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ADAM'S BLOCK WITH FIRST AND SECOND DERIVATIVE FUTURE POINTS FOR INITIAL VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

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Abstract: In this paper, we develop Adam's Block with first and second derivative future points for solving linear and non-linear first order initial value problems in ordinary differential equations. The derivation of the method is based on Taylor series approach. The region of absolute stability of the method is investigated using the boundary locus method and this family of methods have been found to be A-stable for $r = 2, 3, 4$ and 5. Numerical experiments are demonstrated with the method and computational comparisons are presented with some existing numerical methods. The computational comparison depicts the efficiency of the methods on initial value problems in ordinary differential equations (ODEs).

Keywords: second derivative; future point; Taylor series; boundary locus; linear multi-step method; block method.

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1. INTRODUCTION

Many real life problems in science and engineering can be modeled into first order initial value problem of the form:

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$$y'(t) = f(t, y(t)), \;\; y(t_0) = y_0, \; t \in [a,b], \tag{1}$$

where f is continuous within the interval of integration $[a,b]$. We also assume that f satisfies the Lipchitz condition which guarantees the existence and uniqueness of solution of (1) [10]. The author [13] studied the discrete approach of (1) using one step methods and Linear multi-step methods while researchers such as [17], [18] and [5] have studied the continuous approach of (1). Several methods have been in existence for the solutions of (1) proposed by researchers, some of which are the Euler's methods, the Runge-Kutta methods, the multistep methods, the General Linear Methods (GLM) among others. Modifications have been made on some of these methods due to some limitations and deficiencies observed either in terms of stability or implementation cost and mostly when they are needed for certain classes of ODEs. Numerical methods are adopted in situations where analytic solutions are difficult and are generally required to possess high level of accuracy.

Block methods have been found as one of the numerical methods that perform well in most ordinary differential equations and it is self-starting without requiring a predictor or starting method for its computation. The idea of block method was first presented by [14] as a method for solving ordinary differential equations. This has now become popular and it is still in active part of research. The block methods have been modified by researchers. The author [4] proposed a hybrid block method for first order initial value problems in ordinary differential equations. The numerical results show competitiveness with the exact solution. Several of the modified block method can be found in [19], [15], [20], [12], [8], [3], etc.

Our interest in this research work is to derive a class of block method of the form:

$$A^0 Y_{n+k} - A Y_{n+k-1} = h \sum_{j=0}^{k} B_j F_{n+j} + h D F_{n+k+1} + E h^2 F'_{n+k+1}, \tag{2}$$

where

$$Y_{n+k} = \begin{bmatrix} y_{n+k} & y_{n+k+1} & \cdots & y_{n+r} \end{bmatrix}^T, \; Y_{n+k-1} = \begin{bmatrix} y_{n-r+1} & y_{n-r+2} & \cdots & y_{n+k-1} \end{bmatrix}^T, F_{n+k} = \begin{bmatrix} f_{n+k} & f_{n+k+1} & \cdots & f_{n+r} \end{bmatrix}^T,$$

$$F_{n+k-1} = \begin{bmatrix} f_{n-r+k} & f_{n-r+k+1} & \cdots & f_{n+k-1} \end{bmatrix}^T, \; F_{n+k+1} = \begin{bmatrix} f_{n+k+1} & f_{n+k+2} & \cdots & f_{n+r+k} \end{bmatrix}^T, \; F'_{n+k+1} = \begin{bmatrix} f'_{n+k+1} & f'_{n+k+2} & \cdots & f'_{n+r+k} \end{bmatrix}^T$$

and A^0, A, B_j, D and E are properly chosen matrices to ensure stability and improved accuracy of the method.

The method (2) is obtained from the general Adam's method

$$y_{n+k} - _{n+k-1} = h\sum_{j=0}^{k} b_j f_{n+j},\ n \geq k \tag{3}$$

by adding first and second derivative future points. This yields the family of the block methods. Our interest is also to investigate the properties of the proposed method in terms of zero-stability, A-stability, consistency and convergence.

2. Development of the Block Method

The local truncation error associated with the proposed block method (2) is given as

$$T_{n+k} = A^0 Y_{n+k} - AY_{n+k-1} - h\sum_{j=0}^{k} B_j F_{n+j} - hDF_{n+k+1} - Eh^2 F'_{n+k+1} \tag{4}$$

where k is the step-length of the method and h is the step-size

The Taylor series expansion of each of the right terms of (4) and collecting terms in powers of h gives:

$$T_{n+k} = C_1 h y'_n + C_2 h^2 y''_n + C_3 h^3 y'''_n + \cdots + C_p h^p y^p_n + O(h^{p+1}), \tag{5}$$

where C_is are constants and $i = 1(1)p+1$

For $r = 2$, we have

$$Y_{n+1} = \begin{bmatrix} y_{n+1} & y_{n+2} \end{bmatrix}^T,\ Y_n = \begin{bmatrix} y_{n-1} & y_n \end{bmatrix}^T,\ F_{n+1} = \begin{bmatrix} f_{n+1} & f_{n+2} \end{bmatrix}^T,\ F_n = \begin{bmatrix} f_{n-1} & f_n \end{bmatrix}^T,$$
$$F_{n+2} = \begin{bmatrix} f_{n+2} & f_{n+3} \end{bmatrix}^T,\ F'_{n+2} = \begin{bmatrix} f'_{n+2} & f'_{n+3} \end{bmatrix}^T$$

$$A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},\ B_0 = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix},\ B_1 = \begin{bmatrix} c_1 & c_2 \\ 0 & c_4 \end{bmatrix},\ D = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix},\ E = \begin{bmatrix} e_1 & 0 \\ 0 & e_4 \end{bmatrix}$$

Substituting these matrices into (4) and taking $T_{n+k}=0$, we have

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n+1} \\ y_{n+2} \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} y_{n-1} \\ y_n \end{bmatrix} - h\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}\begin{bmatrix} f_{n-1} \\ f_n \end{bmatrix} - h\begin{bmatrix} c_1 & c_2 \\ 0 & c_4 \end{bmatrix}\begin{bmatrix} f_{n+1} \\ f_{n+2} \end{bmatrix} - h\begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}\begin{bmatrix} f_{n+2} \\ f_{n+3} \end{bmatrix} - h^2\begin{bmatrix} e_1 & 0 \\ 0 & e_4 \end{bmatrix}\begin{bmatrix} f'_{n+2} \\ f'_{n+3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Expanding the above matrix equation gives

$$\begin{cases} y_{n+1} - y_n - hb_1 f_{n-1} - hb_2 f_n - hc_1 f_{n+1} - hc_2 f_{n+2} - h^2 e_1 f'_{n+2} = 0 \\ y_{n+2} - y_n - hb_3 f_{n-1} - hb_4 f_n - hc_4 f_{n+2} - hdf_{n+3} - h^2 e_4 f'_{n+3} = 0 \end{cases} \quad (6)$$

Using mathematica software to solve (6) for the values of $b_1, b_2, b_3, b_4, c_1, c_2, c_4, d, e_1, e_4$ gives

the method as

$$\begin{cases} \mathbf{y}_{1+n} = y_n - \frac{23hf_{-1+n}}{1080} + \frac{9hf_n}{20} + \frac{29hf_{1+n}}{40} - \frac{83hf_{2+n}}{540} + \frac{11h^2 f'_{2+n}}{180} \\ \mathbf{y}_{2+n} = y_n - \frac{7hf_{-1+n}}{60} + \frac{127hf_n}{135} + \frac{31hf_{2+n}}{15} + \frac{481hf_{3+n}}{540} - \frac{19h^2 f'_{3+n}}{45} \end{cases} \quad (7)$$

For $r = 3$, we have

$$Y_{n+1} = \left[y_{n+1} \quad y_{n+2} \quad y_{n+3} \right]^T, \; Y_n = \left[y_{n-2} \quad y_{n-1} \quad y_{n-1} \right]^T, \; F_{n+1} = \left[f_{n+1} \quad f_{n+2} \quad f_{n+3} \right]^T,$$
$$F_n = \left[f_{n-2} \quad f_{n-1} \quad f_n \right]^T, \; F_{n+2} = \left[f_{n+2} \quad f_{n+3} \quad f_{n+4} \right]^T, \; F'_{n+2} = \left[f'_{n+2} \quad f'_{n+3} \quad f'_{n+4} \right]^T$$

$$A^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; B_0 = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix}, \; B_1 = \begin{bmatrix} c_1 & c_2 & 0 \\ 0 & c_5 & c_6 \\ 0 & 0 & c_9 \end{bmatrix}, \; D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d \end{bmatrix}, \; E = \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_5 & 0 \\ 0 & 0 & e_9 \end{bmatrix}$$

Substituting these matrices into equation (4) gives

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix} - h \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} \begin{bmatrix} f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix} - h \begin{bmatrix} c_1 & c_2 & 0 \\ 0 & c_5 & c_6 \\ 0 & 0 & c_9 \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \end{bmatrix} - h \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d \end{bmatrix} \begin{bmatrix} f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix} - h^2 \begin{bmatrix} e_1 & 0 & 0 \\ 0 & e_5 & 0 \\ 0 & 0 & e_9 \end{bmatrix} \begin{bmatrix} f'_{n+2} \\ f'_{n+3} \\ f'_{n+4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Expanding the above matrix equation gives

$$\begin{cases} y_{n+1} - y_n - hb_1 f_{n-2} - hb_2 f_{n-1} - hb_3 f_n - hc_1 f_{n+1} - hc_2 f_{n+2} - h^2 e_1 f'_{n+2} = 0 \\ y_{n+2} - y_n - hb_4 f_{n-2} - hb_5 f_{n-1} - hb_6 f_n - hc_5 f_{n+2} - hc_6 f_{n+3} - h^2 e_5 f'_{n+3} = 0 \\ y_{n+3} - y_n - hb_7 f_{n-2} - hb_8 f_{n-1} - hb_9 f_n - hc_9 f_{n+3} - hd_9 f_{n+4} - h^2 e_9 f'_{n+4} = 0 \end{cases} \quad (8)$$

Using Mathematica software to solve (8) for the values of $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, c_1, c_2, c_5,$ $c_6, c_9, d_9, e_2, e_5, e_9$ yields the following block method

$$\begin{cases} y_{1+n} = y_n + \frac{11}{1920}hf_{-2+n} - \frac{7}{135}hf_{-1+n} + \frac{83hf_n}{160} + \frac{19}{30}hf_{1+n} - \frac{1831}{17280}hf_{2+n} + \frac{11}{288}h^2 f'_{2+n} \\ y_{2+n} = y_n + \frac{19}{375}hf_{-2+n} - \frac{59}{180}hf_{-1+n} + \frac{11hf_n}{9} + \frac{74}{45}hf_{2+n} - \frac{2653}{4500}hf_{3+n} + \frac{19}{75}h^2 f'_{3+n} \\ y_{3+n} = y_n + \frac{29}{160}hf_{-2+n} - \frac{477}{500}hf_{-1+n} + \frac{1407hf_n}{640} + \frac{63}{20}hf_{3+n} - \frac{25211}{16000}hf_{4+n} + \frac{609}{800}h^2 f'_{4+n} \end{cases} \tag{9}$$

Again for $r = 4$, we have

$$Y_{n+1} = \begin{bmatrix} y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} \end{bmatrix}^T,\ Y_n = \begin{bmatrix} y_{n-3} & y_{n-2} & y_{n-1} & y_n \end{bmatrix}^T,\ F_{n+1} = \begin{bmatrix} f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} \end{bmatrix}^T,$$
$$F_n = \begin{bmatrix} f_{n-3} & f_{n-2} & f_{n-1} & f_n \end{bmatrix}^T,\ F_{n+2} = \begin{bmatrix} f_{n+2} & f_{n+3} & f_{n+4} & f_{n+5} \end{bmatrix}^T,\ F'_{n+2} = \begin{bmatrix} f'_{n+2} & f'_{n+3} & f'_{n+4} & f'_{n+5} \end{bmatrix}^T$$

$$A^0 = \begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix},\ A = \begin{bmatrix} 0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1 \end{bmatrix},\ B_0 = \begin{bmatrix} b_1&b_2&b_3&b_4\\b_5&b_6&b_7&b_8\\b_9&b_{10}&b_{11}&b_{12}\\b_{13}&b_{14}&b_{15}&b_{16} \end{bmatrix},\ B_1 = \begin{bmatrix} c_1&c_2&0&0\\0&c_6&c_7&0\\0&0&c_{11}&c_{12}\\0&0&0&c_{16} \end{bmatrix},\ D = \begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&d \end{bmatrix},\ E = \begin{bmatrix} e_1&0&0&0\\0&e_6&0&0\\0&0&e_{11}&0\\0&0&0&e_{16} \end{bmatrix}$$

Substituting these matrices into (4) yields

$$\begin{bmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1 \end{bmatrix}\begin{bmatrix} y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4} \end{bmatrix} - \begin{bmatrix} 0&0&0&1\\0&0&0&1\\0&0&0&1\\0&0&0&1 \end{bmatrix}\begin{bmatrix} y_{n-3}\\y_{n-2}\\y_{n-1}\\y_n \end{bmatrix} - h\begin{bmatrix} b_1&b_2&b_3&b_4\\b_5&b_6&b_7&b_8\\b_9&b_{10}&b_{11}&b_{12}\\b_{13}&b_{14}&b_{15}&b_{16} \end{bmatrix}\begin{bmatrix} f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4} \end{bmatrix} - h\begin{bmatrix} c_1&c_2&0&0\\0&c_6&c_7&0\\0&0&c_{11}&c_{12}\\0&0&0&c_{16} \end{bmatrix}\begin{bmatrix} f_{n-3}\\f_{n-2}\\f_{n-1}\\f_n \end{bmatrix} - h\begin{bmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&d \end{bmatrix}\begin{bmatrix} f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5} \end{bmatrix} - h^2\begin{bmatrix} e_1&0&0&0\\0&e_6&0&0\\0&0&e_{11}&0\\0&0&0&e_{16} \end{bmatrix}\begin{bmatrix} f'_{n+2}\\f'_{n+3}\\f'_{n+4}\\f'_{n+5} \end{bmatrix} = \begin{bmatrix} 0\\0\\0\\0 \end{bmatrix}$$

Expanding, we obtain

$$\begin{cases} y_{n+1} - y_n - hb_1 f_{n-3} - hb_2 f_{n-2} - hb_3 f_{n-1} - hb_4 f_n - hc_1 f_{n+1} - hc_2 f_{n+2} - h^2 e_1 f'_{n+2} = 0 \\ y_{n+2} - y_n - hb_5 f_{n-3} - hb_6 f_{n-2} - hb_7 f_{n-1} - hb_8 f_n - hc_6 f_{n+2} - hc_7 f_{n+3} - h^2 e_6 f'_{n+3} = 0 \\ y_{n+3} - y_n - hb_9 f_{n-3} - hb_{10} f_{n-2} - hb_{11} f_{n-1} - hb_{12} f_n - hc_{11} f_{n+3} - hc_{12} f_{n+4} - h^2 e_{11} f'_{n+4} = 0 \\ y_{n+4} - y_n - hb_{13} f_{n-3} - hb_{14} f_{n-2} - hb_{15} f_{n-1} - hb_{16} f_n - hc_{16} f_{n+4} - hdf_{n+5} - h^2 e_{16} f'_{n+5} = 0 \end{cases} \tag{10}$$

Using mathematica software on (10) gives the values of the unknown parameters and the following block method

$$\begin{cases} y_{1+n} = y_n - \frac{773hf_{-3+n}}{28350} + \frac{1039hf_{-2+n}}{5250} - \frac{533hf_{-1+n}}{840} + \frac{4238hf_n}{2835} + \frac{1469hf_{1+n}}{1050} - \frac{9031hf_{2+n}}{21000} + \frac{1621h^2 f'_{2+n}}{9450} \\ y_{2+n} = y_n - \frac{1693hf_{-3+n}}{13720} + \frac{4401hf_{-2+n}}{5600} - \frac{28593hf_{-1+n}}{14000} + \frac{2647hf_n}{896} + \frac{7127hf_{2+n}}{2800} - \frac{6145119hf_{3+n}}{5488000} + \frac{2344h^2 f'_{3+n}}{4725} \\ y_{3+n} = y_n - \frac{1693hf_{-3+n}}{13720} + \frac{4401hf_{-2+n}}{5600} - \frac{28593hf_{-1+n}}{14000} + \frac{26477hf_n}{896} + \frac{7127hf_{3+n}}{2800} - \frac{6145119hf_{4+n}}{5488000} + \frac{19683h^2 f'_{4+n}}{39200} \\ y_{4+n} = y_n - \frac{18hf_{-3+n}}{49} + \frac{10936hf_{-2+n}}{5145} - \frac{22592hf_{-1+n}}{4725} + \frac{4506hf_n}{875} + \frac{14986hf_{4+n}}{3675} - \frac{2551838hf_{5+n}}{1157625} + \frac{11888h^2 f'_{5+n}}{11025} \end{cases} \tag{11}$$

For $r = 5$, we have

$Y_{n+1}=\begin{bmatrix}y_{n+1} & y_{n+2} & y_{n+3} & y_{n+4} & y_{n+5}\end{bmatrix}$, $Y_n=\begin{bmatrix}y_{n-4} & y_{n-3} & y_{n-2} & y_{n-1} & y_n\end{bmatrix}$, $F_{n+1}=\begin{bmatrix}f_{n+1} & f_{n+2} & f_{n+3} & f_{n+4} & f_{n+5}\end{bmatrix}$,
$F_n=\begin{bmatrix}f_{n-4} & f_{n-3} & f_{n-2} & f_{n-1} & f_n\end{bmatrix}$, $F_{n+2}=\begin{bmatrix}f_{n+2} & f_{n+3} & f_{n+4} & f_{n+5} & f_{n+6}\end{bmatrix}$, $F'_{n+2}=\begin{bmatrix}f'_{n+2} & f'_{n+3} & f'_{n+4} & f'_{n+5} & f'_{n+6}\end{bmatrix}$

$$A^0=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},\ A=\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix},\ B_0=\begin{bmatrix}b_1&b_2&b_3&b_4&b_5\\b_6&b_7&b_8&b_9&b_{10}\\b_{11}&b_{12}&b_{13}&b_{14}&b_{15}\\b_{16}&b_{17}&b_{18}&b_{19}&b_{20}\\b_{21}&b_{22}&b_{23}&b_{24}&b_{25}\end{bmatrix},$$

$$B_1=\begin{bmatrix}c_1&c_2&0&0&0\\0&c_7&c_8&0&0\\0&0&c_{13}&c_{14}&0\\0&0&0&c_{19}&c_{20}\\0&0&0&0&c_{25}\end{bmatrix},\ D=\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&d\end{bmatrix},\ E=\begin{bmatrix}e_1&0&0&0&0\\0&e_7&0&0&0\\0&0&e_{13}&0&0\\0&0&0&e_{19}&0\\0&0&0&0&e_{25}\end{bmatrix}.$$

Substituting into (4) gives

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\\y_{n+5}\end{bmatrix}-\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n-4}\\y_{n-3}\\y_{n-2}\\y_{n-1}\\y_n\end{bmatrix}-h\begin{bmatrix}b_1&b_2&b_3&b_4&b_5\\b_6&b_7&b_8&b_9&b_{10}\\b_{11}&b_{12}&b_{13}&b_{14}&b_{15}\\b_{16}&b_{17}&b_{18}&b_{19}&b_{20}\\b_{21}&b_{22}&b_{23}&b_{24}&b_{25}\end{bmatrix}\begin{bmatrix}f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\end{bmatrix}$$
$$-h\begin{bmatrix}c_1&c_2&0&0&0\\0&c_7&c_8&0&0\\0&0&c_{13}&c_{14}&0\\0&0&0&c_{19}&c_{20}\\0&0&0&0&c_{25}\end{bmatrix}\begin{bmatrix}f_{n-4}\\f_{n-3}\\f_{n-2}\\f_{n-1}\\f_n\end{bmatrix}-h\begin{bmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&d\end{bmatrix}\begin{bmatrix}f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\\f_{n+6}\end{bmatrix}-h^2\begin{bmatrix}e_1&0&0&0&0\\0&e_7&0&0&0\\0&0&e_{13}&0&0\\0&0&0&e_{19}&0\\0&0&0&0&e_{25}\end{bmatrix}\begin{bmatrix}f'_{n+2}\\f'_{n+3}\\f'_{n+4}\\f'_{n+5}\\f'_{n+6}\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}.$$

After expanding, we obtain

$$\begin{cases} y_{n+1}-y_n-hb_1f_{n-4}-hb_2f_{n-3}-hb_3f_{n-2}-hb_4f_{n-1}-hb_5f_n-hc_1f_{n+1}-hc_2f_{n+2}-h^2e_1f'_{n+2}=0\\ y_{n+2}-y_n-hb_6f_{n-4}-hb_7f_{n-3}-hb_8f_{n-2}-hb_9f_{n-1}-hb_{10}f_n-hc_7f_{n+2}-hc_8f_{n+3}-h^2e_7f'_{n+3}=0\\ y_{n+3}-y_n-hb_{11}f_{n-4}-hb_{12}f_{n-3}-hb_{13}f_{n-2}-hb_{14}f_{n-1}-hb_{15}f_n-hc_{13}f_{n+3}-hc_{14}f_{n+4}-h^2e_{13}f'_{n+4}=0\\ y_{n+4}-y_n-hb_{16}f_{n-4}-hb_{17}f_{n-3}-hb_{18}f_{n-2}-hb_{19}f_{n-1}-hb_{20}f_n-hc_{19}f_{n+4}-hc_{20}f_{n+5}-h^2e_{19}f'_{n+5}=0\\ y_{n+5}-y_n-hb_{21}f_{n-4}-hb_{22}f_{n-3}-hb_{23}f_{n-2}-hb_{24}f_{n-1}-hb_{25}f_n-hc_{25}f_{n+5}-hdf_{n+6}-h^2e_{25}f'_{n+6}=0 \end{cases} \quad (12)$$

Using Mathematica software on (12) gives the values of the unknown parameters and the following block equation

$$\begin{cases} \mathbf{y}_{1+n} = y_n + \frac{53hf_{-4+n}}{48384} - \frac{341hf_{-3+n}}{33600} + \frac{2393hf_{-2+n}}{53760} - \frac{4033hf_{-1+n}}{30240} + \frac{16789hf_n}{26880} + \frac{3611hf_{1+n}}{6720} - \frac{154913hf_{2+n}}{2419200} + \frac{13h^2 f'_{2+n}}{640} \\ \mathbf{y}_{2+n} = y_n + \frac{3067hf_{-4+n}}{185220} - \frac{128hf_{-3+n}}{945} + \frac{49hf_{-2+n}}{100} - \frac{983hf_{-1+n}}{945} + \frac{6673hf_n}{3780} + \frac{935hf_{2+n}}{756} - \frac{77101hf_{3+n}}{231525} + \frac{92h^2 f'_{3+n}}{735} \\ \mathbf{y}_{3+n} = y_n + \frac{90423hf_{-4+n}}{1003520} - \frac{36913hf_{-3+n}}{54880} + \frac{9549hf_{-2+n}}{4480} - \frac{40959hf_{-1+n}}{11200} + \frac{136021hf_n}{35840} + \frac{33883hf_{3+n}}{15680} - \frac{29786121hf_{4+n}}{35123200} + \frac{44901h^2 f'_{4+n}}{125440} \\ \mathbf{y}_{4+n} = y_n + \frac{2666hf_{-4+n}}{8505} - \frac{229hf_{-3+n}}{105} + \frac{32264hf_{-2+n}}{5145} - \frac{976hf_{-1+n}}{105} + \frac{754hf_n}{105} + \frac{352hf_{4+n}}{105} - \frac{684107hf_{5+n}}{416745} + \frac{1000h^2 f'_{5+n}}{1323} \\ \mathbf{y}_{5+n} = y_n + \frac{122515hf_{-4+n}}{145152} - \frac{1209125hf_{-3+n}}{217728} + \frac{1487125hf_{-2+n}}{100352} - \frac{5848375hf_{-1+n}}{296352} + \frac{86555hf_n}{6912} + \frac{2463805hf_{5+n}}{508032} - \frac{468966265hf_{6+n}}{170698752} + \frac{458975h^2 f'_{6+n}}{338688} \end{cases} \tag{13}$$

3. Properties of the Proposed Block Method

The Order and Error Constants of the proposed Block method

Definition 1. The linear operator L associated with a linear multistep method is given by

$$L[y(x);h] = \sum_{j=0}^{k} [\alpha_j y(x+jh) - h\beta_j y'(x+jh)]$$

where d is the order of the differential equation and $y(x)$ is an arbitrary function that is continuous and differentiable on *[a,b]* Using the Taylor series about point x in $y(x+jh)$ and $y^d(x+jh)$ gives

$$L[y(x);h] = C_0 y(x) + C_1 y'(x) + \ldots + C_q y^q(x) + C_{q+1} y^{q+1}(x) + \ldots,$$

where

$$C_0 = (\alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_k)$$
$$C_1 = (\alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots + k\alpha_k)$$
$$\vdots$$
$$C_q = \frac{1}{q!}(\alpha_1 + 2^q \alpha_2 + 3^q \alpha_3 + \ldots + k^q \alpha_k) - \frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + \ldots + k^{q-1}\beta_k)$$

For $q = 2, 3, \ldots$

A block method is said to have order p if

$C_0 = C_1 = C_2 = \ldots C_p = 0, C_{p+1} \neq 0.$ The first coefficient that does not vanish C_{p+1} is known as the error constant and $C_{p+1} h^{p+1} y^{(p+1)}(x_n)$ is called the *principal local truncation error* [9]. The order and error constants of the proposed family of the block method are listed below.

Table 1: Order and Error Constants of the family of the proposed method

Order (p)	Error Constant
5	$\left[\frac{-11}{2400}, \frac{-19}{225}\right]^T$
6	$\left[\frac{-19}{10080}, \frac{-773}{18900}, \frac{-1693}{5600}\right]^T$
7	$\left[\frac{-53}{56448}, \frac{-3067}{132300}, \frac{-30141}{156800}, \frac{-10664}{11025}\right]^T$
8	$\left[\frac{-7667}{14515200}, \frac{-11531}{793800}, \frac{-23841}{179200}, \frac{-72809}{99225}, \frac{-12141025}{4064256}\right]^T$

Zero Stability of the Proposed Block Method

Definition 2. The block method (2) is said to be *zero-stable* if no root of the first characteristic polynomial $\rho(r) = \det\,[rA^0 - A]$ is having a modulus greater than one and every root of modulus one is simple, where A^0 and A are the coefficients of y - function in our block method. The roots with modulus one is known as the principal roots and the other roots are called *spurious roots* [2]. Therefore for the proposed block method for $r = 2$, we have

$$\rho(\mathrm{r}) = \det[\mathrm{rA}^{(0)} - \mathrm{A}] = \left| r\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right| = 0$$

$$\rho(\mathrm{r}) = \begin{vmatrix} r & -1 \\ 0 & r-1 \end{vmatrix} = 0$$

$$\rho(\mathrm{r}) = r(r-1) = 0$$

$$r = 0; r - 1 = 0$$

$$r = 0,1$$

In this case, the maximum value of r is 1. Hence the method for $r = 2$ is zero-stable.

For $r = 3$

$$\rho(\mathrm{r}) = \det[\mathrm{rA}^{(0)} - \mathrm{A}] = \left| r\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\rho(\mathrm{r}) = \begin{vmatrix} r & 0 & -1 \\ 0 & r & -1 \\ 0 & 0 & r-1 \end{vmatrix} = 0$$

$$\rho(\mathrm{r}) = r[r(r-1)] = 0$$

$$\rho(\mathrm{r}) = r^2(r-1) = 0$$

$$r^2 = 0; r-1 = 0$$

$$r = 0, 0, 1$$

Therefore, the maximum value of r is 1. Hence the method for $r = 3$ is zero-stable.

For $r = 4$

$$\rho(\mathrm{r}) = \det[\mathrm{rA}^{(0)} - \mathrm{A}] = \left| r\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\rho(\mathrm{r}) = \begin{vmatrix} r & 0 & 0 & -1 \\ 0 & r & 0 & -1 \\ 0 & 0 & r & -1 \\ 0 & 0 & 0 & r-1 \end{vmatrix} = 0$$

$$\rho(\mathrm{r}) = r\begin{vmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r-1 \end{vmatrix} - \begin{vmatrix} 0 & r & 0 \\ 0 & 0 & r \\ 0 & 0 & 0 \end{vmatrix} = 0$$

$$\rho(\mathrm{r}) = r[r^2(r-1)] = 0$$

$$\rho(\mathrm{r}) = r^3(r-1) = 0$$

$$r^3 = 0; r-1 = 0$$

$$r = 0, 0, 0, 1$$

In this case, the maximum value of r is 1. Hence the method for $r = 4$ is zero-stable.

For $r = 5$

$$\rho(r) = \det[rA^{(0)} - A] = \left| r\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0$$

$$\rho(r) = \begin{vmatrix} r & 0 & 0 & 0 & -1 \\ 0 & r & 0 & 0 & -1 \\ 0 & 0 & r & 0 & -1 \\ 0 & 0 & 0 & r & -1 \\ 0 & 0 & 0 & 0 & r-1 \end{vmatrix} = 0$$

$$\rho(r) = r\begin{vmatrix} r & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r-1 \end{vmatrix} - \begin{vmatrix} 0 & r & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

$$\rho(r) = r[r^3(r-1)] = 0$$

$$\rho(r) = r^4(r-1) = 0$$

$$r^4 = 0; r - 1 = 0$$

$$r = 0, 0, 0, 0, 1$$

Therefore, the maximum value of r is 1. Hence the method for $r = 5$ is zero-stable.

Region of Absolute Stability

Definition 3. A block method is said to be A-stable if its region of absolute stability or the linear stability domain contains the whole of the left hand half plane i.e. $Re(h\lambda) < 0$, [6].

The boundary locus method proposed by [13] and [11] is adopted in finding the region of absolute stability of our proposed block method.

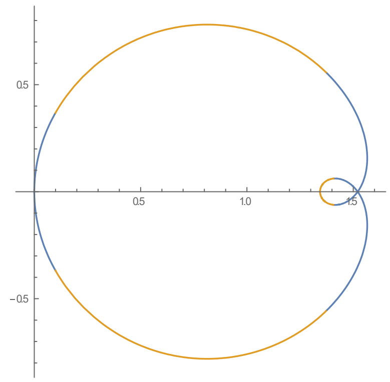

Figure 1: Boundary Locus of the proposed Block Method for $r = 2$

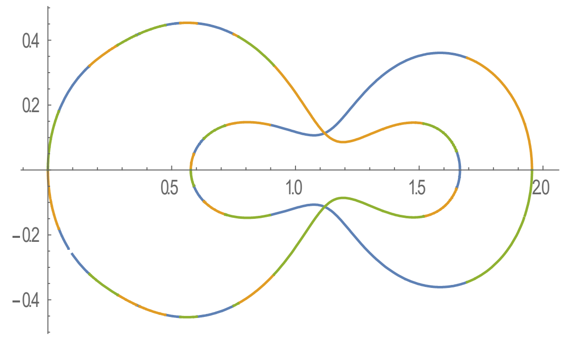

Figure 2: Boundary Locus of the proposed Block Method for $r = 3$

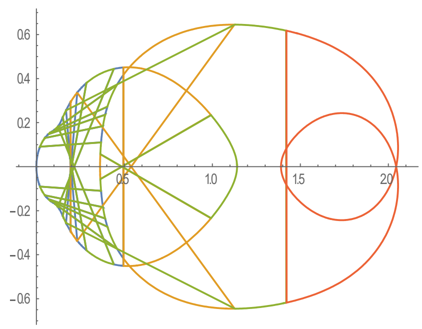

Figure 3: Boundary Locus of the proposed Block Method for $r = 4$

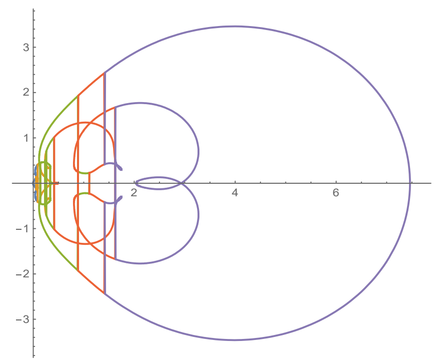

Figure 4: Boundary Locus of the proposed Block Method for $r = 5$

The block methods for $r = 2,3,4,5$ are seen to be A-stable and for values of $r = 6,7,\ldots$ the method become computationally difficult due to large volume of data under processing.

Consistency and Convergence of the Proposed Block Method

Theorem 1. Dahlquist Equivalence Theorem

Consistency and zero stability are sufficient condition for a block method to be convergent [6].

Definition 4. A block method with order p is *consistent* if the following conditions stated below are satisfied [13]:

- $p \geq 1$
- $\sum_{j=0}^{k} \alpha_j = 0$.

Since our proposed method is consistent and zero-stable, it implies that the proposed block method is convergent.

Numerical Experiments

To test the efficiency of our methods, we consider some initial value problems in ODEs that have been solved by some existing methods.

Problem 1: $y'+y=0, y(0)=1, h=0.1, 0\le x\le 1$

Exact solution [1]: $y(x)=e^{-x}$

Problem 2: $y'+60y-10x=\frac{1}{6},\ \ y(0)=\frac{1}{6},\ \ h=0.1,\ \ 0\le x\le 10$

Exact solution [16]: $y(x)=\frac{1}{6}[x+e^{-60x}]$

Problem 3: $y'-1-x+2y=0,\ \ y(0)=2,\ \ h=0.1,\ \ 0\le x\le 1$

Exact solution [16]: $y(x)=\frac{1}{4}[2x+7e^{-2x}+1]$

Problem 4: $2y'=(2x-1)y^3-2y,\ \ y(0)=\sqrt{2},\ \ h=0.1,\ \ 0\le x\le 1$

Exact solution [21]: $y(x)=\frac{1}{\sqrt{x+\frac{1}{2}e^{2x}}}$

Error = |Exact result – Computed Result|

Table 2: Comparison of solutions for problem 1

x	Solution in [1]	Proposed Method	Exact solution
0.1	0.9048374180	0.9048374180	0.9048374180
0.2	0.8187307492	0.8187307540	0.8187307530
0.3	0.7408182137	0.7408182110	0.7408182200
0.4	0.6703200365	0.6703200000	0.6703200460
0.5	0.6065306482	0.6065306600	0.6065306590
0.6	0.5488116230	0.5488116351	0.5488116360
0.7	0.4965852895	0.4965853020	0.4965853030
0.8	0.4493289490	0.4493289630	0.4493289640
0.9	0.4065696441	0.4065696520	0.4065696590
1.0	0.3678794252	0.3678794420	0.3678794410

Table 3: Comparison of solutions for Problem 2

x	Solution in[16]	Proposed Method	Exact solution
0.1	0.015807560	0.017079791	0. 017079792
0.2	0.015807560	0.033334372	0. 033334357
0.3	0. 003436326	0.050000021	0. 050000002
0.4	0. 071562688	0.066666668	0. 066666667
0.5	0. 074687356	0.083333343	0. 083333333
0.6	0. 110312807	0.100000081	0. 100000000
0.7	0. 111594664	0.116666688	0. 116666667
0.8	0. 132309964	0.133333334	0. 133333333
0.9	0. 143308542	0.150000001	0. 150000000
1.0	0. 164574792	0.166666634	0. 166666667

Table 4: Comparison of solutions for Problem 3

x	Solution in[16]	Proposed Method	Exact solution
0	2.000000000	2.000000000	2.000000000
0.1	1.700000000	1.704231478	1.732778818
0.2	1.641084610	1.523896021	1.523960081
0.3	1.448463233	1.360420334	1.360420363
0.4	1.304033029	1.236325710	1.236325687
0.5	1.192758695	1.143789033	1.143789022
0.6	1.112191440	1.082408903	1.077089871
0.7	1.056080923	1.031446870	1.031544687
0.8	1.019872305	1.003318801	1.003318906
0.9	0.999759807	0.989274023	0.989273054
1.0	0.992681275	0.986837404	0.986836745

Table 5: Comparison of solutions for problem 4

x	Solution in[7]	Proposed method	Exact Solution
0.1	1.18608991	1.18618973	1.18619591
0.2	1.02808143	1.02817003	1.02819279
0.3	0.90856202	0.90868210	0.90869320
0.4	0.81303397	0.81304189	0.81304294
0.5	0.73324162	0.73341481	0.73340497
0.6	0.66501098	0.60549433	0.66518150
0.7	0.60536149	0.60549293	0.60549394
0.8	0.55229052	0.55245211	0.55245110
0.9	0.50459061	0.50476560	0.50476580
1.0	0.46142011	0.46153234	0.46153435

4. CONCLUSION

The proposed block method in this paper has been demonstrated on three linear and one non-linear problem in ordinary differential equations. The methods have shown high competitiveness as results can be seen in tables of comparisons (Table 2, 3, 4 and 5). The proposed block method converged almost to the exact solutions of the ordinary differential equations. This family of methods is A-stable, zero-stable, consistent and convergent. These are the characteristics of efficient numerical integrators. Therefore, it can be adopted for both linear and non-linear initial value problems in ordinary differential equations and can as well solve system of stiff initial value problems due to the wide region of absolute stability.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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