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ON CERTAIN TYPES OF CONVERGENCE AND γ-CONTINUITY

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Abstract. In this article, some types of convergence are discussed along with a class of γ-continuous functions. It is known that various classes of generalized continuous functions are closed under the uniform convergence. We show that γ-continuity is closed with respect to a weaker type of convergence. Further properties of such types of convergence related to γ-continuous functions are obtained.

Keywords: quasi-uniform convergence; quasinormal convergence; almost uniform convergence; quasicontinuous; γ-continuous.

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1. INTRODUCTION

The notion of convergence, to gather with the notion of continuity, plays a crucial role in developing the theory of analysis, in particular, the theory of metric spaces and consequently uniform spaces. There are many known types of convergence of nets of functions. The most

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known types are pointwise and uniform convergence. It is known that various types of generalized continuous functions are closed under the uniform convergence but not the pointwise one, see [7] for somewhat continuity, [8] for quasi-continuity and [11] for cliquish functions.

There are several types of convergence in between pointwise and uniform convergence, we mention only those types that are dealt in this work. In 1979, Predio [9] defined a notion of quasi-uniform convergence, which is weaker than uniform convergence but stronger than pointwise one. She showed that quasi-uniform convergence posses similar properties of uniform convergence with respect to continuous functions. In the same year, Császár and Laczkovich [3] introduced another type of convergence, lies between pointwise and uniform convergence, called equal. The notion of equal convergence was used while modifying the Baire classification of real valued functions. In 1991, Bukovská [2] defined an equivalent concept to the equal convergence and named it quasinormal convergence. To reduce the level of confusion about the word "equal", we stick to the word "quasinormal". It is worth saying that quasinormal convergence is independent with the quasi-uniform one. In 1993, Ewert [6] introduced almost uniform convergence of functions. This type of convergence is weaker than uniform convergence but stronger than quasi-uniform one and independent with quasinormal convergence. She found some nice results on almost uniform convergence of functions. Among them, she proved that uniform and almost uniform coincide on compact spaces.

In this work, we mainly consider types of convergence (mentioned above) of nets of γ-continuous functions defined on a topological space with values in a uniform space. We study connections between such types of convergence and preserving of γ-continuity of functions.

2. Preliminaries

Let (X,τ) be a topological space and (Y,u) a uniform space with a family d_u of pseudometrics on Y inducing u. For a subset A of a space X, the closure and interior of A with respect to X are respectively denoted by $\mathrm{Cl}_X(A)$ and $\mathrm{Int}_X(A)$ (or simply $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$). A subset $A \subseteq X$ is called b-open [1] (or γ-open [5]) if $A \subseteq \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$. The family of all b-open sets in X is denoted by $BO(X)$.

For a space X, the intersection of two b-open sets in X need not be b-open ([5, Example 1.1.4]), so $BO(X)$ is not a topology on X. But one can generate a topology with this class in a natural way. We denote by τ_γ the topology generated by $BO(X)$. That is, $\tau_\gamma = \{U \subseteq X : U \cap A \in BO(X)$ whenever $A \in BO(X)\}$. The elements of τ_γ are called γ-open. Since intersection of an open set with a b-open set is b-open ([1, Proposition 2.3]), so all open sets are included in τ_γ. Therefore $\tau \subseteq \tau_\gamma \subseteq BO(X)$. More details on τ_γ can be found in [1, P. 62].

A function $f : X \to Y$ is called γ-continuous at a point $x_0 \in X$ if for each $d \in d_u$ and $\varepsilon > 0$, there exists a γ-open set A that contains x_0 such that $d(f(x), f(x_0)) < \varepsilon$ for all $x \in A$. Evidently, every continuous function is γ-continuous but not conversely. The Dirichlet function is γ-continuous but not continuous.

A space X is γ-compact [5] if every γ-open cover of X has a finite subcover. By the remark [5, Remark 3.1.1], every γ-compact is compact. However, there are spaces which are compact but not γ-compact. Take $X = \mathbb{R}$ with the topology $\tau = \{\emptyset, X, \{0\}\}$, so X is compact that is not γ-compact. A space X is locally γ-compact [5] if every point has a neighborhood which is itself contained in a γ-compact set. For a better view, we define the types of convergence as follow:

Definition 2.1. *Let $(S, \leq)$ be a directed set. Then the net $\{f_s\}_{s \in S}$ of functions $f_s : X \to Y$ is called*

(1) pointwise convergent to $f : X \to Y$ if for each $x \in X$, $d \in d_u$ and $\varepsilon > 0$, there exists $s_0 \in S$ such that

$$d(f_s(x), f(x)) < \varepsilon \quad \text{for } s \in S \text{ and } s_0 \leq s.$$

(2) uniformly convergent to $f : X \to Y$ if for each $d \in d_u$ and $\varepsilon > 0$, there exists $s_0 \in S$ such that

$$d(f_s(x), f(x)) < \varepsilon \quad \text{for all } x \in X \text{ and } s \in S,\ s_0 \leq s.$$

(3) quasi-uniformly convergent [9] *to $f : X \to Y$ if for each point $x_0 \in X$, $d \in d_u$ and $\varepsilon > 0$ there exists $s_0 \in S$ such that for every $s \in S$ and $s_0 \leq s$, there is a neighbourhood H of the point x_0 such that*

$$d(f_s(x), f(x)) < \varepsilon \quad \text{for all } x \in H.$$

(4) almost uniformly convergent [6] *to $f : X \to Y$ if for each point $x_0 \in X$, $d \in d_u$ and $\varepsilon > 0$ there exists neighbourhood V of the point x_0 and $s_0 \in S$ such that*

$$d(f_s(x), f(x)) < \varepsilon \quad \text{for all } x \in V \text{ and } s \in S,\ s_0 \leq s.$$

(5) quasinormally convergent to $f: X \to Y$ *if there exists a net* $\{\varepsilon_s\}_{s\in S}$ *of non-negative real numbers that converges to zero such that for each* $x \in X$ *and* $d \in d_u$ *there exists an index* $s_0 \in S$ *with the property*

$$d\left(f_s(x), f(x)\right) \leq \varepsilon_s \quad \text{for each } s \in S \text{ and } s_0 \leq s.$$

Note that the definition of quasinormal convergence for a sequence of real valued functions was defined in [3, 2].

At this place, a connection between these type of convergence is needed. This diagram is a slight enlargement of the first diagram in [6] and Remark 4.1 in [10].

$$\begin{tikzcd}
 & \text{quasinormal} \arrow[dr, Rightarrow] & \\
\text{uniform} \arrow[ur, Rightarrow] \arrow[d, Rightarrow] & & \text{pointwise} \\
\text{almost uniform} \arrow[rr, Rightarrow] & & \text{quasi-uniform} \arrow[u, Rightarrow]
\end{tikzcd}$$

Diagram I

In general, none of the implications is reversible, as shown in the following examples:

Example 2.2. *(i) Consider the functions* g_n, f *defined in* [6, Example 1.1]*, then* $\{g_n\}_{n\in\mathbb{N}}$ *converges to* f *is almost uniformly but not uniformly.*

(ii) Consider the functions f_n, f *defined in* [6, Example 1.1]*, then* $\{f_n\}_{n\in\mathbb{N}}$ *converges to* f *quasi-uniformly but not almost uniformly.*

(iii) Let $f_n(x) = x^n$ *be defined on* $X = [0,1]$*. Then* $\{f_n\}_{n\in\mathbb{N}}$ *converges to* f *pointwisely but not quasi-uniformly (see* [10, Example 4.1]*), where* $f(x) = 1$ *for* $x = 1$ *and* $f(x) = 0$ *otherwise.*

(iv) The sequence of functions $\{g_m\}_{m\in\mathbb{N}}$ *constructed in* [2, Example 1.7]*, converges to the zero function almost uniformly (by* [4, Theorem 2] *since* g_m *and* f *are continuous) (and consequently, quasi-uniformly and pointwisely) but not quasi-normally.*

(v) Let f_n, f *be such functions given in (iii). Then* $\{f_n\}_{n\in\mathbb{N}}$ *converges to* f *quasinormally but not quasi-uniformly (also, not almost uniformly).*

We remark that almost uniform and quasi-uniform convergence are independent with quasi-normal one.

3. The Results

The γ-continuity with respect to entourages can be stated as follows:

Remark 3.1. *A function $f : X \to Y$ is called γ-continuous at a point $x_0 \in X$ if and only if for each $U \in u$, there exists a γ-open set A containing x_0 such that*

$$(f(x), f(x_0)) \in U \quad \textit{for all } x \in A.$$

We now show that the set of γ-continuous functions is not closed with respect to the pointwise limit. Consider the functions f_n, f in Example 2.2 (iii). Clearly f_n are continuous and so γ-continuous. Then $\{f_n\}_{n\in\mathbb{N}}$ pointwise converges to f but f is not γ-continuous. However, γ-continuity is closed under quasi-uniform convergence.

Theorem 3.2. *If a net $\{f_s\}_{s\in S}$ of γ-continuous functions $f_s : X \to Y$ is quasi-uniformly convergent to a function $f : X \to Y$, then the limit function f is γ-continuous.*

Proof. Let x_0 be any element and $U \in u$ be an arbitrary entourage. Then there exists a symmetric entourage $V \in u$ such that $VoVoV \subseteq U$. Since the net $\{f_s\}_{s\in S}$ converges to a function f quasi-uniformly, then there exists $s_0 \in S$ such that for every $s \in S$ with $s_0 \leq s$, there is an open neighbourhood H of the point x_0 such that

$$(f_s(x), f(x)) \in V \qquad \text{for all } x \in H.$$

Meanwhile, since V is symmetric, then

$$(f(x), f_s(x)) \in V.$$

Also, since $x_0 \in H$, then

$$(f_s(x_0), f(x_0)) \in V.$$

But since f_s is γ-continuous at the point x_0, then for the entourage V there exists a γ-open set A containing x_0 such that

$$(f_s(x), f_s(x_0)) \in V \quad \text{for all } x \in A.$$

Take $G = H \bigcap A$. Since H is an open set and A is a γ-open set, so G is a γ-open set. As a result

$$(f(x), f(x_0)) \in VoVoV \subseteq U \quad \text{for all } x \in G.$$

This implies that the function f is γ-continuous at x_0. But since x_0 is an arbitrary point, hence f is γ-continuous.

□

The converse of the above theorem need not be true in general.

Example 3.3. *Let $X = [0,1]$ with the indiscrete topology τ and let $Y = \mathbb{R}$ with the usual metric d. If $f_n : X \to Y$, $n = 0, 1, 2, 3, \cdots$, are functions defined by $f_n(x) = \frac{1}{nx+1}$, then the sequences $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the function $f(x) = 1$ for $x = 0$ and $f(x) = 0$ for $x \neq 0$. Evidently f and f_n are γ-continuous functions for each n. But f_n does not converge quasi-uniformly to f.*

Theorem 3.2 implies that

Corollary 3.4. *If a net $\{f_s\}_{s \in S}$ of γ-continuous functions $f_s : X \to Y$ is almost uniformly (uniformly) convergent to a function $f : X \to Y$, then the limit function f is γ-continuous.*

Theorem 3.5. *Let X be a γ-compact space and let $\{f_s\}_{s \in S}$ be a monotonic net of γ-continuous functions $f_s : X \to Y$ that converges to a function $f : X \to Y$ pointwise. If the limit function f is γ-continuous, then $\{f_s\}_{s \in S}$ converges to f uniformly.*

Proof. Let $\varepsilon > 0$ be given and $d \in d_u$ be any pseudometric. Put

$$A_s = \{x \in X : \; d(f_s(x), f(x)) \geq \varepsilon\}$$

Since f and f_s are γ-continuous functions, therefore A_s is γ-closed set for each $s \in S$. By assumption, the net $\{f_s\}_{s \in S}$ is monotonic, so $\{A_s\}_{s \in S}$ is monotonically decreasing. Now let $x \in X$ be an element. Since the net $\{f_s\}_{s \in S}$ converges to f pointwise, then there exists $s_0 \in S$ such that

$$d(f_s(x), f(x)) < \varepsilon \quad \text{for each } s \in S \text{ and } s_0 \leq s.$$

It follows that $x \notin A_s$ for each $s \in S$ and $s_0 \leq s$. So $x \notin \bigcap\limits_{s \in S} A_s$, and $\bigcap\limits_{s \in S} A_s = \varnothing$. By γ-compactness of X and monotonicity of $\{A_s\}_{s \in S}$ there is s_0 such that $A_s = \varnothing$ for each $s \in S$ and $s_0 \leq s$. It means

$$d\left(f_s(x), f(x)\right) < \varepsilon \quad \text{for all } x \in X \text{ and } s \in S,\ s_0 \leq s.$$

Hence the net $\{f_s\}_{s\in S}$ converges to f uniformly. □

Theorem 3.6. *Let X be a locally γ-compact space and let $\{f_s\}_{s\in S}$ be a monotonic net of γ-continuous functions $f_s : X \to Y$ that converges to a function $f : X \to Y$ pointwise. If the limit function f is γ-continuous, then $\{f_s\}_{s\in S}$ converges to f almost uniformly.*

Proof. Let $x \in X$ be any element. Then there is a neighbourhood V of the point x and a γ-compact set K such that $x \in V \subseteq K$. Let $\varepsilon > 0$ be given and $d \in d_u$ be any pseudometric. Put

$$B_s = \{x \in K :\ d\left(f_s(x), f(x)\right) < \varepsilon\}.$$

Since f and f_s are γ-continuous functions, therefore B_s is γ-open set for each $s \in S$. Also since the net $\{f_s\}_{s\in S}$ converges to f pointwise, so the collection $\{B_s : s \in S\}$ forms a γ-open cover of K. Due to γ-compacness of K and monotonicity of $\{f_s\}_{s\in S}$ there is s_0 such that $K \subseteq B_s$ for all $s \in S$ and $s_0 \leq s$. Therefore

$$d\left(f_s(x), f(x)\right) < \varepsilon \qquad \text{for all } x \in V \text{ and } s \in S,\ s_0 \leq s.$$

Hence the net $\{f_s\}_{s\in S}$ converges to f almost uniformly. □

Theorem 3.7. *Let X be a discrete space and let $\{f_s\}_{s\in S}$ be a monotonic net of (any) functions $f_s : X \to Y$ that converges to a function $f : X \to Y$ pointwise. Then $\{f_s\}_{s\in S}$ converges to f almost uniformly.*

Proof. Since X is discrete, then open sets and γ-open are equivalent. So all functions defined on X are γ-continuous and X is locally γ-compact. By Theorem 3.6, f_s converges to f almost uniformly. □

Lemma 3.8. *Let $\{f_s\}_{s\in S}$ be a net of functions $f_s : X \to Y$ that converges to a function $f : X \to Y$ pointwise. If X is a locally finite space, then $\{f_s\}_{s\in S}$ converges to f almost uniformly.*

Proof. Let $x_0 \in X$, $d \in d_u$ and $\varepsilon > 0$ be given. By assumption, there is a finite open set G containing x_0 and so for each $x \in G$ there exists $s_x \in S$ such that

$$d\left(f_s(x), f(x)\right) < \varepsilon \quad \text{for each } s \in S \text{ and } s_x \leq s.$$

But since G is finite, we can set $s_0 = \max\{s_x : x \in G\}$. Therefore, we have $s_0 \in S$ such that

$$d\left(f_s(x), f(x)\right) < \varepsilon \quad \text{for all } x \in G \text{ and } s \in S,\, s_0 \leq s.$$

Thus $\{f_s\}_{s\in S}$ converges to f almost uniformly. □

Theorem 3.9. *Let $\{f_s\}_{s\in S}$ be a net of γ-continuous functions $f_s : X \to Y$ that converges to a function $f : X \to Y$ pointwise. If X is a locally finite space, then the limit function f is γ-continuous.*

Proof. The proof is an immediate consequence of Lemma 3.8 and Theorem 3.2. □

The following result is due Császár and Laczkovich [3, Theorem 5.1] and Bukovská [2, Theorem 1.2] for sequences of real valued functions.

Theorem 3.10. *A net $\{f_s\}_{s\in S}$ of functions $f_s : X \to Y$ converges quasinormally to a function $f : X \to Y$ if and only if there exists a monotonically increasing net $\{A_t\}_{t\in T}$ of sets with a directed set $(T, \leq')$ such that $X = \bigcup\limits_{t\in T} A_t$ and the net $\{f_s\}_{s\in S}$ is uniformly convergent to the function f on each A_t.*

Proof. Assume that $\{f_s\}_{s\in S}$ is a net of functions that converges to a function f quasinormally. Then there exists $\{\varepsilon_s\}_{s\in S}$ of non-negative real numbers that converges to zero such that for each $x \in X$ and $d \in d_u$ there exists an index $s_0 \in S$ with the property

$$d\left(f_s(x), f(x)\right) \leq \varepsilon_s \qquad \text{for each } s \in S \text{ and } s_0 \leq s.$$

First, we choose $(T, \leq')$ such that $T = S$ and $\leq'$ is the same as $\leq$. Put

$$A_t = \{x \in X : \; d\left(f_s(x), f(x)\right) \leq \varepsilon_s \text{ for each } s \in S \text{ and } t \leq s\}.$$

We have $A_{t_\alpha} \subseteq A_{t_\beta}$ where $t_\alpha \leq' t_\beta$, thus the net $\{A_t\}_{t\in T}$ of sets is monotonically increasing also $X = \bigcup\limits_{t\in T} A_t$. To show that the net $\{f_s\}_{s\in S}$ is uniformly convergent to f on each set A_t. We put

$$\varepsilon_s^t = \begin{cases} \sup\left\{d\left(f_s(x), f(x)\right) : x \in A_t\right\} & \text{for } s \in S \text{ and } s \leq t \\ \varepsilon_s & \textit{otherwise}. \end{cases}$$

It is obvious that $\{\varepsilon_s^t\}_{s\in S}$ is a net of non-negative real numbers and converges to zero for each $t \in T$. As a consequence,

$$d\left(f_s(x), f(x)\right) \leq \varepsilon_s^t \qquad \text{for each } s \in S \text{ and } x \in A_t.$$

Hence $\{f_s\}_{s\in S}$ converges to f uniformly on a set A_t for each $t \in T$.

Conversely, suppose that $\{A_t\}_{t\in T}$ be a monotonic increasing net of set such that $X = \bigcup_{t\in T} A_t$ and $\{f_s\}_{s\in S}$ is uniformly convergent to a function f on a set A_t for each $t \in T$. Then for each set A_t there exists a net $\{\delta_s^t\}_{s\in S}$ of non-negative real that converges to zero such that

$$(1) \qquad d\left(f_s(x), f(x)\right) \leq \delta_s^t \qquad \text{for each } x \in X \text{ and } s \in S.$$

So there exists a net $\{\delta_t\}_{t\in T}$ of non-negative real that converges to zero such that for each δ_t there exists s_t with the property

$$(2) \qquad \delta_s^t \leq \delta_t \qquad \text{for each } s \in S \text{ and } s_t \leq s.$$

Define

$$(3) \qquad \varepsilon_s = \delta_t \qquad \text{for each } s \in S \text{ and } s_t \leq s < s_{t'}.$$

Then $\{\varepsilon_s\}_{s\in S}$ is a net of non-negative real numbers and converges to zero. Now let $x \in X$, then there exists $t_0 \in T$ such that $x \in A_t$ for each $t_0 \leq' t$, therefore by (1), (2) and (3) we have

$$d\left(f_s(x), f(x)\right) \leq \varepsilon_s \qquad \text{for each } s \in S \text{ and } s_t \leq s$$

Thus $\{f_s\}_{s\in S}$ is quasinormally convergent to f on X. □

Theorem 3.11. *Let $\{f_s\}_{s\in S}$ be a net of γ-continuous functions $f_s : X \to Y$. If the net $\{f_s\}_{s\in S}$ converges quasinormally to a function $f : X \to Y$, then there exists a monotonically increasing net $\{B_t\}_{t\in T}$ of γ-closed sets with a directed set $(T, \leq')$ such that $X = \bigcup_{t\in T} B_t$ and the net $\{f_s\}_{s\in S}$ is uniformly convergent to the function f on each set B_t.*

Proof. Suppose that $\{f_s\}_{s\in S}$ is a net of γ-continuous functions that converges to a function f quasinormally. Then there exists a net $\{\varepsilon_s\}_{s\,\in S}$ of non-negative real numbers that converges to zero such that for each $x \in X$ and $d \in d_u$ there exists an index $s_0 \in S$ with the property

$$d\left(f_s(x), f(x)\right) \leq \varepsilon_s \quad \text{for each } s \in S \text{ and } s_0 \leq s.$$

Take $T = S$ and let $\leq'$ be similar to $\leq$, so $(T, \leq')$ is a directed set. Put

$$B_t = \{x \in X : d\left(f_s(x), f_{s_0}(x)\right) \leq \varepsilon_s + \varepsilon_{s_0} \text{ for each } s, s_0 \in S \text{ and } t \leq s, s_0\}.$$

Since f_s and f_{s_0} are γ-continuous functions for each $s, s_0 \in S$, therefore B_t is a γ-closed set for each $t \in T$. Clearly the net $\{B_t\}_{t\in T}$ of sets is monotonically increasing and $X = \bigcup_{t\in T} B_t$. To show that the net $\{f_s\}_{s\in S}$ is uniformly convergent to f on each set B_t. Put

$$\varepsilon_s^t = \begin{cases} \sup\{d\left(f_s(x), f(x)\right) : x \in B_t\} & \text{for } s \in S \text{ and } s \leq t \\ \varepsilon_s & otherwise. \end{cases}$$

Evidently, $\{\varepsilon_s^t\}_{s\ \in S}$ is a net of non-negative real numbers and converges to zero for each $t \in T$. As a consequence,

$$d\left(f_s(x), f(x)\right) \leq \varepsilon_s^t \quad \text{ for each } s \in S \text{ and } x \in B_t$$

Hence the net $\{f_s\}_{s\in S}$ converges to f uniformly on a set B_t for each $t \in T$. □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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