$$

and

$$x_p = x_q, \text{ for all } e \in E(G). \tag{21}$$

Therefore, the eigenvectors $v = (v_1, v_2, ..., v_{N_1})^T$ corresponding to $\lambda = \frac{1}{n}$ can be determined by Eqns. (19)(20)(21). According to the construction of $Q^{(n)}(G)$, let $\mathbf{x} = (x_q)^T$ be the $E_0$ dimensional vector. From Eqns. (19)(20)(21), we have $B\mathbf{x} = 0$. According to Lemma 2.2, the basic solution system contains $E_0 - N_0$ linearly independent elements, so we have $m_{\mathscr{L}_Q}(\frac{1}{n}) = E_0 - N_0$.

(iv) Let $G$ be bipartite. Combining Eqn. (18) and Eqn. (15), we have

$$\frac{n}{n-1}\sqrt{d_s}v_s = \sum_{e\in E(G) \text{ is incident with } s} x_q, \quad s \in V_O. \tag{22}$$

Let $\frac{nv_1}{(n-1)\sqrt{d_1}} = w_1$. Denote by $X$ and $Y$ the partite sets of the graph $G$ and without loss of generality, let $1 \in X$. Then from Eqn. (18), we have that $\frac{nv_s}{(n-1)\sqrt{d_s}} = w_1$ if $s \in X$, and $\frac{nv_s}{(n-1)\sqrt{d_s}} = -w_1$ if $s \in Y$. According to Eqn. (22), we have that

$$\begin{aligned} \sum_{e\in E(G) \text{ is incident with } s} x_q - d_s w_1 = 0, \text{ if } s \in X, \\ \sum_{e\in E(G) \text{ is incident with } s} x_q + d_s w_1 = 0, \text{ if } s \in Y. \end{aligned} \tag{23}$$

Therefore, the eigenvectors $v = (v_1, v_2, \ldots, v_{N_1})^T$ corresponding to $\lambda = \frac{1}{n}$ can be determined by Eqns. (10)(18) and (23). According to the definition of $Q^{(n)}(G)$, let $\mathbf{x} = (x_q)^T$ be the $E_0$ dimensional vector.

For convenience, assume that the first $|X|$ rows in the incident matrix $B$ of $G$ correspond to the vertices in $X$. Hence the matrix $B$ can be written as $B = \binom{B_X}{B_Y}$. Let $D_X$ and $D_Y$ denote the volume vectors which consist of degree sequences of vertices of $X$ and $Y$, respectively. We denote matrix $C$ by

$$C = \begin{pmatrix} B_X & -D_X \\ B_Y & D_Y \end{pmatrix}.$$

Hence Eqns. (10)(18) and (23) are obviously equivalent to $C\binom{\mathbf{x}}{w_1} = 0$.

By Lemma 2.2, the rank of $B$ is $N_0 - 1$ when $G$ is bipartite. We denote the volume vectors of $C$ by $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{E_0}, \mathbf{e}_0$ from left to right. Assume that $\mathbf{e}_0$ is linearly related to the $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{E_0}$, it means that, there exist constants $c_1, c_2, \cdots, c_{E_0}$ making the followed formula true,

$$\mathbf{e}_0 = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + \cdots + c_{E_0}\mathbf{e}_{E_0}. \tag{24}$$

For every volume of $C$, there are two entries 1 in $B_X$ and $B_Y$, respectively. From Eqn. (24), we have $c_1 + c_2 + \cdots + c_{E_0} = \sum_{i=1}^{|X|} -d_s$ and $c_1 + c_2 + \cdots + c_{E_0} = \sum_{i=|X|+1}^{N_0} d_s$. This implies that $\sum_{i=1}^{|X|}(-d_s) = \sum_{i=|X|+1}^{N_0} d_s$. However, $d_s > 0$ for each $s = 1, 2, \cdots, N_0$. Hence it is obvious that

$\sum_{i=1}^{|X|} -d_s = \sum_{i=|X|+1}^{N_0} d_s$ is impossible. Thus we get a contradiction. So $\mathbf{e}_0$ and $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_{E_0}$ are linearly independent, i.e., the rank of matrix $C$ is $r(C) = r(B) + 1 = N_0$.

Therefore, the basic solution space for $C\binom{\mathbf{x}}{w_1} = 0$ contains $E_0 - N_0 + 1$ linearly independent elements when $G$ is bipartite, i.e., $m_{\mathscr{L}_Q}(\frac{1}{n}) = E_0 - N_0 + 1$.

(v) For $n \geq 2$, substituting $\lambda = \frac{2n-1}{n}$ into Eqns. (5) and (6), we have $v_{p_1^e} = v_{p_2^e}$. By Eqns. (7) and (8), we can get $v_{q_1^e} = v_{q_2^e}$. Easily, we have

$$v_{p_1^e} = v_{p_2^e} = \cdots = v_{p_{n-1}^e}$$

and

$$v_{q_1^e} = v_{q_2^e} = \cdots = v_{q_{n-1}^e}.$$

For convenience, let $v_{p_1^e} = x_p, v_{q_1^e} = x_q$. When $\lambda = \frac{2n-1}{n}$, according to (3)(4)(5) and (7), we have

$$(1-n)v_s = \sum_{t \in N_s} \frac{v_t}{\sqrt{d_s d_t}} + (n-1) \sum_{e \in E(G) \text{ is incident with } s} \frac{x_q}{\sqrt{d_s}}, \tag{25}$$

$$(1-n)v_t = \sum_{s \in N_t} \frac{v_s}{\sqrt{d_t d_s}} + (n-1) \sum_{e \in E(G) \text{ is incident with } t} \frac{x_p}{\sqrt{d_t}},$$

$$(1-n)x_p = \frac{v_t}{\sqrt{d_t}} + (n-1)x_q \tag{26}$$

and

$$(1-n)x_q = \frac{v_s}{\sqrt{d_s}} + (n-1)x_p. \tag{27}$$

Combining Eqns. (26) and (27), we get

$$\frac{v_s}{\sqrt{d_s}} = \frac{v_t}{\sqrt{d_t}}, s \in V_O,\ t \in N_s. \tag{28}$$

Let $\frac{v_s}{\sqrt{d_s}} = w_2$. Substituting Eqn. (28) into Eqns. (25) and (26), we have

$$\sum_{e \in E(G) \text{ is incident with } s} x_q = \frac{n}{1-n} w_2 d_s, \quad s \in V_O. \tag{29}$$

and

$$x_p + x_q = \frac{w_2}{1-n}, \text{ for all } e \in E(G). \tag{30}$$

According to Eqn. (30), we have

$$\sum_{s\in V_O}\sum_{t\in N_s} x_q = (n-1)\sum_{e\in E(G)}(x_p+x_q) = -w_2E_0. \tag{31}$$

On the other hand, using Eqn. (29), we also have

$$\sum_{s\in V_O}\sum_{t\in N_s} x_q = \frac{n}{1-n}w_2\sum_{s\in V_O} d_s = \frac{2nw_2E_0}{1-n} \tag{32}$$

Thus, we have $w_2 = 0$, which means that $v_s = 0$ for any $s \in V_O$. So, $v = (v_1, v_2, \cdots, v_{N1})^T$ respect to $\lambda = \frac{2n-1}{n}$ can be completely obtained by equations below

$$v_s = 0, \quad s \in V_O, \tag{33}$$

$$\sum_{e\in E(G) \text{ is incident with } s} x_q = 0 \tag{34}$$

and

$$x_p + x_q = 0. \tag{35}$$

We can describe the adjacency matrix of $Q^{(n)}(G)$ as

$$A_Q = \begin{pmatrix} A_G & \overbrace{B_1\cdots B_1}^{n-1} & \overbrace{B_2\cdots B_2}^{n-1} \\ B_1^T & 0\cdots 0 & I_{E0}\cdots I_{E0} \\ \vdots & \vdots & \vdots \\ B_1^T & 0\cdots 0 & I_{E0}\cdots I_{E0} \\ B_2^T & 0\cdots 0 & I_{E0}\cdots I_{E0} \\ \vdots & \vdots & \vdots \\ B_2^T & 0\cdots 0 & I_{E0}\cdots I_{E0} \end{pmatrix},$$

where $I_{E_0}$ is an $E_0 \times E_0$ identity matrix. It is routine to check that $B_1 + B_2 = B$, $B_1B_2^T + B_2B_1^T = A_G$ and $B_1B_1^T + B_2B_2^T = D_G$.

From Eqns. (33)(34) and (35) ,we have $(B_1 - B_2)\mathbf{x} = 0$. Combining Lemma (2.3), we have $r(B_1 - B_2) = r[(B_1 - B_2)(B_1 - B_2)^T] = r[(B_1B_1^T + B_2B_2^T) - (B_1B_2^T + B_2B_1^T)] = r(D_G - A_G) = r(L_G) = N_0 - 1$. So the basic solution space contains $E_0 - N_0 + 1$ linearly independent elements, i.e., $m_{\mathscr{L}_Q}(\frac{2n-1}{n}) = E_0 - N_0 + 1$.

(vi) For $n \geq 2$ and $\lambda = 1$ ,from Eqns. (5) and (7), it is clear that

$$v_{q_1^e} + v_{q_2^e} + \cdots + v_{q_{n-1}^e} + \frac{v_t}{\sqrt{d_t}} = 0$$

and

$$v_{p_1^e} + v_{p_2^e} + \cdots + v_{p_{n-1}^e} + \frac{v_s}{\sqrt{d_s}} = 0.$$

For convenience, for each edge $e_i \in E(G)$, $i = 1, 2, \ldots, E_0$, denote by $s_i$ and $t_i$ the end vertices of $e_i$. So, we have the following linear equation system

$$\begin{cases}
v_{q_1^{e_1}} + v_{q_2^{e_1}} + v_{q_3^{e_1}} + \cdots + v_{q_{n-1}^{e_1}} + \dfrac{v_{t_1}^{e_1}}{\sqrt{d_{t_1}^{e_1}}} = 0, \\
v_{q_1^{e_2}} + v_{q_2^{e_2}} + v_{q_3^{e_2}} + \cdots + v_{q_{n-1}^{e_2}} + \dfrac{v_{t_2}^{e_2}}{\sqrt{d_{t_2}^{e_2}}} = 0, \\
\vdots \\
v_{q_1^{e_{E_0}}} + v_{q_2^{e_{E_0}}} + v_{q_3^{e_{E_0}}} + \cdots + v_{q_{n-1}^{e_{E_0}}} + \dfrac{v_{t_{E_0}}^{e_{E_0}}}{\sqrt{d_{t_{E_0}}^{e_{E_0}}}} = 0. \\
v_{p_1^{e_1}} + v_{p_2^{e_1}} + v_{p_3^{e_1}} + \cdots + v_{p_{n-1}^{e_1}} + \dfrac{v_{s_1}^{e_1}}{\sqrt{d_{s_1}^{e_1}}} = 0, \\
v_{p_1^{e_2}} + v_{p_2^{e_2}} + v_{p_3^{e_2}} + \cdots + v_{p_{n-1}^{e_2}} + \dfrac{v_{s_2}^{e_2}}{\sqrt{d_{s_2}^{e_2}}} = 0, \\
\vdots \\
v_{p_1^{e_{E_0}}} + v_{p_2^{e_{E_0}}} + v_{p_3^{e_{E_0}}} + \cdots + v_{p_{n-1}^{e_{E_0}}} + \dfrac{v_{s_{E_0}}^{e_{E_0}}}{\sqrt{d_{s_{E_0}}^{e_{E_0}}}} = 0.
\end{cases} \tag{36}$$

The corresponding coefficient matrix contains the following $2E_0 \times (2n-2)E_0$ submatrix

$$\begin{pmatrix}
\underbrace{1 \ \cdots \ 1}_{n-1} & 0 \ \cdots \ 0 & \cdots & 0 \ \cdots \ 0 \\
0 \ \cdots \ 0 & \underbrace{1 \ \cdots \ 1}_{n-1} & \cdots & 0 \ \cdots \ 0 \\
 & \vdots & & \\
0 \ \cdots \ 0 & 0 \ \cdots \ 0 & \cdots & \underbrace{1 \ \cdots \ 1}_{n-1}
\end{pmatrix}.$$

Clearly, the submatrix above is of rank $2E_0$. Hence the basic solution space for (36) cantains $(2n-4)E_0+N_0$ linearly independent elements. Therefore, $m_{\mathscr{L}_Q}(1)=(2n-4)E_0+N_0$. This completes the proof of the theorem. □

## 4. Some Applications

Let $Q_0^{(n)}(G)=G$ and $Q_r^{(n)}(G)=Q^{(n)}(Q_{r-1}^{(n)}(G))$ for $r\geq 1$. Denote the number of edges of $Q_r^{(n)}(G)(r\geq 0)$ by $E_r$, and denote the number of vertices by $N_r$. By the construction of $Q_r^{(n)}(G)$, we have

$$E_r=n^2E_{r-1},\quad N_r=N_{r-1}+(2n-2)E_{r-1}.$$

Hence

$$E_r=n^{2r}E_0,\quad N_r=N_0+\frac{2(n^{2r}-1)E_0}{n+1}. \tag{37}$$

For convenience, for $Q_r^{(n)}(G)$ and $r\geq 0$, we use $\mathscr{L}_r$ and $\sigma_r$ to denote the normalized Laplacian and its spectrum, respectively. From Theorem 3.2, we have the theorem next.

**Theorem 4.1.** *For $r\geq 2, n\geq 2$,*

(i) *if $G$ is non-bipartite,*

$$\sigma_r=\{\frac{x+2n-1\pm\sqrt{x^2-2x+4n^2-4n+1}}{2n}|x\in\sigma_{r-1}\backslash\{0\}\}\cup\{0,1,\frac{1}{n},\frac{2n-1}{n}\},$$

*where*

$m_{\mathscr{L}_r}(\frac{x+2n-1\pm\sqrt{x^2-2x+4n^2-4n+1}}{2n})=m_{\mathscr{L}_{r-1}}(x)$ *for* $x\in\sigma_{r-1}\backslash\{0\}$,

$m_{\mathscr{L}_r}(0)=1$,

$m_{\mathscr{L}_r}(1)=(2n-4)E_{r-1}+N_{r-1}$,

$m_{\mathscr{L}_r}(\frac{1}{n})=E_{r-1}-N_{r-1}$

*and* $m_{\mathscr{L}_r}(\frac{2n-1}{n})=E_{r-1}-N_{r-1}+1$.

(ii) *if $G$ is bipartite,*

$$\sigma_r=\{\frac{x+2n-1\pm\sqrt{x^2-2x+4n^2-4n+1}}{2n}|x\in\sigma_{r-1}\backslash\{0,2\}\}\cup\{0,1,2,\frac{1}{n},\frac{2n-1}{n}\},$$

*where*

$m_{\mathscr{L}_r}(\frac{x+2n-1\pm\sqrt{x^2-2x+4n^2-4n+1}}{2n})=m_{\mathscr{L}_{r-1}}(x)$ *for* $x\in\sigma_{r-1}\backslash\{0,2\}$,

$m_{\mathscr{L}_r}(0)=1$,

$m_{\mathscr{L}_r}(1)=(2n-4)E_{r-1}+N_{r-1}$,

$m_{\mathscr{L}_r}(2)=1$, $m_{\mathscr{L}_r}(\frac{1}{n})=E_{r-1}-N_{r-1}+1$

*and* $m_{\mathscr{L}_r}(\frac{2n-1}{n})=E_{r-1}-N_{r-1}+1$.

**Theorem 4.2.** *For $r\geq 1$ and $n\geq 2$,*

$$\begin{aligned}Kf^*(Q_r^{(n)}(G)) &= (2n-1)^r n^{2r} Kf^*(G)+\frac{4n^{2r}(n(3n-1)n^{2r}-(n-1)^2-(2n^2+n-1)(2n-1)^r)}{(n-1)(n+1)(2n-1)}{E_0}^2\\ &-\frac{2(n-1)n^{2r}((2n-1)^r-1)}{2n-1}E_0N_0-\frac{n^{2r}((2n-1)^r-1)}{2n-1}E_0.\end{aligned} \tag{38}$$

**Proof:** Since $0=\lambda_1<\lambda_2\leq\cdots\leq\lambda_{N_0}$. Whether $G$ is bipartite or not, from Theorem 4.1 and Lemma 2.4, we have (i)

$$\begin{aligned}Kf^*(Q^{(n)}(G)) &= 2E_1\left[\sum_{i=2}^{N_0}(\frac{1}{f_1(\lambda_s)}+\frac{1}{f_2(\lambda_s)})+N_0+(2n-4)E_0+n(E_0-N_0)+\frac{n}{2n-1}(E_0-N_0+1)\right]\\ &= 2n^2E_0\sum_{i=2}^{N_0}(1+\frac{2n-1}{\lambda_s})+2n^2E_0(\frac{6n^2-10n+4}{2n-1}E_0-\frac{2n^2-2n+1}{2n-1}N_0+\frac{n}{2n-1})\\ &= (2n-1)n^2Kf^*(G)+\frac{4n^2(n-1)(3n-2)}{2n-1}{E_0}^2-\frac{4n^2(n-1)^2}{2n-1}E_0N_0-\frac{2n^2(n-1)}{2n-1}E_0.\end{aligned} \tag{39}$$

It follows from Eqns. (37) and (39) that

$$\begin{aligned}Kf^*(Q_r^{(n)}(G)) &= (2n-1)n^2Kf^*(Q_{r-1}^{n}(G))+\frac{4n^2(n-1)(3n-2)}{2n-1}{E_{r-1}}^2-\frac{4n^2(n-1)^2}{2n-1}E_{r-1}N_{r-1}-\frac{2n^2(n-1)}{2n-1}E_{r-1}.\\ &= (2n-1)^r n^{2r}Kf^*(G)+\frac{4(3n-2)n^{2r}(n^{2r}-(2n-1)^r)}{(n-1)(2n-1)}{E_0}^2-\frac{2(n-1)n^{2r}((2n-1)^r-1)}{2n-1}E_0N_0\\ &+\frac{4n^{2r}(n-1)((2n-1)^{r-1}-1)}{(n+1)(2n-1)}{E_0}^2-\frac{8n^{2r+2}(n^{2r-2}-(2n-1)^{r-1})}{(n+1)(2n-1)}{E_0}^2-\frac{n^{2r}((2n-1)^r-1)}{2n-1}E_0\\ &= (2n-1)^r n^{2r}Kf^*(G)+\frac{4n^{2r}(n(3n-1)n^{2r}-(n-1)^2-(2n^2+n-1)(2n-1)^r)}{(n-1)(n+1)(2n-1)}{E_0}^2\\ &-\frac{2(n-1)n^{2r}((2n-1)^r-1)}{2n-1}E_0N_0-\frac{n^{2r}((2n-1)^r-1)}{2n-1}E_0.\end{aligned}$$

The proof is completed. □

**Theorem 4.3.** *For $r\geq 1$ and $n\geq 2$,*

$$\begin{aligned}K_e(Q_r^{(n)}(G)) &= (2n-1)^rK_e(G)+\frac{2(n(3n-1)n^{2r}-(n-1)^2-(2n^2+n-1)(2n-1)^r)}{(n-1)(n+1)(2n-1)}E_0\\ &-\frac{(n-1)((2n-1)^r-1)}{2n-1}N_0-\frac{(2n-1)^r-1}{2(2n-1)}.\end{aligned}$$

**Proof:**

By Eqns. (37) and (38) and Lemma 2.4 (iv), we can get

$$
\begin{aligned}
K_e(Q_r^{(n)}(G)) &= \frac{1}{2E_r}Kf^*(Q_r^{(n)}(G)) \\
&= \frac{1}{2n^{2r}E_0}((2n-1)^r n^{2r}Kf^*(G) + \frac{4n^{2r}(n(3n-1)n^{2r}-(n-1)^2-(2n^2+n-1)(2n-1)^r)}{(n-1)(n+1)(2n-1)}{E_0}^2 \\
&\quad - \frac{2(n-1)n^{2r}((2n-1)^r-1)}{2n-1}E_0N_0 - \frac{n^{2r}((2n-1)^r-1)}{2n-1}E_0) \\
&= (2n-1)^r K_e(G) + \frac{2(n(3n-1)n^{2r}-(n-1)^2-(2n^2+n-1)(2n-1)^r)}{(n-1)(n+1)(2n-1)}E_0 \\
&\quad - \frac{(n-1)((2n-1)^r-1)}{2n-1}N_0 - \frac{(2n-1)^r-1}{2(2n-1)}.
\end{aligned}
$$

The proof is completed. □

**Theorem 4.4.** *For $r \geq 1$ and $n \geq 2$,*

$$\tau(Q_r^{(n)}(G)) = n^{(2n-4)s_1+2s_2-2r} \cdot (2n-1)^{s_1-s_2+r} \cdot \tau(G),$$

*where $s_1 = \sum_{i=0}^{r-1} E_i = \frac{n^{2r}-1}{n^2-1}E_0$, and $s_2 = \sum_{i=0}^{r-1} N_i = rN_0 + \frac{2}{n+1}(\frac{n^{2r}-1}{n^2-1} - r)E_0$.*

**Proof:** For $Q^{(n)}(G)$, assume that $0 = \lambda_1^{'} < \lambda_2^{'} \leq \cdots \leq \lambda_{N_1}^{'}$. Whether $G$ is bipartite or not, according to Lemma 2.4 (iii), we obtain that

$$\frac{\tau(Q^{(n)}(G))}{\tau(G)} = \frac{n^{N_0+(2n-2)E_0-2} \cdot \prod_{i=2}^{N_1} \lambda_i^{'}}{\prod_{i=2}^{N_0} \lambda_s}. \tag{40}$$

And we can get by Theorem 3.2

$$
\begin{aligned}
\prod_{i=2}^{N_1} \lambda_i^{'} &= (\frac{1}{n})^{E_0-N_0} \cdot (\frac{2n-1}{n})^{E_0-N_0+1} \cdot \prod_{i=2}^{N_0} f_1(\lambda_s)f_2(\lambda_s) \\
&= (\frac{1}{n})^{E_0-N_0} \cdot (\frac{2n-1}{n})^{E_0-N_0+1} \cdot \prod_{i=2}^{N_0} \frac{\lambda_s}{n} \\
&= \frac{(2n-1)^{E_0-N_0+1}}{n^{2E_p0-N_0}} \prod_{i=2}^{N_0} \lambda_s.
\end{aligned} \tag{41}
$$

By Eqns. (40) and (41), we have

$$\tau(Q^{(n)}(G)) = n^{(2n-4)E_0+2N_0-2} \cdot (2n-1)^{E_0-N_0+1} \cdot \tau(G).$$

And from the recursive relation, we have

$$\begin{aligned}\tau(Q_r^{(n)}(G)) &= n^{(2n-4)E_{r-1}+2N_{r-1}-2}(2n-1)^{E_{r-1}-N_{r-1}+1}\tau(Q_{r-1}^{(n)}(G))\\ &= n^{(2n-4)\sum_{i=0}^{r-1}E_i+2\sum_{i=0}^{r-1}N_i-2r}(2n-1)^{\sum_{i=0}^{r-1}E_i-\sum_{i=0}^{r-1}N_i+r}\tau(G)\\ &= n^{(2n-4)s_1+2s_2-2r}(2n-1)^{s_1-s_2+r}\tau(G).\end{aligned}$$

The proof is completed. □

## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

## REFERENCES


[1] R.B. Bapat, Graphs and Matrices. Springer, New York, 2010.

[2] S. Butler, Algebraic aspects of the normalized Laplacian, in: A. Beveridge, J.R. Griggs, L. Hogben, G. Musiker, P. Tetali (Eds.), Recent Trends in Combinatorics, Springer International Publishing, Cham, 2016: pp. 295-315.

[3] H. Chen, F. Zhang, Resistance distance and the normalized Laplacian spectrum, Discrete Appl. Math. 155 (2007), 654?661.

[4] F.R. Chung, Spectral Graph Theory, American Mathematical Society, Providence, RI, 1997.

[5] D. Cvetkovič, P. Rowlinson, S. Simić, An introduction to the theory of graph spectrta, in: London Mathematical Society Student Texts, Cambridge University, London, 2010.

[6] J. Huang, S.C. Li, The normalized Laplacians on both k-triangle graph and k-quadrilateral graph with their applications, Appl. Math. Comput. 320 (2018), 213-225.

[7] C.Y. Wang, Z.L. Guo, S.C. Li, Expected hitting times for random walks on the $k$ -triangle graph and their applications, Appl. Math. Comput. 338 (2018), 698-710.

[8] P.C. Xie, Z.Z. Zhang, F. Comellas, On the spectrum of the normalized Laplacian of iterated triangulations of graphs, Appl. Math. Comput. 273 (2016), 1123-1129.