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SUM OF POWERS OF NATURAL NUMBERS VIA BINOMIAL COEFFICIENTS

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Abstract. In this paper, many properties of several sequences such as Binomial coefficients, the falling numbers, and Stirling numbers of the second kind are discussed. We use these properties to find recurrence equations for the sums of powers $\sigma_m(n) = \sum_{k=1}^n k^m$ in term of Binomial coefficients and Stirling numbers.

Keywords: Stirling numbers; sum of powers; difference operators; sum of powers of positive integers.

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1. INTRODUCTION

Define the sum of the m -th power of the first n positive integers as

$$\sigma_m(n) = \sum_{k=1}^n k^m.$$

For example, one can easily, using mathematical induction, prove that $\sigma_1(n) = \frac{n(n+1)}{2}$, $\sigma_2(n) = \frac{n(n+1)(2n+1)}{6}$, $\sigma_3(n) = \left[\frac{n(n+1)}{2} \right]^2$, and $\sigma_4(n) = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$. Also, it can be proved that if $a = \sigma_1(n) = \frac{n(n+1)}{2}$, then $\sigma_3(n) = a^2$, $\sigma_5(n) = \frac{4a^3 - a^2}{3}$, $\sigma_7(n) = \frac{6a^4 - 4a^3 + a^2}{3}$. Some authors call the polynomials in a on the right-hand sides of these identities Faulhaber polynomials, see [6].

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Faulhaber also knew that if a sum for an odd power is given by

$$\sum_{k=1}^n k^{2m+1} = c_1 a^2 + c_2 a^3 + \cdots + c_m a^{m+1}$$

then the sum for the even power just below is given by

$$\sum_{k=1}^n k^{2m} = \frac{n+1/2}{2m+1} (2c_1 a + 3c_2 a^2 + \cdots + (m+1)c_m a^m).$$

For further properties of $\{\sigma_m(n)\}_{m=0}^{\infty}$, see [5] and [8].

In this paper, we are interested in constructing recurrence equations for $\sigma_m(n)$, this will allow us, for example, to find $\sigma_{m+1}(n)$ using $\sigma_m(n)$. Also, we will be able to find $\sigma_{m+2}(n)$ using $\sigma_{m+1}(n)$ and $\sigma_m(n)$. We will give definitions and needed results in subsections 2.1, 2.2, and 2.3. In section 3, we construct and prove recurrence equations for $\sigma_m(n)$.

2. PRELIMINARIES

2.1. Binomial coefficients and falling numbers. Let \mathbb{N}_0 be the set of nonnegative integers. For $n \in \mathbb{N}_0$, the falling factorial numbers $(x)_n$, introduced by Leo August Pochhammer, are given as

$$(1) \quad (x)_n = x(x-1)(x-2)\cdots(x-n+1) = \frac{\Gamma(x+1)}{\Gamma(x+1-n)}.$$

where $(x)_0 = 1$. It is clear that $(x)_n = \frac{\Gamma(x+1)}{\Gamma(x+1-n)}$. In particular, for $m, n \in \mathbb{N}_0$

$$(2) \quad (m)_n = \frac{m!}{(m-n)!} \text{ where } m \geq n.$$

The binomial coefficient, denoted by $\binom{n}{k}$ is the coefficient of the x^k -term in the polynomial expansion of $(1+x)^n$. It can be found by the formula

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Clearly, using (2), for the binomial coefficients and falling numbers are related as

$$(3) \quad \binom{n}{k} = \frac{(n)_k}{k!}.$$

2.2. The forward difference operator.

Definition 2.1. Let $\mathcal{S}(\mathbb{N})$ be the set of complex valued sequences over \mathbb{N} . For $u \in \mathcal{S}(\mathbb{N})$, the forward difference operators $\Delta : \mathcal{S}(\mathbb{N}) \longrightarrow \mathcal{S}(\mathbb{N}) : u \mapsto \Delta u$ is defined as:

$$(\Delta u)(k) = u(k+1) - u(k).$$

Example 2.2. if $u(k) = (k)_n$, then

$$\begin{aligned} (\Delta u)(k) &= u(k+1) - u(k) \\ &= (k+1)_n - (k)_n \\ &= \frac{\Gamma(k+2)}{\Gamma(k+2-n)} - \frac{\Gamma(k+1)}{\Gamma(k+1-n)} \\ &= \frac{\Gamma(k+1)}{\Gamma(k+1-n)} \left(\frac{k+1}{k+1-n} - 1 \right) \\ &= n \frac{\Gamma(k+1)}{\Gamma(k+1-n)} \frac{1}{k+1-n} \\ &= n \frac{\Gamma(k+1)}{\Gamma(k+1-(n-1))} \\ &= n(k)_{n-1}. \end{aligned}$$

Therefore,

$$(4) \quad \Delta \left((k)_n \right) = n(k)_{n-1}.$$

Consequently,

$$(5) \quad \Delta \left(\binom{k}{n} \right) = \binom{k}{n-1}.$$

Taking the inverse difference operator for both sides of (4), we get $(k)_n = n\Delta^{-1} \left((k)_{n-1} \right)$. Consequently,

$$(6) \quad \Delta^{-1} \left((k)_n \right) = \frac{(k)_{n+1}}{n+1} + C.$$

This also gives

$$(7) \quad \Delta^{-1} \left(\binom{k}{n} \right) = \binom{k}{n+1} + C.$$

Following the same arguments of Example 17, one can prove the following result which is needed later.

Proposition 2.3. For $s \in \mathbb{N}_0$, if $u(k) = (k+s)_n$, then

- (1) $(\Delta u)(k) = n(k+s)_{n-1}$.
- (2) $(\Delta^{-1}u)(k) = \frac{(k+s)_{n+1}}{n+1} + c$.

This proposition leads to the following result.

Proposition 2.4. If $u(k) = \binom{k+s}{n}$, then

- (1) $(\Delta u)(k) = \binom{k+s}{n-1}$.
- (2) $(\Delta^{-1}u)(k) = \binom{k+s}{n+1} + c$.

Also, it is easy to show that the forward difference operator satisfies the following proposition

$$(8) \quad \sum_{k=m}^{n-1} (\Delta f)(k) = f(n) - f(m).$$

For example, using (7) and for $j \in \mathbb{N}$

$$(9) \quad \sum_{k=1}^n \binom{k}{j} = \binom{n+1}{j+1} - \binom{1}{j+1} = \binom{n+1}{j+1}.$$

Now, replacing f in (8) by $\Delta^{-1}F$ we get that,

$$(10) \quad \sum_{k=1}^{n-1} F(k) = (\Delta^{-1}F)(n) - (\Delta^{-1}F)(1).$$

This equations defines the inverse difference operator as

$$(11) \quad (\Delta^{-1}F)(n) = C + \sum_{k=1}^{n-1} F(k).$$

For further properties for difference operators and their applications, see [9], [1], [2], [4] and [3].

2.3. Stirling numbers. For $n \in \mathbb{N}_0$, the sequence $\left\{ \binom{n}{k} \right\}, k = 0, 1, 2, \dots, n$, which satisfies

$$(12) \quad x^n = \sum_{j=0}^n \left\{ \begin{matrix} n \\ j \end{matrix} \right\} (x)_j$$

is called the Stirling numbers of the second kind.

Example 2.5. For $k \in \mathbb{N}$ and using

$$\begin{aligned} x^3 &= x + 3x(x-1) + x(x-1)(x-2) \\ &= (x)_1 + 3(x)_2 + (x)_3 \\ &= \left\{ \begin{matrix} 3 \\ 0 \end{matrix} \right\} (x)_0 + \left\{ \begin{matrix} 3 \\ 1 \end{matrix} \right\} (x)_1 + \left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} (x)_2 + \left\{ \begin{matrix} 3 \\ 3 \end{matrix} \right\} (x)_3, \end{aligned}$$

we get that $\left\{ \begin{matrix} 3 \\ 0 \end{matrix} \right\} = 0$, $\left\{ \begin{matrix} 3 \\ 1 \end{matrix} \right\} = 1$, $\left\{ \begin{matrix} 3 \\ 2 \end{matrix} \right\} = 3$ and $\left\{ \begin{matrix} 3 \\ 3 \end{matrix} \right\} = 1$.

In [10], the following explicit formula is given to calculate these numbers

$$(13) \quad \left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{i=0}^k (-1)^i \binom{k}{i} (k-i)^n.$$

This equation implies that

$$(14) \quad \left\{ \begin{matrix} n \\ 0 \end{matrix} \right\} = \delta_{n,0} = \begin{cases} 1, & n = 0; \\ 0, & n > 0. \end{cases}$$

Thus, for $m \in \mathbb{N}$ and using (14), we have

$$(15) \quad k^m = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (k)_j = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} j! \binom{k}{j}.$$

Taking the sum for both sides of (15) from $k = 1$ to n and using Proposition 2.4 give

$$(16) \quad \sigma_m(n) = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \frac{(n+1)_{j+1}}{j+1} = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} j! \binom{n+1}{j+1}.$$

Example 2.6. Using Example 2.5, we get that

$$\begin{aligned} \sum_{k=1}^n k^3 &= \sigma_3(n) \\ &= \sum_{j=1}^3 \left\{ \begin{matrix} 3 \\ j \end{matrix} \right\} j! \binom{n+1}{j+1} \\ &= 1! \binom{n+1}{2} + 3(2!) \binom{n+1}{3} + 1(3!) \binom{n+1}{4} \\ &= \frac{(n+1)n}{2} \left(1 + 2(n-1) + \frac{(n-1)(n-2)}{2} \right) \\ &= \left(\frac{(n+1)n}{2} \right)^2. \end{aligned}$$

For further properties of the Stirling numbers, see [11].

3. MAIN RESULTS

Proposition 3.1. For $m \in \mathbb{N}_0$, the falling numbers $(x)_n$ satisfies

$$(x+m)_m(x)_j = (x+m)_{m+j}.$$

Proof.

$$\begin{aligned} (x+m)_m(x)_j &= \frac{\Gamma(x+m+1)}{\Gamma(x+1)} \frac{\Gamma(x+1)}{\Gamma(x-j+1)} \\ &= \frac{\Gamma(x+m+1)}{\Gamma(x+m-(j+m)+1)} = (x+m)_{m+j}. \end{aligned}$$

□

For $k \in \mathbb{N}$,

$$\frac{(k+m)_m(k)_j}{m!j!} = \frac{(k+m)_{m+j}(m+j)!}{(m+j)!m!j!}.$$

Therefore,

$$(17) \quad \binom{k+m}{m} \binom{k}{j} = \binom{k+m}{m+j} \binom{m+j}{j} = \binom{k+m}{m+j} \binom{m+j}{m}.$$

Now, we construct recurrence relations for $\sigma_m(n)$. This can be done after proving the following result.

Proposition 3.2. For $m, k \in \mathbb{N}$, the following are true

- (1) $k^{m+1} + k^m = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (j+1)! \binom{k+1}{j+1}$.
- (2) $k^{m+2} + 3k^{m+1} + 2k^m = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (j+2)! \binom{k+2}{j+2}$.
- (3) $k^{m+3} + 6k^{m+2} + 11k^{m+1} + 6k^m = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (j+3)! \binom{k+3}{j+3}$.

Proof. Multiply (12) by $\binom{k+1}{1}$ and use Proposition 3.1 to get

$$\begin{aligned} k^{m+1} + k^m &= (k+1)k^m \\ &= \binom{k+1}{1} k^m \\ &= \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} j! \binom{k+1}{1} \binom{k}{j} \\ &= \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} j! \binom{k+1}{j+1} \binom{j+1}{1} \end{aligned}$$

$$= \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (j+1)! \binom{k+1}{j+1}.$$

Now, use Proposition 3.1 to get

$$\begin{aligned} k^{m+2} + 3k^{m+1} + 2k^m &= (k+2)(k+1)k^m \\ &= 2 \binom{k+2}{2} k^m \\ &= 2! \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} j! \binom{k+2}{2} \binom{k}{j} \\ &= \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (j+2)! \binom{k+2}{j+2}. \end{aligned}$$

Again, use Proposition 3.1 to get

$$\begin{aligned} k^{m+3} + 6k^{m+2} + 11k^{m+1} + 6k^m &= (k+3)(k+2)(k+1)k^m \\ &= 3! \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} j! \binom{k+3}{3} \binom{k}{j} \\ &= 3! \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} j! \binom{k+3}{j+3} \binom{j+3}{3} \\ &= \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (j+3)! \binom{k+3}{j+3}. \end{aligned}$$

□

Taking the sum for the results in Proposition 3.2 from $k = 1$ to n and using (9) to get the following recurrence relations for $\sigma_m(n)$.

Theorem 3.3. For $m \in \mathbb{N}$

- (1) $\sigma_{m+1}(n) = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (j+1)! \binom{n+2}{j+2} - \sigma_m(n)$.
- (2) $\sigma_{m+2}(n) = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (j+2)! \binom{n+3}{j+3} - 3\sigma_{m+1}(n) - 2\sigma_m(n)$
- (3) $\sigma_{m+3}(n) = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (j+3)! \binom{n+4}{j+4} - 6\sigma_{m+2}(n) - 11\sigma_{m+1}(n) - 6\sigma_m(n)$.

Proof. Taking the sum for each of the statements in Proposition 3.2 from $k = 1$ to n and using Proposition 2.4, we will get the result. For example, taking the sum for the statement

$$k^{m+1} + k^m = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (j+1)! \binom{k+1}{j+1}$$

gives that

$$(18) \quad \sigma_{m+1}(n) + \sigma_m(n) = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (j+1)! \left(\sum_{k=1}^n \binom{k+1}{j+1} \right).$$

Now, using (9), Equation (18) becomes $\sigma_{m+1}(n) + \sigma_m(n) = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} (j+1)! \binom{n+2}{j+2}$. Same steps prove the second and the third parts. \square

Theorem 3.3 implies that $\sigma_{m+1}(n)$, $\sigma_{m+2}(n)$, $\sigma_{m+3}(n)$, and $\sigma_{m+4}(n)$ are given explicitly as

Corollary 3.4. (1) $\sigma_{m+1}(n) = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} ((j+1)! \binom{n+2}{j+2} - j! \binom{n+1}{j+1})$.
 (2) $\sigma_{m+2}(n) = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} ((j+2)! \binom{n+3}{j+3} - 3(j+1)! \binom{n+2}{j+2} + j! \binom{n+1}{j+1})$.
 (3) $\sigma_{m+3}(n) = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} ((j+3)! \binom{n+4}{j+4} - 6(j+2)! \binom{n+3}{j+3} + 7(j+1)! \binom{n+2}{j+2} - j! \binom{n+1}{j+1})$.

Example 3.5. Using $\left\{ \begin{matrix} 2 \\ j \end{matrix} \right\} = 1$ for $j = 1, 2$, we get

$$\begin{aligned} \sum_{k=1}^n k^5 &= \sigma_5(n) = \sum_{j=1}^2 \left\{ \begin{matrix} 2 \\ j \end{matrix} \right\} (j+3)! \binom{n+4}{j+4} - 6\sigma_4(n) - 11\sigma_3(n) - 6\sigma_2(n) \\ &= 4! \binom{n+4}{5} + 5! \binom{n+4}{6} \\ &\quad - \frac{6}{30} n(n+1)(2n+1)(3n^2+3n-1) - 11 \left(\frac{n(n+1)}{2} \right)^2 - 6 \frac{n(n+1)(2n+1)}{6} \\ &= \frac{(n+4)(n+3)(n+2)(n+1)n}{5} + \frac{(n+4)(n+3)(n+2)(n+1)n(n-1)}{6} \\ &\quad - \frac{6}{30} n(n+1)(2n+1)(3n^2+3n-1) - 11 \left(\frac{n(n+1)}{2} \right)^2 - 6 \frac{n(n+1)(2n+1)}{6} \\ &= \frac{1}{12} n^2 (n+1)^2 (2n^2+2n-1). \end{aligned}$$

Now, one can construct other recurrence relations using Corollary 3.4. To achieve that, the following result is needed:

Proposition 3.6.

$$(19) \quad \sum_{n=1}^N \sigma_m(n) = (N+1)\sigma_m(N) - \sigma_{m+1}(N).$$

Proof. interchanging the sums gives

$$\begin{aligned} \sum_{n=1}^N \sigma_m(n) &= \sum_{n=1}^N \sum_{k=1}^n k^m = \sum_{k=1}^N \sum_{n=k}^N k^m = \sum_{k=1}^N (N-k+1)k^m \\ &= (N+1)\sigma_m(N) - \sigma_{m+1}(N). \end{aligned}$$

□

By taking the sum from $n = 1$ to N and using (19) and (8), new recurrence relations for $\sigma_m(n)$ are generated. For example, applying these steps to part(3) of Corollary 3.4 gives that:

$$(N+1)\sigma_{m+3}(N) - \sigma_{m+4}(N) = \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \left((j+3)! \binom{N+5}{j+5} - 6(j+2)! \binom{N+4}{j+4} \right. \\ \left. + 7(j+1)! \binom{N+3}{j+3} - j! \binom{N+2}{j+2} \right).$$

Therefore,

$$\sigma_{m+4}(N) = (N+1)\sigma_{m+3}(N) - \sum_{j=1}^m \left\{ \begin{matrix} m \\ j \end{matrix} \right\} \left((j+3)! \binom{N+5}{j+5} - 6(j+2)! \binom{N+4}{j+4} \right. \\ \left. + 7(j+1)! \binom{N+3}{j+3} - j! \binom{N+2}{j+2} \right).$$

Example 3.7. Using Example 3.5,

$$\sigma_6(N) = (N+1)\sigma_5(N) - \sum_{j=1}^2 \left\{ \begin{matrix} 2 \\ j \end{matrix} \right\} \left((j+3)! \binom{N+5}{j+5} - 6(j+2)! \binom{N+4}{j+4} \right. \\ \left. + 7(j+1)! \binom{N+3}{j+3} - j! \binom{N+2}{j+2} \right) \\ = \frac{1}{12}N^2(N+1)^2(2N^2+2N-1) - \left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} \left(4! \binom{N+5}{6} - 6(3!) \binom{N+4}{5} \right) \\ + 7(2!) \binom{N+3}{4} - 1! \binom{N+2}{3} - \left\{ \begin{matrix} 2 \\ 2 \end{matrix} \right\} \left(5! \binom{N+5}{7} - 6(4!) \binom{N+4}{6} \right) \\ + 7(3!) \binom{N+3}{5} - 2! \binom{N+2}{4} \\ = \frac{1}{12}N^2(N+1)^2(2N^2+2N-1) - \left(4! \binom{N+5}{6} - 6(3!) \binom{N+4}{5} \right) \\ + 7(2!) \binom{N+3}{4} - 1! \binom{N+2}{3} - \left(5! \binom{N+5}{7} - 6(4!) \binom{N+4}{6} \right) \\ + 7(3!) \binom{N+3}{5} - 2! \binom{N+2}{4} \\ = \frac{1}{42}N(1+N)(1+2N)(1-3N+6N^3+3N^4).$$

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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