Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 1, 1053-1062
https://doi.org/10.28919/jmcs/5277
ISSN: 1927-5307

# NONSTANDARD ANALYSIS OF UNIFORM SPACES

S. ALAGU[1], R. KALA[2]

[1]Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli 627012, India

[2]Department of Mathematics, Manonmaniam Sundaranar University, Tirunelveli 627012, India



**Abstract.** There are nonstandard characterizations of classical topological concepts like total boundedness, completeness, comapactness etc., in metric spaces. In this article we present nonstandard extensions of these characterizations to uniform spaces. We also furnish completion of uniform spaces. We accomplish this by extending the methods in metric spaces by way of passing from sequences to filters and nets.

**Keywords:** standard; non-standard; uniform structure; uniform spaces; filter; nets; Cauchy filter; Cauchy net; completeness; compactness.

**2010 AMS Subject Classification:** 54J05, 54E15, 54E50.

## 1. Introduction

Non-standard analysis is a branch of Mathematics introduced by Abraham Robinson in 1966[1]. Abraham Robinson constructed a superstructure to work in any given structure like the Euclidean spaces, topological spaces, algebraic structures (rings, fields etc.,.), graphs and so on. The basic idea is not to study the superstructure but to study the classical spaces by getting on to a higher platform, namely a superstructure, and get a microscopic view of the classical

*Corresponding author
E-mail address: alagu391@gmail.com
Received November 28, 2020

space below. For instance, we get the completion of a uniform space $X$ with the tools available in a superstructure $^{*}X$.

## 2. Preliminaries

First we collect all the basics of Uniform spaces which are required in this article. For this our references are [3], [4], [5]. For the fundamentals of Nonstandard analysis by Abraham Robinson, we refer to [1], [2].

**Definition 2.1.**

Let $X$ be a set. A uniform structure on $X$ is a filter $\Psi \subseteq X \times X$ such that

(i) $\forall\, U \in \Psi, \Delta(X) \subseteq U$, where $\Delta(X) = \{(x,x) : x \in X\}$ being the diagonal of $X$.

(ii) $\forall\, U \in \Psi, U^{-1} \in \Psi$, where $U^{-1} = \{(x,y) : (y,x) \in U\}$

(iii) $\forall\, U \in \Psi, \exists\, V \subseteq U$ such that $V \circ V \subseteq U$,

where $V \circ W = \{(x,z) : (x,y) \in V \wedge (y,z) \in W\}$, for general $V, W \subseteq X \times X$.

**Definition 2.2.**

Let $X$ be a set with uniform structure $\Psi$. For $V \in \Psi, x \in X$ define $V(x) = \{y \in X : (x,y) \in V\}$. There exists a topology on $X$ such that $\forall\, x \in X,\ \{V(x) : V \in \Psi\}$ is a neighbourhood base for $x$. Henceforth $X$ with this induced topology will be referred to as the uniform space $X$.

**Definition 2.3.**

Let $X$ be a uniform space with uniform structure $\Psi$.

(i) A cauchy filter on $X$ is a filter $\Omega$ on $X$ such that $\forall\, V \in \Psi,\ \exists\, A \in \Omega$ with $A \times A \subseteq V$.

(ii) A cauchy net in $X$ is a net $(x_\alpha)_{\alpha \in D}$, $D$ being the directed set, satisfying the following : $\forall\, V \in \Psi, \exists\, \alpha_0 \in D$ such that $\alpha, \beta \geq \alpha_0 \Rightarrow \left(x_\alpha, x_\beta\right) \in V$.

Let us now recall the following definitions.

**Definition 2.4.**

(i) A filter $\Omega$ on a topological space $X$ is said to converge to $x \in X$ if for every neighbourhood $G$ of $x$, $\exists\, A \in \Omega$ such that $A \subseteq G$.

(ii) A net $(x_\alpha)_{\alpha \in D}$ in a topological space $X$ is said to converge to $x \in X$ if given any neighbourhood $G$ of $x$, $\exists \alpha_0 \in D$ such that $\alpha \geq \alpha_0 \Rightarrow (x_\alpha, x) \in G$

**Definition 2.5.**

A uniform space $X$ is said to be complete if every cauchy filter on $X$ converges.

Equivalently, we have the following :

**Proposition 2.6.**

A uniform space $X$ is complete if and only if every cauchy net on $X$ converges.

As a common notation as in [1],[2], ${}^*X$ denotes a nonstandard extension of $X$, $V(X), V({}^*X)$ the corresponding superstructures on $X$, ${}^*X$ respectively. We assume $V({}^*X)$ is an enlargement of ${}^*X$, as defined in [2].
We now give the definition of concurrence.

**Definition 2.7.**

A binary relation $P$ is said to be concurrent on $A \subseteq domP$ if for each finite set $\{x_1, x_2 \ldots . x_n\}$ in $A$ there is a $y \in rangeP$ so that $\langle x_i, y \rangle \in P, 1 \leq i \leq n$. $P$ is concurrent if it is concurrent on $domP$.

The following proposition is from [2].

**Proposition 2.8.**

The following are equivalent.
(i) $V({}^*X)$ is an enlargement of $V(X)$.
(ii) For each concurrent relation $P \in V(X)$ there is an element $b \in range\ {}^*P$ so that $\langle {}^*x, b \rangle \in {}^*P$ for all $x \in domP$

We now recall the notion of nearness in the case of a topological space $X$.

**Definition 2.9.**

$y \in {}^*X$ is said to be near-standard if there exists $x \in X$ such that for every neighbourhood $G$ of $x$, $y \in {}^*G$. We denote it by $y \simeq x$.

If $X$ is a metric space, it is equivalent to saying $\forall\, \varepsilon > 0$, ${}^*d(y, x) < \varepsilon$.

In the case of a uniform space $X$ with a uniform structure $\Psi, y \in {}^*X$ is near-standard if

$\exists\, x \in X$ such that $\forall\, V \in \Psi,\ y \in {}^*(V(x))$ which is equivalent to $(x,y) \in {}^*V$.

When $X$ is a metric space, we have the following notion of pre-near-standardness.

**Definition 2.10.**

$y \in {}^*X$ is said to be pre-near-standard if $\forall\, \varepsilon > 0,\ \exists\, x \in X$ such that ${}^*d(y,x) < \varepsilon$.

For a uniform space, the notion of pre-near-standradness extends as follows.

**Definition 2.11.**

Let $X$ be a uniform space with a uniform structure $\Psi$. $y \in {}^*X$ is said to be pre-near-standard if $\forall\, V \in \Psi, \exists\, x \in X$ such that $y \in {}^*(V(x))$ which is equivalent to $(x,y) \in {}^*V$.

**Definition 2.12.**

Let $X$ be a uniform space with uniform structure $\Psi$. We say $x,y \in {}^*X$ are near if $x,y \in {}^*V$ for every $V \in \Psi$. We denote it by $x \simeq y$.

Clearly $\simeq$ is an equivalence relation.

## 3. Main Results

Here we present nonstandard characterizations of the classical topological concepts namely, completeness, total boundedness and compactness in uniform spaces.

**Proposition 3.1.**

A uniform space is complete if and only if every pre-near-standard point $y \in {}^*X$ is near-standard to some $x \in X$.

*Proof.* Let $X$ be complete and $y \in {}^*X$ be pre-near-standard.

$\forall\, V \in \Psi$, fix $x_V \in X$ such that $(x_V, y) \in {}^*V$

$\forall\, W \in \Psi$, define $\Gamma_W = \{x_V : V \subseteq W\}$

Given $W_1, W_2 \in \Psi$, we have $\Gamma_{W_1 \cap W_2} = \Gamma_{W_1} \cap \Gamma_{W_2}$

Therefore there exists a filter $\Lambda$ containing all $\Gamma_W$'s, $W \in \Psi$.

Claim (i) : $\Lambda$ is a cauchy filter.

Let $V \in \Psi$ and $W$ be symmetric with $W \subseteq V$ and $W \circ W \subseteq V$

Enough to prove : $\Gamma_W \times \Gamma_W \subseteq V$

Let $(x_G, x_H) \in \Gamma_W \times \Gamma_W$, where $G \subseteq W, H \subseteq W$

Now $(x_G, y) \in {}^*G$ and hence $(x_G, y) \in {}^*W$

Similarly $(x_H, y) \in {}^*W$

Let $a, b \in X$

$(\forall\, z \in X)\,[(a,z) \in W \wedge (b,z) \in W \Rightarrow (a,b) \in V]$, since $W \circ W \subseteq V$

$(\forall\, z \in {}^*X)\,[(a,z) \in {}^*W \wedge (b,z) \in {}^*W \Rightarrow (a,b) \in {}^*V]$, by Transfer.

Now $(x_G, y) \in {}^*W$ and $(x_H, y) \in {}^*W$

Therefore $(x_G, x_H) \in {}^*V$

Since $x_G, x_H \in X$, we get $(x_G, x_H) \in V$

Therefore $\Gamma_W \times \Gamma_W \subseteq V$

Therfore $\Lambda$ is a cauchy filter on $X$ as claimed and hence $\Lambda$ converges to some $x \in X$.

Claim (ii) : $y \simeq x$

Take any ${}^*(V(x))$, where $V \in \Psi$. We want to show $y \in {}^*(V(x))$

That is, to show $(x,y) \in^* V$

Fix a symmetric $W$ with $W \subseteq V$ and $W \circ W \subseteq V$

Then $\Gamma_W$ intersects $W(x)$, since $\Lambda$ converges to $x$.

Therefore for some $U \subseteq W, x_U \in W(x)$

That is, $(x, x_U) \in W$

Also $(x_U, y) \in {}^*U \subseteq {}^*W$

Now, $(\forall\, z \in X)\,[(x,a) \in W \cap (a,z) \in W \Rightarrow (x,z) \in V]$

Therefore by Transfer, $(\forall\, z \in {}^*X)\,[(x,a) \in {}^*W \cap (a,z) \in {}^*W \Rightarrow (x,z) \in {}^*V]$

Taking $a = x_U, z = y$, we get $(x,y) \in {}^*V$

This completes the proof of the 'if' part.

Conversely let every pre-near standard point of $X^*$ be near standard.

To prove : $X$ is complete.

Let $(x_\alpha)_{\alpha \in D}$ be a cauchy net in $X$.

By cofinality of the directed set $D, \forall\ \alpha_1, \alpha_2, ...\alpha_n \in D, \exists\ \gamma \in D$ such that $\gamma \geq \alpha_i\ for\ i = 1, 2, ...n$.

By concurrence, $\exists\ \gamma \in\ ^*D$ such that $\gamma \geq\ ^*\alpha = \alpha\ \forall \alpha \in D$.

Claim : $^*x_\gamma$ is pre-near standard in $^*X$.

Let $V \in \Psi$.

$\exists\ \alpha_0 \in D$ such that $\alpha, \beta \in D$ and $\alpha, \beta \geq \alpha_0 \Rightarrow \left(x_\alpha, x_\beta\right) \in V$.

Therefore $\alpha, \beta \in\ ^*D$ and $\alpha, \beta \geq \alpha_0 \Rightarrow \left(^*x_\alpha,\ ^*x_\beta\right) \in\ ^*V$

In particular, $\alpha \in\ ^*D$ and $\alpha \geq \alpha_0 \Rightarrow (^*x_\alpha, x_{\alpha_0}) \in\ ^*V$

Therefore $\left(^*x_\gamma, x_{\alpha_0}\right) \in\ ^*V$

$^*x_\gamma$ is pre-near standard in $^*X$.

By hypothesis, $^*x_\gamma$ is near standard to some $y \in X$.

Claim : $(x_\alpha) \to y$

Let $U(y)$ be any basic neighbourhood of $y$, where $U \in \Psi$ and $U$ is symmetric.

First $^*x_\gamma \in\ ^*(U(y))$

Equivalently, $\left(^*x_\gamma, y\right) \in\ ^*U$ —-(1)

$\exists\ \beta_0 \in D$ such that $\alpha, \beta \in D$ and $\alpha, \beta \geq \beta_0 \Rightarrow \left(x_\alpha, x_\beta\right) \in U$

Therefore $\alpha, \beta \in\ ^*D$ and $\alpha, \beta \geq \beta_0 \Rightarrow \left(^*x_\alpha, ^*x_\beta\right) \in\ ^*U$

In particular, $\alpha \geq \beta_0 \Rightarrow \left(^*x_\alpha, ^*x_\gamma\right) \in\ ^*U$ —(2)

From (1) and (2), $\alpha \in\ ^*D$ and $\alpha \geq \beta_0 \Rightarrow (^*x_\alpha, y) \in\ ^*U$

By Downward Transfer, $\alpha \in D$ and $\alpha \geq \beta_0 \Rightarrow (x_\alpha, y) \in U \Rightarrow x_\alpha \in (U(y))$

Therefore $(x_\alpha) \to y$ and so $X$ is complete.

Hence the theorem. □

**Proposition 3.2.**

A uniform space $X$ is totally bounded if and only if every $y \in\ ^*X$ is pre-near-standard.

*Proof.* Let $X$ be a totally bounded uniform space with uniform structure $\Psi$.

Let $V \in \Psi$

$\exists\ x_1, x_2, ...x_n \in X$ such that $X = \cup_{i=1}^n V(x_i)$

$(\forall\ x \in X)\,(\exists\ i \in \{1, 2, ...n\})\,[x \in V(x_i)]$

$(\forall\ x \in\ ^*X)\,(\exists\ i \in \{1, 2, ...n\})\,[x \in\ ^*V(x_i)]$

Therefore every point of $^*X$ is pre-near-standard.

Conversly suppose $X$ is not totally bounded.

Then $\exists\, V \in \Psi$ such that for every finite set $\{x_1, x_2, \ldots x_n\} \subseteq X$, $\exists\, y \in X$ such that $y \notin V(x_i)$ for $i = 1, 2, \ldots n$.

By concurrence of this relation, $\exists\, y \in {}^*X$ such that $y \notin {}^*(V(x))$ for every $x \in X$.

Then $y$ is not pre-near-standard.

This completes the proof. □

**Proposition 3.3.**

A uniform space $X$ is compact if and only if every point of ${}^*X$ is near-standard.

*Proof.* Let $X$ be a compact uniform space.

Suppose $y \in {}^*X$ is not near-standard.

Then $\forall\, x \in X$, $\exists\, V_x \in \Psi$ such that $y \notin {}^*(V_x(x))$

$\exists\, x_1, \ldots . x_n \in X$ such that $X = \cup_{i=1}^{n} V_{x_i}(x_i)$

Therefore ${}^*X = \cup_{i=1}^{n}\, {}^*(V_{x_i}(x_i))$

Now $y \in {}^*X$ but $y \notin {}^*(V_{x_i}(x_i))$ for $i = 1, 2 \ldots n$

This contradiction shows that every $y \in {}^*X$ is near-standard.

Conversely suppose $X$ is not compact.

Then there exists an open cover $G_\alpha$ of $X$ which does not admit a finite subcover.

$\forall\, \alpha_1, \ldots \alpha_n, \exists\, y \in X$ such that $y \notin G_{\alpha_i}$ for $i = 1, \ldots n$

By concurrence, $\exists\, y \in {}^*X$ such that $y \notin {}^*G_\alpha\ \forall \alpha$

That is, $\exists\, y \in {}^*X$ such that $y$ is not near-standard. □

From the above results, we may state the following classical result:

**Proposition 3.4.**

A uniform space is compact if and only if it is totally bounded and complete.

Now we present a nonstandard method of completion of a uniform space $X$.

**Theorem 3.5.**

A uniform space $(X,\Psi)$ has a completion $\left(\widehat{X},\widehat{\Psi}\right)$

*Proof.* Let $X'$ be the set of pre-near-standard points of $^*X$ and $\widehat{X}$ be the set of equivalence classes of $X'$ under the relation $x' \simeq y'$ if $x'$ is near $y'$.

Let the elements of $\widehat{X}$ be denoted by $m(x')$ for $x' \in X'$. For $V \in \Psi$, we define $\widehat{V} \subseteq \widehat{X} \times \widehat{X}$ as $\widehat{V} = \{(m(x'), m(y')) : (x', y') \in {}^*V\}$

Let $\widehat{\Psi} = \left\{\widehat{V} : V \in \Psi\right\}$

Define $\phi : X \to \widehat{X}$ by $\phi(x) = m(x)$

$m(x) = m(y) \Rightarrow x \simeq y \Rightarrow x = y$, since $x, y \in X$

$\phi$ is one-one.

Since $(x,y) \in {}^*U \Leftrightarrow (m(x), m(y)) \in \widehat{U}$, we get that $\phi$ is a homeomorphism of $X$ onto $\phi(X)$

Let $m(y') \in \widehat{X}$, where $y' \in X'$ and let $\widehat{U} \in \widehat{\Psi}$ with $U \in \Psi$

$\exists\, x \in X$ such that $(x, y') \in {}^*U$, since $y'$ is pre-near-standard.

Therefore $(m(x), m(y')) \in \widehat{U}$ and hence $\phi(X)$ is dense in $\widehat{X}$

To show : $\widehat{X}$ is complete

Let $(m(x'_\alpha))_{\alpha \in D}$ be a cauchy net in $\widehat{X}$, where $x'_\alpha \in X'$

By cofinality of the directed set $D, \forall\, \alpha_1, \alpha_2, ...\alpha_n \in D, \exists\, \gamma \in D$ such that $\gamma \geq \alpha_i \; for \; i = 1,2,...n$.

By concurrence, $\exists\, \gamma \in {}^*D$ such that $\gamma \geq {}^*\alpha = \alpha \; \forall \alpha \in D$.

Fix $G \in {}^*\Psi$ such that $G \subseteq {}^*V \; \forall\, V \in \Psi$. This again is by concurrence.

Now $\forall\, V \in \Psi, \alpha \in D$, $\exists\, x_{\alpha,V} \in X$ such that $(x'_\alpha, x_{\alpha,V}) \in {}^*V$

Claim : ${}^*x_{\gamma,G} \in X'$ and $(m(x'_\alpha)) \to m\left({}^*x_\gamma, G\right)$

Let $U \in \Psi$.

Let $V \circ V \subseteq U$ and $W_0 \circ W_0 \subseteq V$ ; $V \subseteq U, W_0 \subseteq V$ where $V, W_0 \in \Psi$ and are symmetric.

$\exists\, \alpha_0 \in D$ such that $\alpha, \beta \geq \alpha_0 \Rightarrow \left(m(x'_\alpha), m\left(x'_\beta\right)\right) \in \widehat{W_0} \Rightarrow \left(x'_\alpha, x'_\beta\right) \in {}^*W_0$

$(x_{\alpha,W_0}, x'_\alpha) \in {}^*W_0$

$\left(x'_\alpha, x'_\beta\right) \in {}^*W_0$

Therefore $\left(x_{\alpha,W_0}, x'_\beta\right) \in {}^*V$ (1)

Also $\left(x'_\beta, x_{\beta,W_0}\right) \in {}^*W_0 \subseteq {}^*V$ (2)

(1) and (2) are true for every $W \subseteq W_0$

Thus $\alpha,\beta \geq \alpha_0,\ W_1 \subseteq W_0$ and $W_2 \subseteq W_0 \Rightarrow \left(x_{\alpha,W_1}, x'_\beta\right) \in {}^*V$ and $\left(x'_\beta, x_{\beta,W_2}\right) \in {}^*V$

Hence $\alpha,\beta \geq \alpha_0$ and $W_1 \subseteq W_0$ and $W_2 \subseteq W_0 \Rightarrow \left(x_{\alpha,W_1}, x_{\beta,W_2}\right) \in {}^*U$

$\Rightarrow \left(x_{\alpha,W_1}, x_{\beta,W_2}\right) \in U$, since $x_{\alpha,W_1}, x_{\beta,W_2} \in X$

By Transfer we have $\forall\ \alpha,\beta \in {}^*D;\ W_1,W_2 \in {}^*\Psi;$

$$\alpha,\beta \geq \alpha_0, W_1 \subseteq W_0, W_2 \subseteq W_0 \Rightarrow \left({}^*x_{\alpha,W_1},\ {}^*x_{\beta,W_2}\right) \in {}^*U \tag{3}$$

In particular, $\forall\ \alpha \in {}^*D,\ W_1 \in {}^*\Psi$

$\alpha \geq \alpha_0, W_1 \subseteq W_0 \Rightarrow \left({}^*x_{\alpha,W_1},\ {}^*x_{\gamma,G}\right) \in {}^*U$

Therefore $\alpha \in D, W_1 \in \Psi, \alpha \geq \alpha_0, W_1 \subseteq W_0 \Rightarrow \left(x_{\alpha,W_1},\ {}^*x_{\gamma,G}\right) \in {}^*U$

That is, $\forall\ U \in \Psi\ \exists\ x_{\alpha,W_1} \in X$ such that $\left(x_{\alpha,W_1},\ {}^*x_{\gamma,G}\right) \in {}^*U$

Therefore ${}^*x_{\gamma,G} \in X'$

To prove : $(m(x'_\alpha)) \rightarrow m\left({}^*x_{\gamma,G}\right)$

Let $U \in \Psi$

From (2) we have the following :

$$\beta \in D, W_2 \in \Psi, \beta \geq \beta_0, W_2 \subseteq W_0 \Rightarrow \left(x'_\beta, x_{\beta,W_2}\right) \in {}^*U, \text{ since } V \subseteq U \tag{4}$$

From (3), $\beta \in D, W_2 \in \Psi, \beta \geq \beta_0, W_2 \subseteq W_0, \alpha \in {}^*D, W_1 \in {}^*\Psi, \alpha \geq \alpha_0, W_1 \subseteq W_0$

$$\Rightarrow \left({}^*x_{\alpha,W_1}, x_{\beta,W_2}\right) \in {}^*U \tag{5}$$

From (4) and (5),

$\alpha \in {}^*D,\ \beta \in D, \alpha \geq \alpha_0,\ \beta \geq \alpha_0, W_1 \in {}^*\Psi,\ W_1 \subseteq W_0 \Rightarrow \left(x'_\beta,\ {}^*x_{\alpha,W_1}\right) \in {}^*U$

In particular, $\beta \in D,\ \beta \geq \alpha_0 \Rightarrow \left(x'_\beta,\ {}^*x_{\gamma,G}\right) \in {}^*U \Rightarrow \left(m\left(x'_\beta\right), m\left({}^*x_{\gamma,G}\right)\right) \in \widehat{U}$

Therefore $(m(x'_\alpha)) \rightarrow m\left({}^*x_{\gamma,G}\right)$

Therefore $\widehat{X}$ is complete.

Hence the theorem. □

## ACKNOWLEDGEMENT

The first author thanks Department of Science and Technology (DST), India, for its financial support through INSPIRE fellowship.

## CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

## REFERENCES

[1] A. Robinson, Non-standard analysis, North-Holland Publishing Company, Amsterdam, 1966.

[2] A.E. Hurd, P.A. Loeb, An introduction to nonstandard real analysis, Academic Press, New York, 1985.

[3] J. Dugundji, Topology, Prince Hall of India Private Limited, New Delhi, 1975.

[4] J. Munkres, Topology, A First Course, Prentice-Hall, Englewood Cliffs, NJ, 1975.

[5] S. Willard, General Topology, Dover Publications, New York, 1998.