DYNAMICAL BEHAVIOURS OF PREY-PREDATOR FISHERY MODEL WITH TWO RESERVED AREA FOR PREY IN THE PRESENCE OF TOXICITY AND RESPONSE FUNCTION HOLLING TYPE IV

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Abstract. This paper proposes a model with two type of preys and one predator of fishery with Holling type IV function response. The effect of harvesting was incorporated to both populations and thoroughly analysed. We study the dynamics as the prey-predator system of fishing in two fishing zones: a free zone and a reserved zone. The equilibrium points are calculated and the local and global stability conditions of the system are obtained. The local stability conditions were obtained by the Routh-Hurwitz criterion. In addition, the global stability of the coexistence equilibrium point is proved by defining an appropriate Lyapunov function. The optimal harvesting policy is discussed by using the Pontryagin's Maximal Principle. Finally, numerical simulations are carried to verify the analytical results.

Keywords: predator–prey system; toxicity; equilibria; stability; functional response; optimal harvesting policy.

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1. INTRODUCTION

Currently, the relationship between prey and predator has become a very important topic to discuss in ecology. The prey-predator system has attracted many researchers to study the interaction between species (see [4, 11, 14, 24, 28, 34]). For this, we use the aspect of mathematical

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ecology to study the interacting species. The current study considers fishing between two areas: a free area and another area reserved for fishing as a case study. However, current trends in species dynamics between the two areas have been obtained by reading various publications such as [13, 15, 18, 23, 25]. This literature clearly explains the current trend on this kind of models and fisheries management.

Dubey et al. [8] proposed and analyzed the dynamics of a predator-prey model where one prey species is reserved and the other free for fishing. In addition, it assumes that the predator is not authorized to enter the reserved zone, however it is authorized to access the free zone. Kar et al. [13] they proposed and analyzed a mathematical prey-predator model made up of two prey in two areas (free fishing area, reserved area) and a predator authorized to enter the free area. The dynamics follow the response of the type I Holling function. Yang et al. [31] proposed and studied a predator-prey model with harvest and reserve area for prey in the presence of toxicity where the dynamics follow a Holling type II response. They studied the boundedness of solutions and the existence of the equilibria of this system. The optimal harvest policy was also discussed. Thus, the Holling II type functional response is fairly representative in the case of the model presented by Yang et al. [31].

In theoretical ecology, the mathematicians and ecologists study more often the interactions between the model prey-predator using the response functions of Holling type (I, II and III) [10]. The functional response of Holling type IV comes in the form : Sokol et al. [27] proposed this function in this form : $\frac{ax}{b+cx+x^2}$. However, in this paper, the prey-predator interaction modeling using a simplified type IV Holing functional response is expressed by : $\frac{ax}{b+x^2}$.

When the prey population is low, predators who follow a type IV functional response consume very little. For such predators, the highest consumption of prey occurs when the size of the prey is intermediate. When their numbers is low, the prey therefore undergoes a relaxation of the predation pressure. Therefore, they are opportunistic predators, capable of diversifying

the prey they hunt.

The specific study aims to analyze the impact of the type IV functional response on fishing activities between the two zones on the model proposed by Yang et al. [31]. Thus, we are studying a prey-predator system of two prey and a single predator in a two-zone environment : one zone accessible to fishing and the other is prohibited. We assumed that the prey migrated randomly between the two plots. The growth of prey in each plot in the absence of a predator is assumed to be logistic and linear respectively. The Holling response to predation from predators is type IV.

Considering the hypotheses below, we formulate the system of equations of the model as :

(1)
$$\begin{cases} \dfrac{dx}{dt} = r_1 x \left(1 - \dfrac{x}{K}\right) - \sigma_1 x + \sigma_2 y - ux^2 - \dfrac{axz}{b+x^2} - q_1 Ex, \\ \dfrac{dy}{dt} = r_2 y + \sigma_1 x - \sigma_2 y - vy^2, \\ \dfrac{dz}{dt} = \dfrac{\beta axz}{b+x^2} - dz - wz - q_2 Ez. \end{cases}$$

with initial conditions $x(0) > 0$, $y(0) > 0$, $z(0) > 0$. Here, $x \equiv x(t)$, $y \equiv y(t)$, $z \equiv z(t)$ are the population densities of prey and predator respectively. Model system (1) is defined on the set $\mathbb{R}^3_+ = \{(x,y,z) \in \mathbb{R}^3 / x \geq 0, \ y \geq 0, \ z \geq 0 \}$ and $r_1 > 0$, $r_2 > 0$, $K > 0$, $\sigma_1 > 0$, $\sigma_2 > 0$, $q_1 > 0$, $q_2 > 0$, $d > 0$, $\beta > 0$ Further, the biological meanings of parameters are described in Table 1. The prey x in an unreserved area, reproduces in logistic terms of carrying capacity K and intrinsic growth rate r_1 according to the term $r_1 x \left(1 - \dfrac{x}{K}\right)$. For the prey y in the reserve zone, it increases linearly according to the term $r_2 y$. It is assumed that there is a migration from the unreserved area to the reserved area of the prey species x according to the term $\sigma_1 x$ and vice versa for the prey y following the term $\sigma_2 y$. The Interaction between prey x and predator z is expressed by Holling type-IV functional response (see [29, 30]) that is $\dfrac{axz}{b+x^2}$. The explanations of the parameters are presented on this table :

Parameters	Biological meaning
r_1	Intrinsic growth rate of the prey inside unreserved area
r_2	Intrinsic growth rate of the prey inside reserved area
K	The environmental carrying capacity of prey species in the unreserve area
σ_1, σ_2	migration rate from unreserved area to reserved area and vice versa
q_1	The coefficient of catchability of prey in the unreserved area
q_2	The coefficient of catchability of predator in the unreserved area
d	The death rate of the predator species
β	The conversion rate of predator due to prey

TABLE 1. Parameters descriptions

If there is no migration of fish population from reserved area to unreserved area ($\sigma_2 = 0$) and ($r_1 - \sigma_1 - q_1 E < 0$), we find that $\dot{x} < 0$. Similarly, if there is no migration of fish population from unreserved area to reserved area ($\sigma_1 = 0$) and $r_2 - \sigma_2 < 0$, then $\dot{y} < 0$. If $\frac{\beta a}{2\sqrt{b}} < d + w + q_2 E$, then $\dot{z} < 0$.

So, from our analysis, we suppose that :

$$(2) \qquad r_1 - \sigma_1 - q_1 E > 0, \ r_2 - \sigma_2 > 0 \text{ and } \frac{\beta a}{2\sqrt{b}} > d + w + q_2 E.$$

What follows is organized in the following way. In the next section, we show the bournitude system solutions (1). In section 3-4, we study the existence and stability of all the equilibria of our model. Then, we discuss the optimal harvesting policy of the system (1) in section 5. Finally, we present the numerical simulations to study the stability of the equilibria.

2. BASIC RESULTS AND EXISTENCE OF EQUILIBRIA

In the first part of this section, we demonstrate the boundedness of the solutions of our system (1). From the point of view of biology, we are only interested in the dynamics of system (1) in the first octant $\mathbb{R}^3_+$.

Lemma 2.1. *The set* $\Omega = \left\{ (x, y, z) \in \mathbb{R}^3_+ : x + y + \frac{1}{\beta} z \leq \frac{H}{d + w + q_2 E} \right\}$ *is a region of attraction for all solutions initiating in the interior of the positive octant, where*

$$H = \frac{K(r_1 - q_1 + d + w + q_2E)^2}{4(r_1 + Ku)} + \frac{(r_2 + d + w + q_2E)^2}{4v}.$$

Proof. Let us consider $X(t) = x(t) + y(t) + \frac{1}{\beta}z(t)$, then the time derivative along the solutions of the system (1) is given by

$$
\begin{aligned}
\frac{dX(t)}{dt} + (d + w + q_2E)X(t) &= (r_1 - q_1E + d + w + q_2E)x - (\frac{r_1}{K} + u)x^2 - vy^2 \\
&\quad + (r_2 + d + w + q_2E)y, \\
&\leq \frac{K(r_1 - q_1 + d + w + q_2E)^2}{4(r_1 + Ku)} + \frac{(r_2 + d + w + q_2E)^2}{4v} = H.
\end{aligned}
$$

Applying the theory of differential inequality [3, 17], we get

$$X(t) \leq \frac{H}{d + w + q_2E} - \left(\frac{H}{d + w + q_2E} - (x(0) + y(0) + \frac{1}{\beta}z(0)) \right) \exp\left(-(d + w + q_2E)t\right)$$

and when $t \to \infty, 0 < X(t) \leq \frac{H}{d+w+q_2E}$, proving the Lemma. □

In this section, we discuss the existence of positive equilibria of system (1).

Equilibria of model (1) is obtained by solving $dx/dt = dy/dt = dz/dt = 0$. It can be checked that model (1) has four positive equilibria.

(1) $P_0(0,0,0)$ there is a trivial equilibrium.

(2) The equilibrium point $\bar{P}(\bar{x}, \bar{y}, 0)$, where $(\bar{x}, \bar{y})$ is the positive solution of the following equations :

(3)
$$
\begin{aligned}
(r_1 - \sigma_1 - q_1E)x - (\frac{r_1 + Ku}{K})x^2 + \sigma_2 y &= 0, \\
(r_2 - \sigma_2)y + \sigma_1 x - vy^2 &= 0.
\end{aligned}
$$

After the calculations, x is satisfied by the following cubic equation,

(4)
$$a_1x^3 + a_2x^2 + a_3x + a_4 = 0,$$

where

$$
\begin{aligned}
a_1 &= -\frac{v}{\sigma_2^2}\left(\frac{r_1 + Ku}{K}\right)^2, \\
a_2 &= \frac{2v(r_1 + Ku)(r_1 - \sigma_1 - q_1E)}{K\sigma_2^2}, \\
a_3 &= \frac{(r_2 - \sigma_2)(r_1 + Ku)}{K\sigma_2} - \frac{v(r_1 - \sigma_1 - q_1E)^2}{\sigma_2^2}, \\
a_4 &= \sigma_1 - \frac{(r_2 - \sigma_2)(r_1 - \sigma_1 - q_1E)}{\sigma_2}.
\end{aligned}
$$

The above equation (4) had a unique positive solution if the following inequalities hold. We find that $a_1 < 0$ and, according to the criteria of Descartes [7] it is necessary to impose that :

$$a_3 > 0 \quad \text{if} \quad E > \frac{1}{q_1} \left((r_1 - \sigma_1) - \sqrt{\frac{(r_2 - \sigma_2)(r_1 + Ku)\sigma_2}{Kv}} \right),$$

$$a_4 > 0 \quad \text{if} \quad E > \frac{1}{q_1} \left(r_1 - \sigma_1 - \frac{\sigma_1 \sigma_2}{r_2 - \sigma_2} \right).$$

Then, we obtain

(5) $$E > \max \left(\frac{1}{q_1}(r_1 - \sigma_1 - \frac{\sigma_1 \sigma_2}{r_2 - \sigma_2}), \frac{1}{q_1} \left((r_1 - \sigma_1) - \sqrt{\frac{(r_2 - \sigma_2)(r_1 + Ku)\sigma_2}{Kv}} \right) \right).$$

Thus, from the first equation of (3), we deduce

$$\bar{y} = \frac{\bar{x}}{\sigma_2} \left(\left(\frac{r_1 + Ku}{K} \right) \bar{x} - (r_1 - \sigma_1 - q_1 E) \right) > 0,$$

if

(6) $$\bar{x} > \frac{(r_1 - \sigma_1 - q_1 E)K}{r_1 + Ku}.$$

(3) For the interior equilibrium $P^*(x^*, y^*, z^*)$, (x^*, y^*, z^*) is the positive solution of the following equations :

(7) $$\begin{aligned} r_1 x \left(1 - \frac{x}{K} \right) - \sigma_1 x + \sigma_2 y - ux^2 - \frac{axz}{b + x^2} - q_1 Ex &= 0, \\ r_2 y + \sigma_1 x - \sigma_2 y - vy^2 &= 0, \\ \frac{\beta axz}{b + x^2} - dz - wz - q_2 Ez &= 0. \end{aligned}$$

From the last equation of (7), we get two positive solutions :

(8) $$x_1^* = \frac{\beta a - \sqrt{(\beta a)^2 - 4b(d + w + q_2 E)^2}}{2(d + w + q_2 E)},$$

(9) $$x_2^* = \frac{\beta a + \sqrt{(\beta a)^2 - 4b(d + w + q_2 E)^2}}{2(d + w + q_2 E)}.$$

Substitute x_i^* for $i = 1, 2$ in the second equation of (7), we get :

(10) $$y_i^* = \frac{(r_2 - \sigma_2) + \sqrt{(r_2 - \sigma_2)^2 + 4v\sigma_1 x_i^*}}{2v} > 0.$$

Using the third equation of (7), we obtain :

$$(11) \qquad z_i^* = \frac{b+x_i^{*2}}{ax_i^*}\left((r_1-\sigma_1-q_1E)x_i^* - \left(\frac{r_1+Ku}{K}\right)x_i^{*2}+\sigma_2 y_i^*\right),$$

which is positive for $i=1,2$, if

$$(12) \qquad 0<x_i^* < \frac{K(r_1-\sigma_1-q_1E)+\sqrt{(r_1-\sigma_1-q_1E)^2+\frac{4\sigma_2(r_1+Ku)y_i^*}{K}}}{2(r_1+uK)}.$$

We therefore state the existence of the positives equilibrium points in the following theorem.

Theorem 2.2.   *(1) The trivial equilibrium point $P_0(0,0,0)$ exists.*

    (2) The positive predator equilibrium $\bar{P}(\bar{x},\bar{y},0)$ will exist if the following conditions (5) and (6) are satisfied,

    (3) The positive interior equilibrium $P_i^*(x_i^*,y_i^*,z_i^*)$ for $i=1,2$ exists if (12) is realized.

3. LOCAL STABILITY OF EQUILIBRIUM POINTS

In this section, we will study local stability by finding the eigenvalues of the Jacobian matrix

$$J(x,y,z) = \begin{pmatrix} J_{11} & \sigma_2 & -\frac{ax}{b+x^2} \\ \sigma_1 & r_2-\sigma_2-2vy & 0 \\ \frac{\beta az(b-x^2)}{b+x^2} & 0 & \frac{\beta ax}{b+x^2}-(d+w+q_2E) \end{pmatrix},$$

where $J_{11}=r_1-\sigma_1-q_1E-2(\frac{r_1}{K}+u)x-\frac{az(b-x^2)}{b+x^2}$.

Theorem 3.1. *The equilibrium $P_0(0,0,0)$ of the system (1) is unstable.*

Proof. The characteristic equation of $P_0(0,0,0)$ is :

$$(\lambda+d+w+q_2E)(\lambda^2+(r_1-\sigma_1-q_1E+r_2-\sigma_2)\lambda-\sigma_2\sigma_1)=0.$$

It is clear that $\lambda_1=-(d+w+q_2E)<0$. Let λ_2 and λ_3 be the two other eigenvalues. Evidently $\lambda_2+\lambda_3=r_1-\sigma_1-q_1E+r_2-\sigma_2>0$. Therefore λ_2 and λ_3 have one positive value. Hence, $P_0(0,0,0)$ is unstable.   □

Theorem 3.2. *The equilibrium point $\bar{P}(\bar{x},\bar{y},0)$ of the system (1) is locally asymptotically stable if (6) and $\bar{x}\notin[x_1^*,x_2^*]$.*

Proof. The characteristic equation at $\bar{P}$ is written as :

$$\left(\frac{\beta a \bar{x}}{b+\bar{x}^2} - (d+w+q_2E) - \lambda\right) \times \left((r_2 - \sigma_2 - 2v\bar{y} - \lambda)(r_1 - \sigma_1 - q_1E - \frac{2(r_1+Ku)}{K}\bar{x} - \lambda) - \sigma_1\sigma_2\right) = 0$$

Obviously, $\lambda_1 = \frac{\beta a \bar{x}}{b+\bar{x}^2} - (d+w+q_2E) < 0$ if $\bar{x} \notin [x_1^*, x_2^*]$. Let λ_2 and λ_3 be two other eigenvalues.
Then λ_2 and λ_3 are the root of the equation : $\lambda^2 - n_1\lambda + n_2 = 0$,

where

$$n_1 = (r_2 - \sigma_2 - 2v\bar{y}) + \left(r_1 - \sigma_1 - q_1E - \frac{2(r_1+Ku)}{K}\bar{x}\right),$$

$$n_2 = (r_2 - \sigma_2 - 2v\bar{y}) \times \left(r_1 - \sigma_1 - q_1E - \frac{2(r_1+Ku)}{K}\bar{x} - \sigma_1\sigma_2\right).$$

We have $\lambda_2 + \lambda_3 = n_1 < 0$ and $\lambda_2\lambda_3 = n_2 > 0$. So $\lambda_1, \lambda_2, \lambda_3 < 0$. Thus the equilibrium point
$\bar{P}(\bar{x}, \bar{y}, 0)$ is locally asymptotically stable. $\qquad\square$

Theorem 3.3. *If the interior equilibrium point $P_1^*(x_1^*, y_1^*, z_1^*)$ exists, then P_1^* is locally asymptotically stable if (13) and (14) are realized, which are given in the proof.*

Proof. The characteristic equation of the system (1) at P_1^* is :

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0,$$

where

$$a_0 = -(r_2 - \sigma_2 - 2vy_1^*)\frac{\beta a^2 x_1^* z_1^*(b - x_1^{*2})}{(b+x_1^{*2})^3} = (\sigma_1\frac{x_1^*}{y_1^*} + vy_1^*)\frac{\beta a^2 x_1^* z_1^*(b - x_1^{*2})}{(b+x_1^{*2})^3}.$$

Using (8) we get $b - x_1^{*2} > 0$, which proves $a_0 > 0$.

$$a_1 = (r_1 - \sigma_1 - q_1E) - 2\left(\frac{r_1+Ku}{K}\right)x_1^* - \frac{az_1^*(b-x_1^{*2})}{(b+x_1^{*2})^2}(r_2 - \sigma_2 - 2vy_1^*) + \frac{\beta a^2 x_1^* z_1^*(b-x_1^{*2})}{(b+x_1^{*2})^3} - \sigma_1\sigma_2,$$

$$= \frac{r_1+Ku}{K}x_1^*(\sigma_1\frac{x_1^*}{y_1*} + vy_1^*) + \sigma_2\frac{vy_1^{*2}}{x_1^*} + \frac{ax_1^*z_1^*}{(b+x_1^{*2})}\left(\frac{\beta a(b-x_1^{*2})}{b+x_1^{*2}} - 2x_1^*(\sigma_1\frac{x_1^*}{y_1^*} + vy_1^*)\right).$$

Then,

(13) $$a_1 > 0 \Longrightarrow \frac{\beta a(b - x_1^{*2})}{b+x_1^{*2}} > 2x_1^*(\sigma_1\frac{x_1^*}{y_1^*} + vy_1^*).$$

$$a_2 = -\left(\left((r_1 - \sigma_1 - q_1E) - 2(\frac{r_1+Ku}{K})x_1^* - \frac{az_1^*(b-x_1^{*2})}{(b+x_1^{*2})^2} + (r_2 - \sigma_2 - 2vy_1^*)\right),\right.$$

$$= \frac{r_1+Ku}{K}x_1^* + \sigma_2\frac{y_1^*}{x_1^*} - \frac{2az_1^*x_1^{*2}}{(b+x_1^{*2})^2} + \sigma_1\frac{x_1^*}{y_1^*} + vy_1^*.$$

(14) $$a_2 > 0 \Longrightarrow \frac{r_1+Ku}{K} > \frac{2az_1^*x_1^*}{(b+x_1^{*2})^2}.$$

$$
\begin{aligned}
a_2 a_1 - a_0 &= \left(\tfrac{r_1 + Ku}{K} x_1^*(\sigma_1 \tfrac{x_1^*}{y_{1*}} + vy_1^*) + \sigma_2 \tfrac{vy_1^{*2}}{x_1^*} + \tfrac{ax_1^* z_1^*}{(b+x_1^{*2})} \left(\tfrac{\beta a(b - x_1^{*2})}{b + x_1^{*2}} - 2x_1^*(\sigma_1 \tfrac{x_1^*}{y_1^*} + vy_1^*) \right) \right) \\
&\quad \times \left(\tfrac{r_1 + Ku}{K} x_1^* + \sigma_2 \tfrac{y_1^*}{x_1^*} - \tfrac{2az_1^* x_1^{*2}}{(b+x_1^{*2})^2} + \sigma_1 \tfrac{x_1^*}{y_1^*} + vy_1^* \right) - (\sigma_1 \tfrac{x_1^*}{y_1^*} + vy_1^*) \tfrac{\beta a^2 x_1^* z_1^*(b - x_1^{*2})}{(b+x_1^{*2})^3}, \\
&= \left((\sigma_1 \tfrac{x_1^*}{y_{1*}} + vy_1^*) x_1^* (\tfrac{r_1 + Ku}{K} - \tfrac{2az_1^* x_1^{*2}}{(b+x_1^{*2})^2}) + \sigma_2 \tfrac{vy_1^{*2}}{x_1^*} \right) \\
&\quad \times \left(\tfrac{r_1 + Ku}{K} x_1^* + \sigma_2 \tfrac{y_1^*}{x_1^*} - \tfrac{2az_1^* x_1^{*2}}{(b+x_1^{*2})^2} + \sigma_1 \tfrac{x_1^*}{y_1^*} + vy_1^* \right) \\
&\quad + \tfrac{\beta x_1^* z_1^* a^2 (b - x_1^{*2})}{(b+x_1^{*2})^{''}} \left(\tfrac{r_1 + Ku}{K} x_1^* + \sigma_2 \tfrac{y_1^*}{x_1^*} - \tfrac{2az_1^* x_1^{*2}}{(b+x_1^{*2})^2} \right).
\end{aligned}
$$

Using the Routh-Hurwitz criteria, it is to check that all roots of the equation had a negative real parts if (13) and (14) realized, then P_1^* is locally asymptotically stable. $\qquad\square$

Theorem 3.4. *If the interior equilibrium point $P_2^*(x_2^*, y_2^*, z_2^*)$ exists, then P_2^* is unstable.*

Proof. The characteristic equation of the system (1) at P_2^* is :

$$
\lambda^3 + b_2 \lambda^2 + b_1 \lambda + b_0 = 0,
$$

where

$$
\begin{aligned}
b_0 &= -(r_2 - \sigma_2 - 2vy_2^*) \tfrac{\beta a^2 x_2^* z_2^*(b - x_2^{*2})}{(b + x_2^{*2})^3} = (\sigma_1 \tfrac{x_1^*}{y_1^*} + vy_1^*) \tfrac{\beta a^2 x_2^* z_2^*(b - x_2^{*2})}{(b + x_2^{*2})^3}, \\
b_1 &= \left((r_1 - \sigma_1 - q_1 E) - 2\left(\tfrac{r_1 + Ku}{K} \right) x_2^* - \tfrac{az_2^*(b - x_2^{*2})}{(b + x_2^{*2})^2} \right)(r_2 - \sigma_2 - 2vy_2^*) + \tfrac{\beta a^2 x_2^* z_2^*(b - x_2^{*2})}{(b + x_2^{*2})^3} - \sigma_1 \sigma_2, \\
b_2 &= -\left((r_1 - \sigma_1 - q_1 E) - 2\left(\tfrac{r_1 + Ku}{K} \right) x_2^* - \tfrac{az_2^*(b - x_2^{*2})}{(b + x_2^{*2})^2} + (r_2 - \sigma_2 - 2vy_2^*) \right).
\end{aligned}
$$

Using the Routh-Hurwitz criteria, based on (9) it is easy to check that $b_0 < 0$. Then P_2^* is unstable. $\qquad\square$

4. GLOBAL STABILITY OF THE EQUILIBRIUM POINTS $\bar{P}$ AND P_1^*

To prove the global stability of the interiors equilibrium points $\bar{P}$ and P_1^*, we state and prove the following theorems.

Theorem 4.1. *The equilibrium $\bar{P}(\bar{x}, \bar{y}, 0)$ is globally asymptotically stable.*

Proof. We consider the following positive definite function about $\bar{P}$:

$$
V(x, y) = \left(x - \bar{x} - \bar{x} \ln \left(\tfrac{x}{\bar{x}} \right) \right) + l \left(y - \bar{y} - \bar{y} \ln \left(\tfrac{y}{\bar{y}} \right) \right),
$$

where l is a positive constant, to be chosen later on.

Differentiating V with respect to time along the solutions of model (1), After the calculations we obtain :

$$\frac{dV}{dt} = (x - \bar{x})\left(-\left(\frac{r_1 + Ku}{K}\right)(x - \bar{x}) + \sigma_2\left(\frac{y}{x} - \frac{\bar{y}}{\bar{x}}\right)\right) + l(y - \bar{y})\left(-v(y - \bar{y}) + \sigma_1\left(\frac{x}{y} - \frac{\bar{x}}{\bar{y}}\right)\right).$$

Choosing $l = \frac{\sigma_2 \bar{y}}{\sigma_1 \bar{x}}$. So

$$\frac{dV}{dt} = -\left(\frac{r_1 + Ku}{K}\right)(x - \bar{x})^2 - v(y - \bar{y})^2 - \frac{\sigma_2}{x\bar{x}y}(y\bar{x} - \bar{y}x)^2 < 0.$$

Therefore, the equilibrium $\bar{P}$ is globally asymptotically stable. $\square$

Theorem 4.2. *The equilibrium $P_1(x_1^*, y_1^*, z_1^*)$ is globally asymptotically stable if (16) are realized.*

Proof. We utilize the geometric approach [16] to prove a global stability of $P_1(x_1^*, y_1^*, z_1^*)$

The variational matrix $J(x, y, z)$ of the system (1) is :

$$A = \begin{pmatrix} r_1 - \sigma_1 - q_1 E - \frac{2(r_1 + Ku)x}{K} - \frac{az(b - x^2)}{(b + x^2)^2} & \sigma_2 & -\frac{ax}{b + x^2} \\ \sigma_1 & r_2 - \sigma_2 - 2vy & 0 \\ \frac{\beta az(b - x^2)}{(b + x^2)^2} & 0 & \frac{\beta ax}{b + x^2} - (d + w + q_2 E) \end{pmatrix}$$

and its second additive compound matrix $A^{[2]}$ is :

$$A^{[2]} = \begin{pmatrix} A_{11} & 0 & \frac{ax}{b + x^2} \\ 0 & A_{22} & \sigma_2 \\ -\frac{\beta az(b - x^2)}{(b + x^2)^2} & \sigma_1 & \left(r_2 - \sigma_2 - 2vy + \frac{\beta ax}{b + x^2} - (d + w + q_2 E)\right) \end{pmatrix},$$

where

$$A_{11} = r_1 - \sigma_1 - q_1 E - \frac{2(r_1 + Ku)x}{K} - \frac{az(b - x^2)}{(b + x^2)^2} + r_2 - \sigma_2 - 2vy,$$

$$A_{22} = r_1 - \sigma_1 - q_1 E - -\frac{2(r_1 + Ku)x}{K} - \frac{az(b - x^2)}{(b + x^2)^2} + \frac{\beta ax}{b + x^2} - (d + w + q_2 E).$$

We set B as the following diagonal matrix :

$$B(x, y, z) = diag\left(\frac{x}{z}, \frac{x}{z}, \frac{x}{z}\right).$$

So we get :

$$B^{-1} = diag\left(\frac{z}{x}, \frac{z}{x}, \frac{z}{x}\right), \quad B_f = \frac{dB}{dX} = diag\left(\frac{\dot{x}}{z} - \frac{x}{z^2}\dot{z}, \frac{\dot{x}}{z} - \frac{x}{z^2}\dot{z}, \frac{\dot{x}}{z} - \frac{x}{z^2}\dot{z}\right),$$

$B_f B^{-1} = diag\left(\frac{\dot{x}}{x} - \frac{\dot{z}}{z}, \frac{\dot{x}}{x} - \frac{\dot{z}}{z}, \frac{\dot{x}}{x} - \frac{\dot{z}}{z}\right)$ and the matrix $C = B_f B^{-1} + BA^{[2]}B^{-1}$ can be written in the following block form :

$$C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

where :

$$C_{11} = \frac{\dot{x}}{x} - \frac{\dot{z}}{z} + r_1 - \sigma_1 - q_1 E - 2\left(\frac{r_1+Ku}{K}\right)x - \frac{az(b-x^2)}{(b+x^2)^2} + r_2 - \sigma_2 - 2vy,$$

$$C_{12} = \left(0 \quad \frac{ax}{b+x^2}\right),$$

$$C_{21} = \begin{pmatrix} 0 \\ -\frac{\beta az(b-x^2)}{(b+x^2)^2} \end{pmatrix},$$

$$C_{22} = \begin{pmatrix} \frac{\dot{x}}{x} + r_1 - \sigma_1 - q_1 E - \frac{2(r_1+Ku)x}{K} - \frac{az(b-x^2)}{(b+x^2)^2} & \sigma_2 \\ \sigma_1 & \frac{\dot{x}}{x} + r_2 - \sigma_2 - 2vy \end{pmatrix}.$$

We consider the norm in $\mathbb{R}^3$ as :

$$|(u,v,w)| = \max\{|u|, |v| + |w|\}$$

Where $|(u,v,w)|$ is a vector in $\mathbb{R}^3$ and denote by $\eta(B)$ the Lozinskiĭ measure with respect to this norm is given by :

(15) $$\eta(C) \leq \sup\{C_{11} + |C_{12}|, \eta_1(C_{22}) + |C_{21}|\} = \sup\{g_1, g_2\}$$

where

$$|C_{12}| = \max\left\{0, \frac{ax}{b+x^2}\right\} = \frac{ax}{b+x^2},$$

$$|C_{21}| = \max\left\{0, \left|-\frac{\beta az(b-x^2)}{(b+x^2)^2}\right|\right\} = \frac{\beta az(b-x^2)}{(b+x^2)^2},$$

$$\eta_1(C_{22}) = \frac{\dot{x}}{x} + \max\left\{r_1 - q_1 E - 2\left(\frac{r_1+Ku}{K}\right)x - \frac{az(b-x^2)}{(b+x^2)^2}; r_2 - 2vy\right\}$$

Then, g_1 and g_2 are expressed in this form :

$$g_1 = \frac{\dot{x}}{x} + \frac{ax(1-\beta)}{b+x^2} + (d+w+q_2 E) + r_1 - \sigma_1 - q_1 E - \frac{2(r_1+Ku)x}{K} - \frac{az(b-x^2)}{(b+x^2)^2} + r_2 - \sigma_2 - 2vy,$$

$$g_2 = \frac{\dot{x}}{x} + \frac{\beta az(b-x^2)}{(b+x^2)^2} + \max\left\{r_1 - q_1 E - 2\left(\frac{r_1+Ku}{K}\right)x - \frac{az(b-x^2)}{(b+x^2)^2}; r_2 - 2vy\right\}.$$

Hence, from (15) we obtain

$\eta(C) \leq \sup\{g_1, g_2\} = \frac{\dot{x}}{x} + \max\{\xi_1, \xi_2\}$ where :

$$\xi_1 = \frac{a\varepsilon(1-\beta)}{b+\varepsilon^2} + (d+w+q_2 E) + r_1 - \sigma_1 - q_1 E - 2\left(\frac{r_1+Ku}{K}\right)\varepsilon - \frac{a\varepsilon(b-\varepsilon^2)}{(b+\varepsilon^2)^2} + r_2 - \sigma_2 - 2v\varepsilon,$$

$$\xi_2 = \frac{\beta a\varepsilon(b-\varepsilon^2)}{(b+\varepsilon^2)^2} - \min\left\{-r_1 + q_1 E + 2\left(\frac{r_1+Ku}{K}\right)\varepsilon + \frac{a\varepsilon(b-\varepsilon^2)}{(b+\varepsilon^2)^2}; -r_2 + 2v\varepsilon\right\}.$$

where $\varepsilon = inf(x,y,z)$ is the constant of uniform persistence.

Now, we impose that

(16)
$$\frac{a\varepsilon(1-\beta)}{b+\varepsilon^2} + (d+w+q_2E) + r_1 - \sigma_1 - q_1E + r_2 - \sigma_2 - 2v\varepsilon < 2(\frac{r_1+Ku}{K})\varepsilon + \frac{a\varepsilon(b-\varepsilon^2)}{(b+\varepsilon^2)^2},$$
$$\frac{\beta a\varepsilon(b-\varepsilon^2)}{(b+\varepsilon^2)^2} < \min\{-r_1 + q_1E + 2(\frac{r_1+Ku}{K})\varepsilon + \frac{a\varepsilon(b-\varepsilon^2)}{(b+\varepsilon^2)^2}; -r_2 + 2v\varepsilon\}.$$

Then, from the above calculation we get

$\eta(C) \leq \frac{\dot{x}}{x} - \min\{-\xi_1, -\xi_2\}$

$i.e. \frac{1}{t}\int_0^t \eta(C)ds \leq \frac{1}{t}\ln\left(\frac{x(t)}{x(0)}\right) - \min\{-\xi_1, -\xi_2\}$

$\lim_{t\to+\infty} \sup \frac{1}{t}\int_0^t \eta(B)\,ds \leq -\min\{-\xi_1, -\xi_2\}$

The model (1) is globally asymptotically stable if (16) realized.   □

5. OPTIMAL HARVESTING POLICY

Recently, the optimal harvesting policy has played an important role in the field of biomathematics. Increasingly the inclusion of economic factors in population models, optimal harvesting policies is one of the most important problems taking into account both biological and economic. In this section, we wish to evaluate the economic performance of reserve-based management with the employment of the Pontryagin's Principle. The objective is to select a harvest method which can maximize the net profit. The present value I of a continuous time-stream of revenues is given by :

$$I = \int_0^{t_f} \pi(x,y,z,E)e^{-\delta t}\,dt = \int_0^{t_f} (p_1q_1x(t) + p_2q_2z(t) - D)Ee^{-\delta t}dt.$$

where D be the constant Harvesting cost per unit effort, p_1 is the constant price per unit biomass of the prey in the unreserved zone, p_2 is the constant price per unit biomass of the predator and δ is the instantaneous discount rate.

Thus, our objective is to maximize I subject to state equations (1) and to the control constraints :

$$0 \leq E \leq E_{max}$$

To solve this optimization problem, we utilize the Pontryagin's maximal Principle (see [6, 26, 33]). The associated Hamiltonian of the problem is given by :

$$
\begin{aligned}
H &= (p_1 q_1 x + p_2 q_2 z - D) E e^{-\delta t} \\
&+ \lambda_1 \left((r_1 - \sigma_1 - q_1 E) x - (\tfrac{r_1}{K} + u) x^2 + \sigma_2 y - \tfrac{axz}{b+x^2} \right) \\
&+ \lambda_2 \left((r_2 - \sigma_2) y + \sigma_1 x - vy^2 \right) + \lambda_3 \left(\tfrac{\beta axz}{b+x^2} - (d + w + q_2 E) z \right).
\end{aligned}
$$

where λ_1, λ_2 and λ_3 are the adjoint variables and

$$
\frac{\partial H}{\partial E} = (p_1 q_1 x + p_2 q_2 z - D) e^{-\delta t} - \lambda_1 q_1 x - \lambda_3 q_2 z.
$$

Now, the adjoint equation are :

(17)
$$
\begin{aligned}
\frac{\partial \lambda_1}{\partial t} &= -\frac{\partial H}{\partial x} = -p_1 q_1 e^{-\delta t} E - \lambda_1 \left((r_1 - \sigma_1 - q_1 E) - 2 \left(\tfrac{r_1 + Ku}{K} \right) x - az \tfrac{b - x^2}{(b+x^2)^2} \right) \\
&\quad - \lambda_2 \sigma_1 - \lambda_3 \beta az \tfrac{b - x^2}{(b+x^2)^2}, \\
\frac{\partial \lambda_2}{\partial t} &= -\frac{\partial H}{\partial y} = -\lambda_1 \sigma_2 - \lambda_2 (r_2 - \sigma_2 - 2vy), \\
\frac{\partial \lambda_3}{\partial t} &= -\frac{\partial H}{\partial z} = -e^{-\delta t} p_2 q_2 E + \lambda_1 \tfrac{ax}{b+x^2} - \lambda_3 \left(\tfrac{ax\beta}{b+x^2} - (d + w + q_2 E) \right).
\end{aligned}
$$

And then since H is linear in the control variable E. The optimal control E which maximizes H must satisfy the following conditions :

(18)
$$
\begin{cases}
E = E_{max}, & \text{for} \quad \frac{\partial H}{\partial E} > 0, \\
E = 0, & \text{for} \quad \frac{\partial H}{\partial E} < 0.
\end{cases}
$$

The function $\lambda_i e^{\delta t}$, $(i = 1, 2, 3)$ is the usual shadow price and $p_1 q_1 x + p_2 q_2 z - D$ is the net economic revenue on a unit harvest. Economically, if the first condition of (18) is satisfied, after the payment of all expenditure, we obtains a positive profit. According to the second condition of (18), we have a negative profits that is to say to a loss, then the fisherman will not exert any effort.

When $\frac{\partial H}{\partial E} = 0$, i.e. when the shadow price equals the net economic revenue on a unit harvest, the Hamiltonian H becomes independent of the variable E. This is necessary condition for the singular control E^* to be optimal over the control set $0 < E^* < E_{max}$.

Thus, the optimal harvesting policy is :

$$E(t) = \begin{cases} E_{max} & , \quad \frac{\partial H}{\partial E} > 0, \\ 0 & , \quad \frac{\partial H}{\partial E} < 0, \\ E^* & , \quad \frac{\partial H}{\partial E} = 0. \end{cases}$$

When $\frac{\partial H}{\partial E} = 0$, it follows that :

$$\lambda_1 e^{\delta t} q_1 x + \lambda_3 e^{\delta t} q_2 z = p_1 q_1 x + p_2 q_2 z - D = \frac{\partial \pi}{\partial E} e^{-\delta t}.$$

This involves that the user's cost of harvest per unit of effort equals the discounted value of the future marginal profit of the effort at the steady-state level. In the interior of equilibria P_1^*. The third equation of (17) in the following form :

$$\frac{d\lambda_3}{dt} = M_1 \lambda_3 - M_2 e^{-\delta t},$$

where :

$$\begin{aligned} M_1 &= -\frac{q_2 a z^*}{q_1 (b + x^{*2})}, \\ M_2 &= p_2 q_2 E - \frac{a(p_1 q_1 x^* + p_2 q_2 z^* - D)}{q_1 (b + x^{*2})}. \end{aligned}$$

By calculation, we get $\lambda_3(t) = \frac{M_2}{M_1 + \delta} e^{-\delta t}$.

Similarly,

$$\frac{d\lambda_2}{dt} = N_1 \lambda_2 - N_2 e^{-\delta t}$$

where :

$$\begin{aligned} N_1 &= -(r_2 - \sigma_2 - 2vy^*), \\ N_2 &= \frac{\sigma_2}{q_1 x^*} \left(\frac{q_2 z^* M_2}{M_1 + \delta} - (p_1 q_1 x^* + p_2 q_2 z^* - D) \right). \end{aligned}$$

Then $\lambda_2(t) = \frac{N_2}{N_1 + \delta} e^{-\delta t}$.

The first equation of (17) can be written as

$$\frac{d\lambda_1}{dt} = T_1 \lambda_1 - T_2 e^{-\delta t}.$$

where

$$\begin{aligned} T_1 &= -(r_1 - \sigma_1 - q_1 E) + 2\left(\frac{r_1 + Ku}{K}\right)x^* + \frac{az^*(b - x^{*2})}{(b + x^{*2})^2}, \\ T_2 &= p_1 q_1 E + \frac{N_2 \sigma_1}{N_1 + \delta} + \frac{M_2 \beta a z^*(b - x^{*2})}{(M_1 + \delta)(b + x^{*2})^2}. \end{aligned}$$

It easy to verify that : $\lambda_1 = \frac{T_2}{T_1+\delta}e^{-\delta t}$.

According to the above calculation, we have :

(19) $$\frac{T_2}{T_1+\delta}q_1x^* + \frac{M_2}{M_1+\delta}q_2z^* = p_1q_1x^* + p_2q_2z^* - D.$$

Thus, we denote the function by :

$$
\begin{aligned}
G(x) &= \frac{T_2}{T_1+\delta}q_1x + \frac{M_2}{M_1+\delta}q_2z - (p_1q_1x + p_2q_2z - D) \\
&= \frac{Kp_1q_1^2E(b+x^2)^2x}{(\delta-(r_1-\sigma_1-q_1E))K(b+x^2)^2+2(r_1+Ku)(b+x^2)+Kaz(b-x^2)} + \\
&\quad \frac{\sigma_1\sigma_2K(b+x^2)^2[p_2q_2^2Eq_1z(b+x^2)-q_2za(p_1q_1x+p_2q_2z-D)]}{[\delta q_1(b+x^2)-q_2az][\delta-(r_2-\sigma_2-2vy)][\delta-(r_1-\sigma_1-q_1E))K(b+x^2)^2+2(r_1+Ku)(b+x^2)+Kaz(b-x^2)]} - \\
&\quad \frac{\sigma_1\sigma_2(p_1q_1x+p_2q_2z-D)[\delta q_1(b+x^2-q_2az)]}{[\delta q_1(b+x^2)-q_2az][\delta-(r_2-\sigma_2-2vy)][\delta-(r_1-\sigma_1-q_1E))K(b+x^2)^2+2(r_1+Ku)(b+x^2)+Kaz(b-x^2)]} + \\
&\quad \frac{q_1K\beta azx(b-x^2)(b+x^2)^2[p_2q_2Eq_1(b+x^2)-a(p_1q_1x+p_2q_2z-D)]}{[\delta q_1(b+x^2)-q_2az][\delta-(r_1-\sigma_1-q_1E))K(b+x^2)^2+2(r_1+Ku)(b+x^2)+Kaz(b-x^2)]} + \\
&\quad \frac{p_2q_2^2Eq_1(b+x^2)-aq_2z(p_1q_1x+p_2q_2z-D)}{\delta q_1(b+x^2)-q_2az} - (p_1q_1x + p_2q_2z - D).
\end{aligned}
$$

Hence (19) can be written as $G(x_1^*) = 0$. After the calculations we get $G(0) > 0$ There exists a unique positive root $x_1^* = x_\delta$ of $G(x_1^*) = 0$ in the interval $0 < x_1^* < K$ if the following inequalities hold :

(20) $$G(K) < 0, \ \dot{G}(x) < 0 \quad for \quad 0 < x < K$$

Therefore, we summarize the analysis above by the following theorem.

Theorem 5.1. *If $E > 0$ and (20) is satisfied, then the optimal harvesting control E_δ and the corresponding solutions x_δ, y_δ, z_δ exist that maximize. Where :*

$$
\begin{aligned}
x_\delta &= x_1^*, \\
y_\delta &= \frac{r_2-\sigma_2+\sqrt{(r_2-\sigma_2)^2+4v\sigma_1x_\delta}}{2v}, \\
z_\delta &= \frac{b+x_\delta^2}{ax_\delta}[(r_1-\sigma_1-q_1E)x_\delta - (\frac{r_1+Ku}{K})x_\delta^2 + \sigma_2y_\delta].
\end{aligned}
$$

Furthermore, the adjoint functions λ_1, λ_2 and λ_3 exist that satisfy equations (17) with the the conditions $\lambda_i(t_f) = 0$, $i = 1,2,3$. from equation (19), we have

$$\frac{T_2}{T_1+\delta}q_1x_1^* + \frac{M_2}{M_1+\delta}q_2z_1^* = p_1q_1x_1^* + p_2q_2z_1^* - D \to 0 \ as \ \delta \to \infty.$$

Thus, the net economic revenue $\pi(x,y,z,E,t) = 0$.

So, if the discount rate δ is infinite, the economic gain is zero, which leads to the closure of the

fishery. Consequently, the interest rate rises, which leads to an increase in the rate of inflation. Consequently, the real value of the resource is reduced.

6. NUMERICAL SIMULATIONS

It is difficult to find realistic data in order to validate the results of our model (1). For this reason, we take some hypothetical data in order to illustrate the results that we have already established in the previous sections.

(1) In order to ensure the local stability of the equilibrium $\bar{P}$, we consider the following parameters :

$$
\begin{aligned}
&r_1 = 2, r_2 = 1, K = 4, \sigma_1 = 1, \sigma_2 = 0.5, a = 1, b = 1, q_1 = 0.5, \\
&q_2 = 0.8, E = 0.8, \beta = 0.8, u = 0.4, v = 0.4, w = 0.4, d = 0.3.
\end{aligned}
$$
(21)

The system (1) with the initial conditions $(x(0), y(0), z(0)) = (20, 20, 20)$. For this set of parameters (21), the conditions of existence (2) and (6) are established. Consequently, we find the equilibrium point following $\bar{P}(1.60, 3.38, 0)$. As well as, the local stability conditions mentioned in theorem 3.2 are realized, which proves the effectiveness of our theoretical results. So, the solution of our model (1) approaches to the equilibrium point $\bar{P}$ which is locally asymptotically stable. In the same way, there are few oscillations in the beginning for the populations of prey x and y before reaching the equilibrium. Although, the predator populations rapidly decrease to reach equilibrium. (see Figure 1)

(2) We consider the following parameters to prove the local stability of the equilibrium P_1^*:

$$
\begin{aligned}
&r_1 = 4, r_2 = 2, K = 2, \sigma_1 = 2, \sigma_2 = 1, a = 1, b = 1, q_1 = 0.5, \\
&q_2 = 0.5, E = 1, \beta = 5.7, u = 0.4, v = 0.4, w = 0.4, d = 0.01.
\end{aligned}
$$
(22)

The system (1) with the initial conditions $(x(0), y(0), z(0)) = (0.1, 1, 1)$. For this set of parameters (22), the conditions of existence (2) and (12) are established. Consequently, we find the equilibrium point following $P_1^*(0.16, 2.79, 18.63)$. As well as, the

local stability conditions mentioned in theorem 3.3 are realized, which proves the effectiveness of our theoretical results. So, the solution of our model (1) approaches to the equilibrium point P_1^* which is locally asymptotically stable. We remark that there are a few oscillations of prey populations x and y, then they decreases to converge towards the equilibrium. So, the predators population z can attack the prey x which implies the diminution of prey. After that, it converges to the equilibrium state. (see Figure 2)
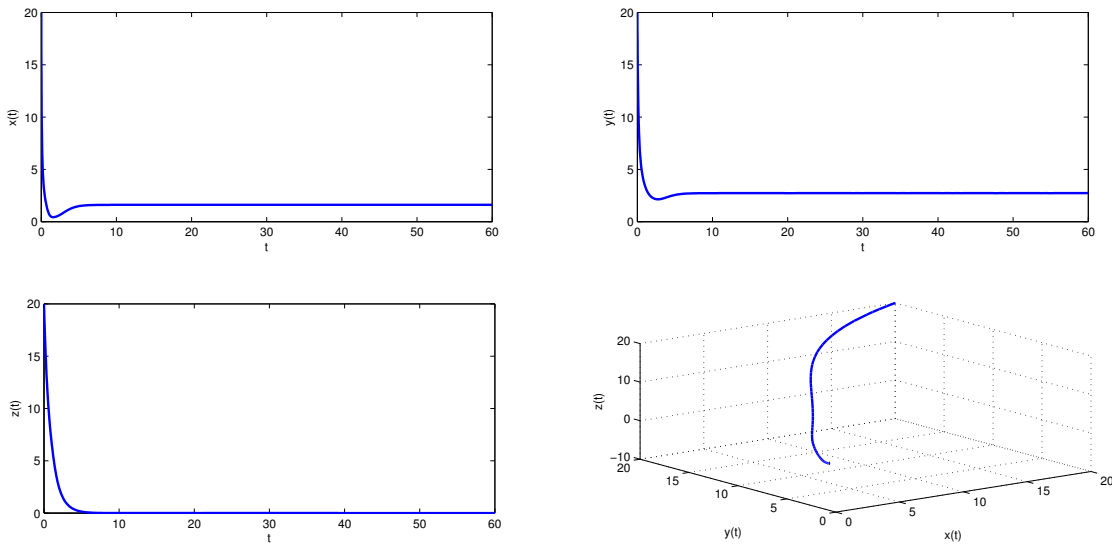

FIGURE 1. (a) Time series of $x(t)$ of the system, (b) Time series of $y(t)$ of the system, (c) Time series of $z(t)$ of the system, (d) The phase trajectory of the system (1)
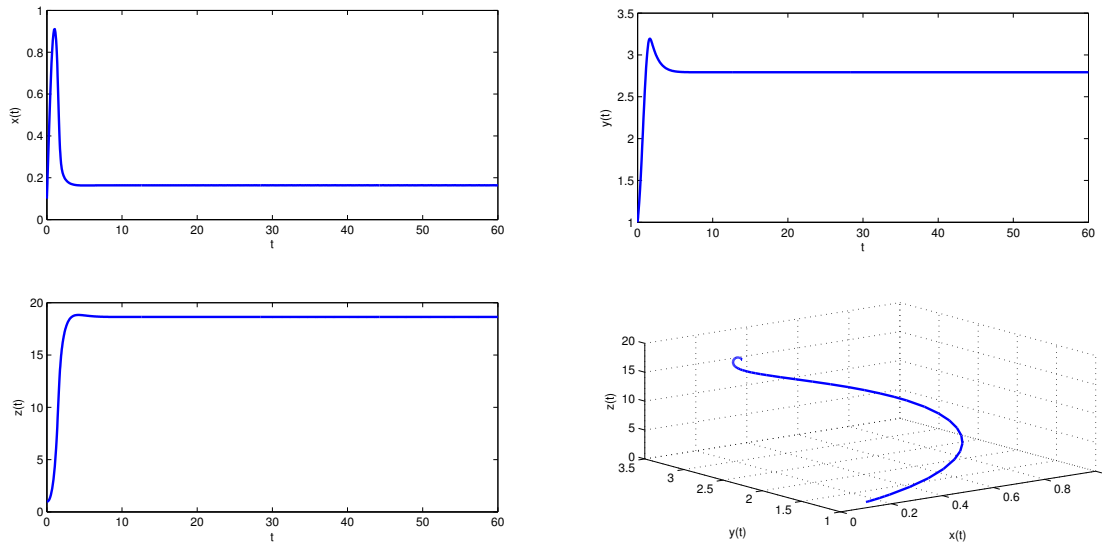
FIGURE 2. (a) Time series of $x(t)$ of the system, (b) Time series of $y(t)$ of the system, (c) Time series of $z(t)$ of the system, (d) The phase trajectory of the system (1)

7. CONCLUSION AND DISCUSSION

In recent years, researchers have proposed various prey-predator models to describe the interaction between the prey and predator population using the holling II function response. In this paper, we have proposed a mathematical model analyze and study the dynamics of prey-predator fishing using the function response holling type IV and toxic substances. Interactions between species are based on the following assumptions :

- The prey can move between the two fishing zones (free zone and prohibited fishing zone).
- The populations of predators are prohibited in the prohibited fishing area.
- The interaction between species of prois and predators is linked by the Holling type IV functional response.

To describe the behavior of our system (1), we proved the boundedness of the solutions. Then, we studied the existence of equilibria as well as their local and global stability. We studied the global stability of the equilibrium $\bar{P}$ using the Lyapunov function, and we used the geometric

approach to prove the global stability of the equilibrium P_1^*. To study the economic side, we formulated an optimal control problem which is solved by the Pontryagin's Maximum Principle. Finally, in numerical simulations, we choose the parameters to ensure the overall stability of the system.

In this article, there are factors that are not taken into account. As predator species can access the prohibited fishing area, predators can be divided into several types (Infected and susceptible), and competition factors, etc. So, we will try to take those factors into consideration in a future research.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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