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SOLVING SPDDE USING FOURTH ORDER NUMERICAL METHOD

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Abstract: In this paper we present a fourth order numerical method to solve singularly perturbed differential-difference equations. The solution of this problem exhibits layer behaviour at one end. A fourth order finite difference scheme on a uniform mesh is developed. The effect of delay and advance parameters on the boundary layer(s) has also been analyzed and depicted in graphs. The applicability of the proposed scheme is validated by implementing it on model examples. To show the accuracy of the method, the results are presented in terms of maximum absolute errors.

Keywords: differential-difference equations; central differences; fourth order.

2010 AMS Subject Classification: 65L10, 65L11, 65L12.

1. INTRODUCTION

Mathematically, any ordinary differential equation in which the highest derivative is multiplied by a small positive parameter and containing at least one shift term(delay or advance) is known as singularly perturbed differential-difference equation(SPDDE). Such problems arise frequently in the study of human pupil light reflex [1], control theory [2], mathematical biology [3], etc. The mathematical modelling of the determination of the expected time for the generation of action

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potentials innerve cells by random synaptic inputs in dendrites includes a general boundary value problem for singularly perturbed differential-difference equation with small shifts. Different numerical methods were proposed to solve singularly perturbed problems by Roberts [4], Bender and Orszag [5], O'Malley [6], and Miller et al. [7]. In [8], Lange and Miura considered boundary-value problems for singularly perturbed linear second-order differential-difference equations with small shifts. The analyses of the layer equations using Laplace transform lead to novel results. Numerical study for approximating the solution of SPDDE given by Kadalbajoo and Sharma[9] with mixed shifts. In [10] Kadalbajoo and Kumar presented a technique based on piecewise uniform mesh and quasilinearization process for SPDDE with small shifts. Chakravarthy and Rao [11] proposed a modified fourth order Numerov method is presented for solving singularly perturbed differential-difference equations of mixed type. Authors constructed a special type of mesh, so that the terms containing shift lie on nodal points after discretization. This finite difference method works nicely when the delay parameter is smaller or bigger to perturbation parameter. In [12], Ravi Kanth and Murali has given a numerical method based on parametric cubic spline for a class of nonlinear singularly perturbed delay differential equations. Quasilinearization process is applied to reduce the nonlinear singularly perturbed delay differential equations into a sequence of linear singularly perturbed delay differential equations. To handle the delay term, they have constructed a special type of mesh in such a way that the term containing delay lies on nodal points after discretization. RaviKanth and Murali [13] discussed an exponentially fitted spline method for singularly perturbed convection delay problems

2. DESCRIPTION OF THE METHOD

Think about SPDDE along with little delay and additionally advance parameters of the kind:

$$\varepsilon u''(x) + p(x)u'(x) + q(x)u(x-\delta) + r(x)u(x)+s(x)u(x+\eta) = f(x) \tag{1}$$

$\forall x \in (0,1)$ and under the boundary conditions

$$u(x) = \varphi(x) \quad on -\delta \le x \le 0 \tag{2}$$

$$u(x) = \gamma(x) \quad on \quad 1 \le x \le 1+\eta \tag{3}$$

Here $p(x), q(x), r(x), s(x), \varphi(x)$ and $\gamma(x)$ are sufficiently smooth functions on (0, 1), the perturbation parameter ε is small positive parameter (0 $< \varepsilon <<$1), and $0 < \delta = o(\varepsilon)$ and $0 < \eta = o(\varepsilon)$ are the delay(negative shift) and the advance(positive shift) parameters respectively.Typically, the solution for Eq.(1)-Eq.(3) reveals layer behaviour at one end of the interval [0,1] depending upon the sign of $p(x) + s(x)\eta - q(x)\delta$.

By utilizing Taylor series almost the aspect x, the deviating argument conditions may be taken as

$$u(x - \delta) \approx u(x) - \delta u'(x) \tag{4}$$

$$u(x + \eta) \approx u(x) + \eta u'(x) \tag{5}$$

Using Eq.(4) and Eq.(5) in Eq.(1) we receive an asymptotically equal singularly perturbed boundary value problem of the type:

$$\varepsilon u''(x) + \alpha(x)u'(x) + \beta(x)u(x) = f(x) \tag{6}$$

$$y(0) = \varphi(0) = \varphi_0 \tag{7}$$

$$y(1) = \gamma(1) = \gamma_1 \tag{8}$$

where $\alpha(x) = p(x) + s(x)\eta - q(x)\delta$

$$\beta(x) = q(x) + r(x) + s(x)$$

Since$0 < \delta << 1$ and $0 < \eta << 1$, The transition from Eq.(1) to Eq.(6) is admissible. Further details on the validity of this transition is found in El'sgol'ts and Norkin[14].

On discretizing the interval [0,1] into N equal subintervals of step size $h = \frac{1}{N}$ to make sure that $x_i = ih, i =$0, 1, 2, … ,N.

Let $u_i = u(x_i)$ for $x_i \in [0,1]$

Assuming that $u(x)$ has continuous derivatives on [0,1] and making use of Taylor's series expansions of u_{i+1} and u_{i-1} upto $O(h^7)$, we get the finite difference approximations for u'_i and u''_i as

$$u'_i = \frac{u_{i+1} - u_{i-1}}{2h} - \frac{h^2}{6}u'''_i - \frac{h^4}{120}u_i^{(5)} + \tau_1 \tag{9}$$

$$where \quad \tau_1 = -\frac{h^6}{7!}u^{(7)}(\xi_1), \qquad \text{for} \quad \xi_1 \in [x_{i-1}, x_i]$$

$$u''_i = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} - \frac{h^2}{12} u_i^{(4)} - \frac{h^4}{360} u_i^{(6)} + \tau_2 \tag{10}$$

where $\tau_2 = -\frac{h^8}{8!} u^{(8)}(\xi_2)$, for $\xi_2 \in [x_{i-1}, x_i]$

Substituting Eqs. (9) and (10) into Eq. (6) and simplifying, we obtain:

$$\frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}) + \frac{\alpha_i}{2h}(u_{i+1} - u_{i-1}) - \frac{h^2}{6}\alpha_i u'''_i - \frac{h^2}{12} u_i^{(4)} - \frac{h^4}{120}\alpha_i u_i^{(5)} + \beta_i u_i = r_i + \tau \tag{11}$$

$\tau = \frac{h^4}{360} u^{(6)}(\xi_2) - \alpha_i \tau_i - \tau_2$ is the local truncation error and

$\alpha(x_i)/\varepsilon = \alpha_i, \beta(x_i)/\varepsilon = \beta_i, f(x_i)/\varepsilon = r_i$

By successively differentiating both sides of Eq. (6) and evaluating at x_i , and using into Eq.(11), we obtain:

$$\frac{1}{h^2}(u_{i+1} - 2u_i + u_{i-1}) + \frac{\alpha_i}{2h}(u_{i+1} - u_{i-1}) + P_i u''_i + Q_i u'_i + \mathrm{R}_i u_i = S_i,$$

for $i = 1,2,---N-1$ (12)

where

$$P_i = \frac{h^2}{6}\alpha_i^2 - \frac{h^2}{12}(\alpha_i^2 - 2\alpha'_i - \beta_i) - \frac{h^4}{120}\alpha_i(2\alpha_i\alpha'_i - 3\alpha''_i - 3\beta'_i + \alpha_i(\alpha'_i + \beta_i) - \alpha_i(\alpha_i^2 - 2\alpha'_i - \beta_i))$$

$$Q_i = \frac{h^2}{6}\alpha_i(\alpha'_i + \beta_i) - \frac{h^2}{12}\left[\alpha_i(\alpha'_i + \beta_i) - \alpha''_i - 2\beta'_i\right] - \frac{h^4}{120}\alpha_i\left[\alpha'_i(\alpha'_i + \beta_i) + \alpha_i(\alpha''_i + \beta'_i) - \alpha'''_i - 3\beta''_i + \alpha_i\beta'_i - (\alpha'_i + \beta_i)(\alpha_i^2 - 2\alpha'_i - \beta_i)\right]$$

$$R_i = \frac{h^2}{6}\alpha_i\beta'_i - \frac{h^2}{12}(\alpha_i\beta'_i - \beta''_i) - \frac{h^4}{120}\alpha_i\left[\alpha'_i\beta'_i + \alpha_i\beta''_i - \beta'''_i - \beta'_i(\alpha_i^2 - 2\alpha'_i - \beta_i)\right] + \beta_i$$

$$S_i = r_i + \left[\frac{h^2}{12}\alpha_i + \frac{h^4}{120}\alpha_i(\alpha_i^2 - 3\alpha'_i - \beta_i)\right] r'_i + \left[\frac{h^2}{12} - \frac{h^4}{120}\alpha_i^2\right] r''_i + \frac{h^4}{120}\alpha_i r'''_i$$

Now, using central difference approximation for u''_i and u'_i in Eq.(12) and further simplifying, we get:

$E_i y_{i-1} - F_i y_i + G_i y_{i+1} = H_i$, for $i = 1,2,----N-1$

where

$$E_i = \frac{1}{h^2} - \frac{\alpha_i}{2h} + \frac{P_i}{h^2} - \frac{Q_i}{2h}$$

$$F_i = \frac{2}{h^2} + \frac{2P_i}{h^2} - R_i$$

$$G_i = \frac{1}{h^2} + \frac{\alpha_i}{2h} + \frac{P_i}{h^2} + \frac{Q_i}{2h}$$

3. NUMERICAL EXAMPLES

The exact solution of Eq. (1) to Eq. (3) with constant coefficients (i.e,

$p(x) = p, q(x) = q, r(x) = r, s(x) = s, f(x) = f, \varphi(x) = \varphi$ *and* $\gamma(x) = \gamma$)

is given by $u(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x} + f/c_3$

where

$$m_1 = \frac{-(p+s\eta-q\delta)+\sqrt{(p+s\eta-q\delta)^2-4\varepsilon c_3}}{2\varepsilon}, \quad m_2 = \frac{-(p+s\eta-q\delta)-\sqrt{(p+s\eta-q\delta)^2-4\varepsilon c_3}}{2\varepsilon}$$

$$c_1 = \frac{-f+\gamma c_3+e^{m_2}(f-\varphi c_3)}{(e^{m_1}-e^{m_2})c_3}, \quad c_2 = \frac{f-\gamma c_3+e^{m_1}(-f+\varphi c_3)}{(e^{m_1}-e^{m_2})c_3}, \quad c_3 = q+r+s$$

Example 1. Consider the SPDDE with left end boundary layer:

$\varepsilon u''(x) + u'(x) + 2u(x-\delta) - 3u(x) = 0$, $\varphi(x) = 1$, $\gamma(x) = 1$

The numerical results are given in Table 1 & Fig.1.

Table 1. The maximum absolute errors in the solution of Example 1 for $\varepsilon = 0.1$ and $\eta = 0$

δ \ N	8	32	128	512
0.00	6.9153e-03	2.5271e-05	9.7873e-08	3.8222e-10
0.05	5.0195e-03	1.8142e-05	7.0494e-08	2.7596e-10
0.09	3.7678e-03	1.3493e-05	5.2858e-08	2.0778e-10
	Results in Kadalbajoo and Sharma [15]			
0.00	0.09907804	0.03700736	0.00954678	0.00214501
0.05	0.09659609	0.03640566	0.00924661	0.00202998
0.09	0.09277401	0.03556652	0.00895172	0.00192488

Example 2. Consider the SPDDE with left end boundary layer:

$\varepsilon u''(x) + u'(x) - 3u(x) + 2u(x+\eta) = 0\,, \qquad \varphi(x) = 1, \qquad \gamma(x) = 1$

The numerical results are given in Table 2 & Fig. 2.

Table 2. The maximum absolute errors in the solution of Example 2 for $\boldsymbol{\varepsilon = 0.1}$ and $\boldsymbol{\delta = 0}$

η \N	8	32	128	512
0.00	6.9153e-03	2.5271e-05	9.7873e-08	3.8222e-10
0.05	9.1907e-03	3.3936e-05	1.3196e-07	5.1523e-10
0.09	1.1286e-02	4.2002e-05	1.6457e-07	6.4247e-10
	Results in Kadalbajoo and Sharma [15]			
0.00	0.09907804	0.03700736	0.00954678	0.00214501
0.05	0.09977501	0.03727087	0.00979659	0.00224472
0.09	0.10031348	0.03723863	0.00996284	0.00458698

Example 3. Consider the SPDDE with left end boundary layer:

$\varepsilon u''(x) + u'(x) - 2u(x-\delta) - 5u(x) + u(x+\eta) = 0\,, \varphi(x) = 1, \gamma(x) = 1$

The numerical results are given in Tables 3 and 4 & Figs. 3 and 4.

Table 3. The maximum absolute errors in the solution of Example 3 for $\boldsymbol{\varepsilon = 0.1}$ and $\boldsymbol{\eta = 0.05}$

δ \N	8	32	128	512
0.00	2.2405e-02	8.9231e-05	3.4466e-07	1.3457e-09
0.05	2.8631e-02	1.1750e-04	4.5156e-07	1.7633e-09
0.09	3.4338e-02	1.4437e-04	5.5404e-07	2.1617e-09
	Results in Kadalbajoo and Sharma [15]			
0.00	0.09190267	0.03453494	0.01164358	0.00300463
0.05	0.10233615	0.03823132	0.01295871	0.00335137
0.09	0.11018870	0.04110846	0.01400144	0.00362925

Table 4. The maximum absolute errors in the solution of Example 3 for $\varepsilon = 0.1$ and $\delta = 0.05$

η \N	8	32	128	512
0.00	2.5394e-02	1.0266e-04	3.9499e-07	1.5440e-09
0.05	2.8631e-02	1.1750e-04	4.5156e-07	1.7633e-09
0.09	3.1402e-02	1.3043e-04	5.0093e-07	1.9547e-09
	Results in Kadalbajoo and Sharma [15]			
0.00	0.09720029	0.03640446	0.01229476	0.00317786
0.05	0.10233615	0.03823132	0.01295871	0.00335137
0.09	0.10632014	0.03965833	0.01348348	0.00349050

Example 4. Consider SPDDE with right end boundary layer:

$\varepsilon u''(x) - u'(x) - 2u(x-\delta) + u(x) = 0\,, \varphi(x) = 1, \gamma(x) = -1$

The numerical results are given in Table 5 & Fig. 5.

Table 5. The maximum absolute errors in the solution of Example 4 for $\varepsilon = 0.1$ and $\eta = 0$

δ \N	8	32	128	512
0.00	1.6153e-02	5.9019e-05	2.2858e-07	8.9266e-10
0.05	1.0877e-02	3.9305e-05	1.5273e-07	5.9597e-10
0.09	7.6714e-03	2.7464e-05	1.0761e-07	4.1931e-10
	Results in Kadalbajoo and Sharma [15]			
0.00	0.07847490	0.04678972	0.01727912	0.00443086
0.05	0.09222560	0.03828329	0.01487799	0.00380679
0.09	0.10509460	0.03149275	0.01299340	0.00331935

Example 5. Consider the SPDDE with right end boundary layer:

$\varepsilon u''(x) - u'(x) + u(x) - 2u(x+\eta) = 0\,, \varphi(x) = 1, \gamma(x) = -1$

The numerical results are given in Table 6 & Fig. 6.

Table 6. The maximum absolute errors in the solution of Example 5 for $\varepsilon = 0.1$ and $\delta = 0$

η \N	8	32	128	512
0.00	1.6153e-02	5.9019e-05	2.2858e-07	8.9266e-10
0.05	2.3063e-02	8.5147e-05	3.3109e-07	1.2927e-09
0.09	2.9918e-02	1.1133e-04	4.3622e-07	1.7029e-09
	Results in Kadalbajoo and Sharma [15]			
0.00	0.07847490	0.04678972	0.01727912	0.00443086
0.05	0.06834579	0.05516436	0.01972508	0.00506769
0.09	0.08328237	0.06168267	0.02169662	0.00558451

Example 6. Consider the SPDDE with right end boundary layer:

$\varepsilon u''(x) - u'(x) - 2u(x-\delta) + u(x) - 2u(x+\eta) = 0\,, \varphi(x) = 1, \gamma(x) = -1$

The numerical results are given in Tables 7 and 8 & Figs. 7 and 8.

Table 7. The maximum absolute errors in the solution of Example 6 for $\varepsilon = 0.1$ and $\eta = 0.05$

δ \N	8	32	128	512
0.00	2.2592e-02	8.5192e-05	3.3407e-07	1.3046e-09
0.05	1.6657e-02	6.1800e-05	2.4154e-07	9.4339e-10
0.09	1.2806e-02	4.7196e-05	1.8344e-07	7.1611e-10
	Results in Kadalbajoo and Sharma [15]			
0.00	0.09930002	0.03685072	0.01331683	0.00342882
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.10044578	0.02850398	0.01038902	0.00266379

Table 8. The maximum absolute errors in the solution of Example 6 for $\varepsilon = 0.1$ and $\delta = 0.05$

$\eta \backslash$N	8	32	128	512
0.00	1.1956e-02	4.3990e-05	1.7080e-07	6.6680e-10
0.05	1.6657e-02	6.1800e-05	2.4154e-07	9.4339e-10
0.09	2.1298e-02	7.9762e-05	3.1372e-07	1.2248e-09
	Results in Kadalbajoo and Sharma [15]			
0.00	0.10055269	0.02759534	0.01007834	0.00258299
0.05	0.09997296	0.03218424	0.01167102	0.00299572
0.09	0.09944067	0.03591410	0.01297367	0.00334044

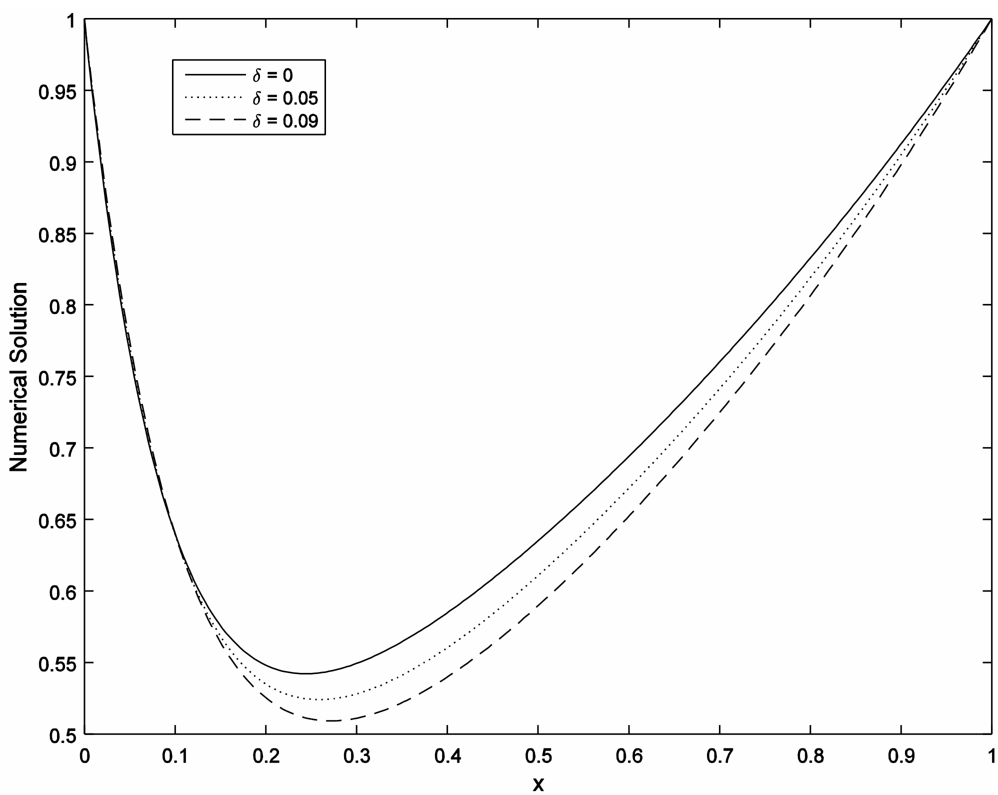

Fig. 1. Numerical solution of Example 1 for $\varepsilon = 0.1$ and $\eta = 0$

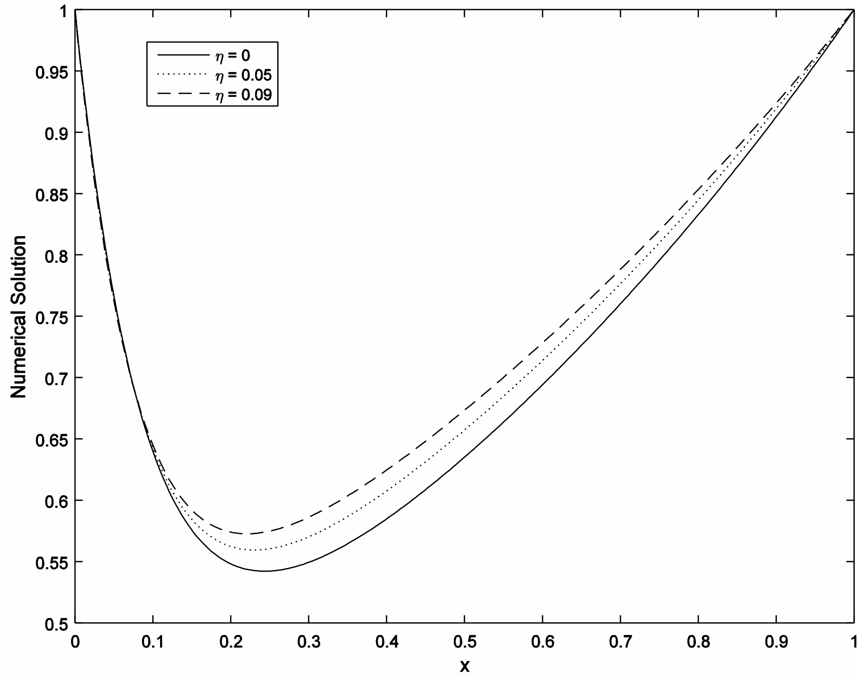

Fig. 2. Numerical solution of Example 2 for $\varepsilon = 0.1$ and $\delta = 0$

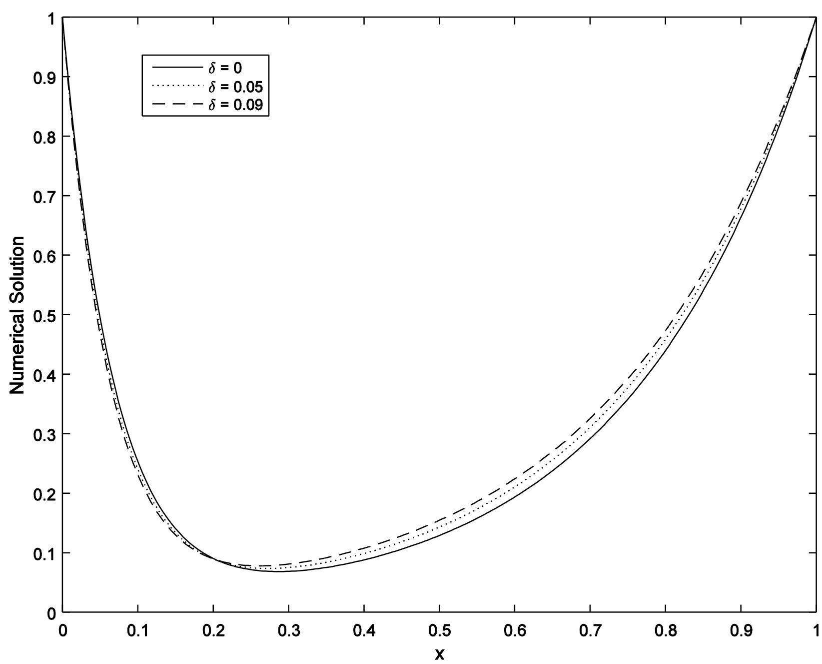

Fig. 3. Numerical solution of Example 3 for $\varepsilon = 0.1$ and $\eta = 0.05$

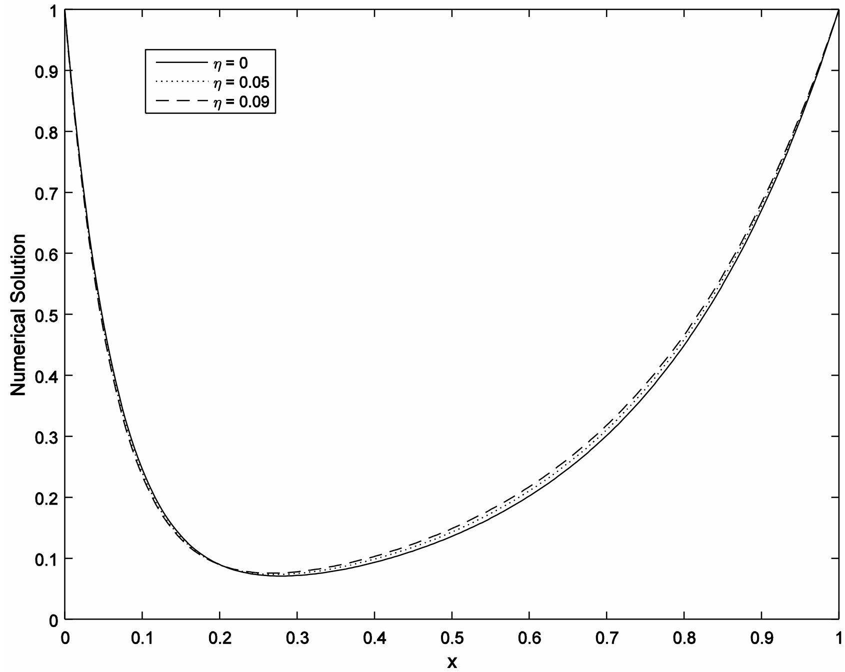

Fig. 4. Numerical solution of Example 3 for $\varepsilon = 0.1$ and $\delta = 0.05$

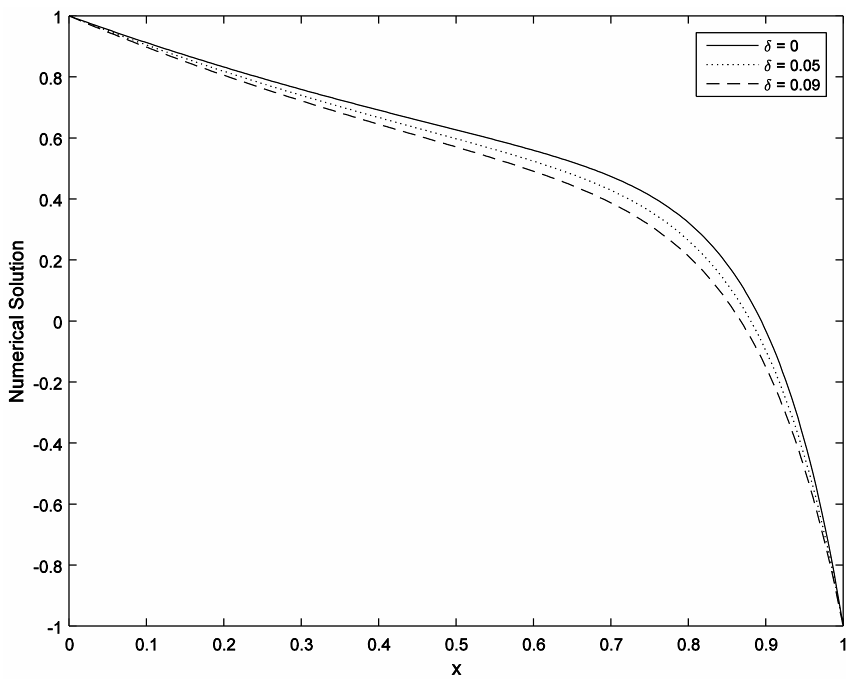

Fig. 5. Numerical solution of Example 4 for $\varepsilon = 0.1$ and $\eta = 0$

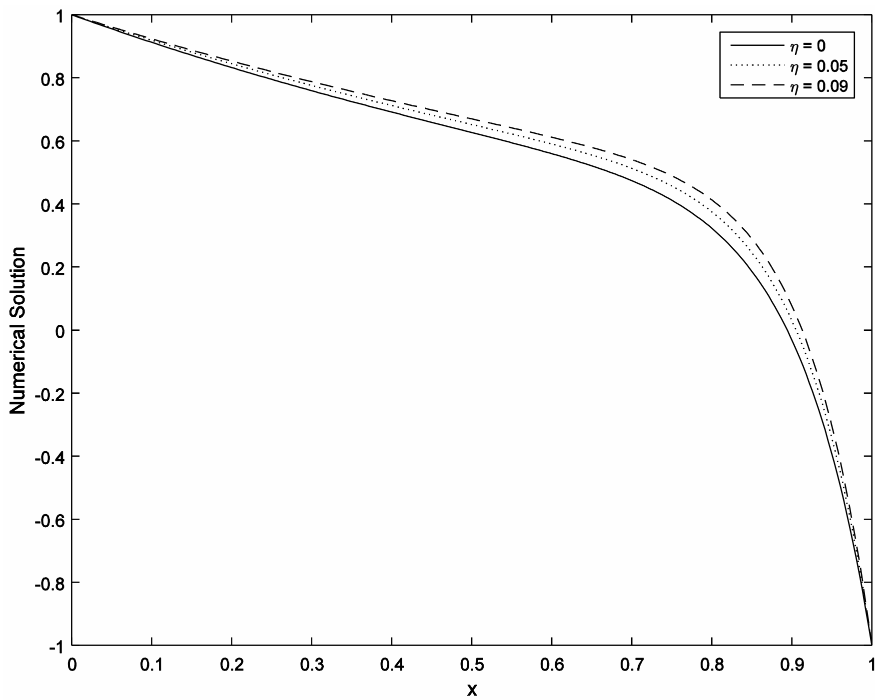

Fig. 6. Numerical solution of Example 5 for $\varepsilon = 0.1$ and $\delta = 0$

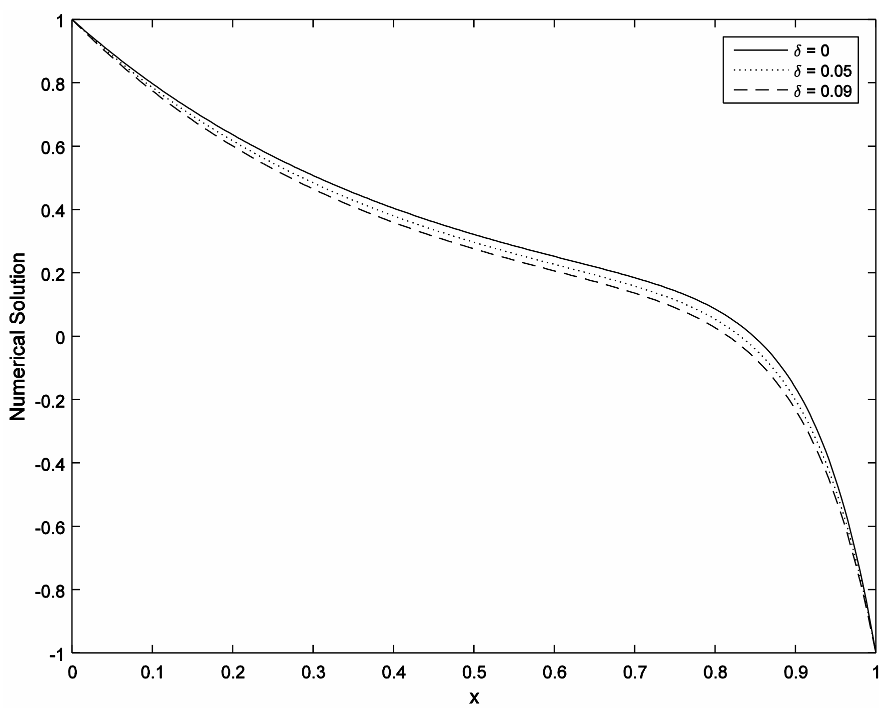

Fig. 7. Numerical solution of Example 6 for $\varepsilon = 0.1$ and $\eta = 0.05$

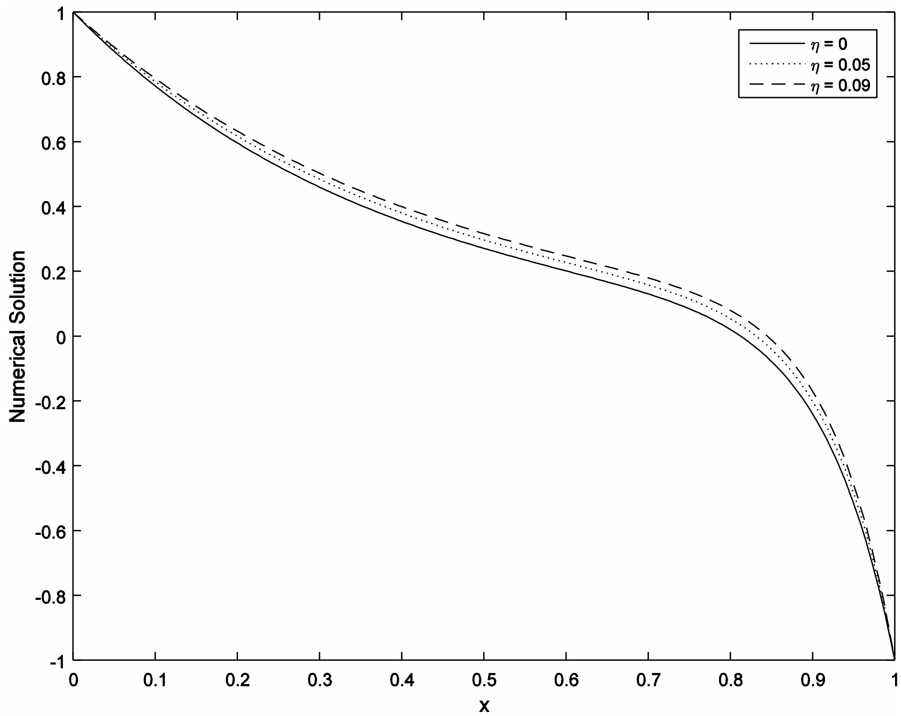

Fig. 8. Numerical solution of Example 6 for $\varepsilon = 0.1$ and $\delta = 0.05$

4. DISCUSSIONS AND CONCLUSION

We have discussed fourth order numerical method to solve singularly perturbed differential-difference equations exhibiting one end layer behaviour. To discuss the applicability of the method we have solved model examples by taking different values of N, ε, δ *and* η. The numerical solution is compared with the exact solution to test the proposed method. To support and strengthen the method, numerical results are compared with the results of Kadalbajoo and Sharma [15]. We have presented maximum absolute errors for the standard examples chosen from the literature. From the tables, the results demonstrate that the present method produced good approximation to the exact solution. To analyze the effect of the parameters on the solution, the numerical results have been plotted using graphs. From the graphs (Fig.1 - Fig. 4), we observed that when the solution of the boundary value problem exhibits layer behaviour on the left side, the affect of delay or advance on the solution in the boundary layer region is negligible, while in the outer region it is considerable and the change in advance affects the solution similarly as the change in delay effects, but reversely. From the graphs (Fig. 5 – Fig. 8), we observed that when the solution of the boundary value problem exhibits layer behaviour on the right side, the changes in delay or advance affect the solution in the boundary layer region as well as the outer region. The thickness of the layer increases as the size of the delay increases while it decreases as the size of the advance increases.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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