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CONFORMAL SEMI-SLANT SUBMERSIONS FROM COSYMPLECTIC MANIFOLDS

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Abstract. In this paper, we introduce conformal semi-slant submersions from Cosymplectic manifolds onto Riemannian manifolds. We investigate integrability of distributions and the geometry of leaves of such submersions from Cosymplectic manifolds onto Riemannian manifolds. Moreover, we examine necessary and sufficient conditions for such submersions to be totally geodesic where characteristic vector field ξ is vertical.

Keywords: cosymplectic manifolds; semi-slant submersion; conformal semi-slant submersion.

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1. INTRODUCTION

Firstly, O' Neill [26] and Grey [19] defined and studied the theory of Riemannian submersion between Riemannian manifolds. Later, this notion was widely studied in differential geometry. The Riemannian submersions have several important applications both in mathematics and in physics, because of their applications in Yang-Mills theory [8], Kaluza-Klein theory [9, 20],

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robotic theory [5], supergravity and superstring theories [21, 25] etc. On the other hand Riemannian submersions are very useful in Riemannian geometry for studying the geometry of Riemannian manifolds equipped with differentiable structures.

By using the concept of Riemannian submersion and the condition of almost complex mapping, Watson [32] introduced the notion of almost Hermitian submersions. Sahin [30] introduced the notion of anti-invariant Riemannian submersions from almost Hermitian manifolds. Afterwards, he also defined slant submersions from almost Hermitian manifolds onto a Riemannian manifold in [31]. As a generalization of slant, semi-invariant, and anti-invariant submersions, Park and Prasad [27] defined and studied the notion of semi-slant submersions from an almost Hermitian manifold onto a Riemannian manifold. Considering different conditions on Riemannian submersions many geometers studied this area and obtained lots of results on this ([14, 7, 15, 17], [29]).

In 1985, D. Chinea [11] extended the notion of almost Hermitian submersion to different subclasses of almost contact metric manifolds. He investigated some geometric properties between base manifold and total manifold as well as fibers. Recently, considering different conditions on Riemannian submersions many studies have been done([10, 12, 16, 22]).

A related topic of growing interest deals with the study of Riemannian submersion so-called horizontally conformal submersions: these maps, which provide a natural generalization of Riemannian submersion, were introduced independently by Fuglede [13] and Ishihara [23]. As a generalization of holomorphic submersions, Gudmundsson and Wood [15] introduced the notion of conformal holomorphic submersions and obtained necessary and sufficient conditions for conformal holomorphic submersions to be a harmonic morphism. Recently, Akyol and Sahin studied the notion of conformal anti-invariant submersions and conformal semi-invariant submersions from almost Hermitian manifolds onto Riemannian manifolds ([3], [4]). Akyol introduced the concept of conformal semi-slant submersions from almost Hermitian manifolds onto Riemannian manifolds [1]. In 2019, Prasad and Kumar defined and studied the notion of conformal semi-invariant submersions from almost contact metric manifolds onto Riemannian manifolds [28] and conformal semi-slant submersions from Lorentzian para Sasakian manifolds onto Riemannian manifolds [24] (see also [2, 18]).

In the present paper, we study conformal semi-slant submersion from Cosymplectic manifolds onto Riemannian manifolds. The paper is organized as follows: In the second section, we gather main notions and formulae for other sections. In the third section, we give the definition of conformal semi-slant submersions and some results. We also study the integrability of distributions and the geometry of leaves of vertical distribution. Finally, we obtain certain conditions for such submersions to be totally geodesic.

2. Preliminaries

In this section, we recall main definitions and properties of Cosymplectic manifolds and submersions.

We consider M_1 is a $(2n+1)-$dimensional almost contact manifold [11] which carries a tensor field ϕ of the tangent space, $1-$form η and characteristic vector field ξ satisfying

$$\begin{aligned} \phi^2 &= -I+\eta\otimes\xi, \quad \eta(\xi)=1, \\ \phi\xi &= 0, \qquad \eta\circ\phi=0, \end{aligned} \tag{1}$$

where $I : TM_1 \longrightarrow TM_1$ is the identity map.

Since any almost contact manifold (M_1,ϕ,ξ,η) admits a Riemannian metric g such that

$$g(\phi X_1,\phi X_2)=g(X_1,X_2)-\eta(X_1)\eta(X_2), \tag{2}$$

for any vector fields $X_1,X_2\in\Gamma(TM_1)$, where $\Gamma(TM_1)$ represents the Lie algebra of vector fields on M_1. The manifold M_1 together with the structure (ϕ,ξ,η,g) is called an almost contact metric manifold.

The immediate consequence of (2), we have

$$\eta(X_1)=g(X_1,\xi) \quad \text{and} \quad g(\phi X_1,X_2)+g(X_1,\phi X_2)=0, \tag{3}$$

for all vector fields $X_1,X_2\in\Gamma(TM_1)$.

An almost contact structure (ϕ,ξ,η) is said to be normal if the almost complex structure J on the product manifold $M_1\times R$ is given by

$$J(U,\ f\frac{d}{dt})=(\phi U-f\xi,\ \eta(U)\frac{d}{dt}),$$

where $J^2 = -I$ and f is the differentiable function on $M_1 \times R$ has no torsion i.e., J is integrable. The condition for normality in terms of ϕ, ξ and η is $[\phi,\phi]+2d\eta\otimes\xi=0$ on M_1, where $[\phi,\phi]$ is the Nijenhuis tensor of ϕ. Now, the fundamental 2-form is defined by $\Phi(X_1,X_2)=g(X_1,\phi X_2)$.

An almost contact metric manifold is said to be a Cosymplectic manifold if it is normal and both Φ and η are closed. The structure equation of a Cosymplectic manifold is given by

$$(\nabla_{X_1}\phi)X_2=0, \tag{4}$$

for all vector fields $X_1,X_2\in\Gamma(TM_1)$, where ∇ represents the Levi-Civita connection of (M_1,g). Moreover, for a Cosymplectic manifold, we have

$$\nabla_{X_1}\xi=0, \tag{5}$$

for every vector field $X_1\in\Gamma(TM_1)$.

Example 1. *We consider R^{2k+1} with Cartesian coordinates (x_i,y_i,z) $(i=1,.....,k)$ and its usual contact form $\eta=dz$.*

The characteristic vector field ξ is given by $\frac{\partial}{\partial z}$, and its Riemannian metric g and tensor field ϕ are given by

$$g=\sum_{i=1}^{k}((dx_i)^2+(dy_i)^2)+(dz)^2,\ \phi=\begin{bmatrix}0 & \delta_{ij} & 0\\ -\delta_{ij} & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\ i=1,.....k.$$

This gives a Cosymplectic structure on R^{2k+1}. The vector fields $E_i=\frac{\partial}{\partial y_i},E_{k+i}=\frac{\partial}{\partial x_i},\xi=\frac{\partial}{\partial z}$ form a $\phi-$basis for the Cosymplectic structure. On the other hand, it can be shown that $(R^{2k+1},\phi,\xi,\eta,g)$ is a Cosymplectic manifold.

Let (M_1,g_1) and (M_2,g_2) be two Riemannian manifolds of dimension m and n respectively, where g_1 and g_2 are the Riemannian metrics on M_1 and M_2. Let $f:(M_1,g_1)\to(M_2,g_2)$ be a differentiable map. We call the map f a differentiable submersion if f is surjective and the differential $(f_*)_p$ has a maximal rank for any $p\in M_1$. The map f is said to be a Riemannian submersion if f is a differentiable submersion and $(f_*)_p:((\ker(f_*)_p)^\perp,(g_1)_p)\to(T_{f(p)}M_2,(g_2)_{f(p)})$ is a linear isometry for each $p\in M_1$, where $(\ker(f_*)_p)^\perp$ is the orthogonal complement of the space $\ker(f_*)_p$ in the tangent space T_PM_1 of M_1 at p.

Let $f:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ be a differentiable map from almost contact metric manifold (M_1,ϕ,ξ,η,g_1) to Riemannian manifold (M_2,g_2). We call the map f slant submersion if f is a Riemannian submersion and the angle $\theta=\theta(X_1)$ between ϕX_1 and the space $ker(f_*)_p-\{\xi\}_p$ is constant for non-zero vector fields $X_1\in\ ker(f_*)_p-\{\xi\}_p$ if $\xi\in\ ker(f_*)_p$ and $p\in M_1$. We call the angle θ a slant angle.

Let $f:(M_1,g_1)\to(M_2,g_2)$ be a differentiable map between Riemannian manifolds. The second fundamental form of f is given by

$$(\nabla f_*)(X_1,X_2)=\nabla^f_{X_1}f_*X_2-f_*(\nabla_{X_1}X_2), \tag{6}$$

for all $X_1,X_2\in\Gamma(TM_1)$, where ∇^f is the pullback connection and we denote conveniently by ∇ the Levi-Civita connections of the metrics g_1 and g_2.

Define O'Neill's tensors $\mathscr{T}$ and $\mathscr{A}$ by

$$\mathscr{A}_E F=\mathscr{H}\nabla_{\mathscr{H}E}\mathscr{V}F+\mathscr{V}\nabla_{\mathscr{H}E}\mathscr{H}F, \tag{7}$$

$$\mathscr{T}_E F=\mathscr{H}\nabla_{\mathscr{V}E}\mathscr{V}F+\mathscr{V}\nabla_{\mathscr{V}E}\mathscr{H}F, \tag{8}$$

for any vector field E,F on M_1, where ∇ is the Levi-Civita connection of g_1. It is easy to see that $\mathscr{T}_E$ and $\mathscr{A}_E$ are skew-symmetric operators on the tangent bundle of M_1 reversing the vertical and the horizontal distributions. We summarize the properties of the tensor fields $\mathscr{T}$ and $\mathscr{A}$.

On the other hand, from equations (7) and (8), we have

$$\nabla_{X_1}X_2=\mathscr{T}_{X_1}X_2+\mathscr{V}\nabla_{X_1}X_2, \tag{9}$$

$$\nabla_{X_1}V_1=\mathscr{T}_{X_1}V_1+\mathscr{H}\nabla_{X_1}V_1, \tag{10}$$

$$\nabla_{V_1}X_1=\mathscr{A}_{V_1}X_1+\mathscr{V}\nabla_{V_1}X_1, \tag{11}$$

$$\nabla_{V_1}V_2=\mathscr{H}\nabla_{V_1}V_2+\mathscr{A}_{V_1}V_2, \tag{12}$$

for all $X_1,X_2\in\Gamma(\ker f_*)$ and $V_1,V_2\in\Gamma(\ker f_*)^{\perp}$, where $\mathscr{H}\nabla_{X_1}V_2=\mathscr{A}_{V_2}X_1$, if V_2 is basic. It is not difficult to observe that $\mathscr{T}$ acts on the fibers as the second fundamental form, while $\mathscr{A}$

acts on the horizontal distribution and measures of the obstruction to the integrability of this distribution.

It is seen that for $q \in M_1$, $X_1 \in \mathscr{V}_q$ and $V_1 \in \mathscr{H}_q$ the linear operators

$$\mathscr{A}_{V_1},\ \mathscr{T}_{X_1} : T_qM_1 \to T_qM_2$$

are skew-symmetric, that is

$$g_1(\mathscr{A}_{V_1}E,F) = -g_1(E,\mathscr{A}_{V_1}F) \text{ and } g_1(\mathscr{T}_{X_1}E,F) = -g_1(E,\mathscr{T}_{X_1}F) \tag{13}$$

for each $E,F \in T_qM_1$. We have also defined the restriction of $\mathscr{T}$ to the vertical distribution $\mathscr{T}|_{\mathscr{V}\times\mathscr{V}}$ is precisely the second fundamental form of the fibres of f. Since $\mathscr{T}_V$ is skew-symmetric we get: f has totally geodesic fibres if and only if $\mathscr{T} \equiv 0$ [6].

Next, we find necessary and sufficient condition for conformal semi-slant Riemannian submersion to be totally geodesic. We recall that a differentiable map f between two Riemannian manifolds is called totally geodesic if

$$(\nabla f_*)(V_1,V_2) = 0, \text{ for all } V_1,V_2 \in \Gamma(TM_1).$$

A geometric clarification of a totally geodesic map is that it maps every geodesic in the total space into a geodesic in the base space in proportion to arc lengths.

Lemma 1. *Let (M_1,ϕ,ξ,η,g_1) be an $m-$dimensional Cosymplectic manifold and (M_2,g_2) be an $n-$dimensional Riemannian manifold. Let $f : M_1 \to M_2$ be a differentiable map between them and $p \in M_1$. Then f is called horizontally weakly conformal or semi-conformal at p if either $df_p = 0$, or df_p maps the horizontal space $\mathscr{H} = ((\ker f_*)_p)^\perp$ conformally onto $T_{f(p)}M_2$.*

The second condition in the above definition exactly is the same as df_p is symmetric and there exists a number $\lambda(p) \neq 0$ such that

$$g_2(f_*X_1,f_*X_2) = \lambda^2(p)g_1(X_1,X_2), \text{ for all } X_1,X_2 \in ((\ker f_*)_p)^\perp. \tag{14}$$

Here $\lambda(p)$ is called the square dilation of f at p. The map f is called horizontally weakly conformal or semi-conformal on M_1 if it is horizontally weakly conformal at every point on M_1. If f has no critical point, then it is said to be a (horizontally) conformal submersion [26].

We should mention that a horizontally conformal submersion $f : M_1 \to M_2$ is called horizontally homothetic if the gradient of its dilation λ is vertical, i.e.,

$$\mathscr{H}(grad\lambda) = 0, \tag{15}$$

at $p \in M_1$, where $\mathscr{H}$ is the complement orthogonal distribution to $\mathscr{V} = \ker f_*$ in $\Gamma(T_pM_1)$.

Again, we recall the following definition from [6].

Let $f : M_1 \to M_2$ be a conformal submersion. A vector field E on M_1 is called projectiable if there exist a vector field $\widehat{E}$ on M_2 such that $f_*(E_p) = \widehat{E}_{f(p)}$ for any $p \in M_1$. In this case E and $\widehat{E}$ are called $f-$related. A horizontal vector field X_2 on M_1 is called basic, if it is projectiable. It is a well known fact that if $\widehat{Z}$ is a vector field on M_2, then there exists a unique basic vector field Z which is called the horizontal lift of $\widehat{Z}$.

Lemma 2. *Let $f : M_1 \to M_2$ be a horizontal conformal submersion. Then, for any horizontal vector fields X_1, X_2 and vertical vector fields V_1, V_2, we have*

(i) $(\nabla f_*)(X_1, X_2) = X_1(\ln\lambda) f_* X_2 + X_2(\ln\lambda) f_* X_1 - g_1(X_1, X_2) f_*(grad \ln\lambda)$,
(ii) $(\nabla f_*)(V_1, V_2) = -f_*(\mathscr{T}_{V_1} V_2)$,
(iii) $(\nabla f_*)(X_1, V_1) = -f_*(\nabla^{M_1}_{X_1} V_1) = -f_*(\mathscr{A}_{X_1} V_1)$.

3. Conformal Semi-Slant Submersions

In this section, we define and study conformal semi-slant submersion from Cosymplectic manifolds.

Definition 1. *Let $(M_1, \phi, \xi, \eta, g_1)$ be a Cosymplectic manifold and (M_2, g_2) be a Riemannian manifold. A horizontal conformal submersion $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ is called conformal semi-slant submersion if there is a distribution $D_1 \subset (\ker f_*)$ such that*

$$\ker f_* = D_1 \oplus D_2 \oplus < \xi >, \ \phi(D_1) = D_1, \tag{16}$$

and the angle $\theta = \theta(V_1)$ between ϕV_1 and the space $(D_2)_p$ is constant for non-zero vector field $V_1 \in (D_2)_p$ and $p \in M_1$, where D_2, D_1 and $< \xi >$ are mutually orthogonal in $(\ker f_*)$. As it is, the angle θ is called the semi-slant angle of the horizontally conformal submersions.

It is known that the distribution $\ker f_*$ is integrable. Hence above definition (1) implies that the integral manifold (fiber) $f^{-1}(q), q \in M_2$ of $\ker f_*$ is a semi-slant submanifold.

Now, we shall give some examples of conformal semi-slant submersion from Cosymplectic manifold onto Riemannian manifold.

Example 2. *Let $\mathbb{R}^7$ has a Cosymplectic structure as in Example 1. Let $f : \mathbb{R}^7 \to \mathbb{R}^2$ be a submersion defined by*

$$f(x_1, x_2, x_3, y_1, y_2, y_3, z) = e^3(\frac{x_2 - y_3}{\sqrt{2}}, y_2).$$

Then, by direct calculations, we obtain the Jacobian matrix of f as

$$\begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

After a straightforward computation, we obtain

$$\begin{aligned} (\ker f_*) &= span\{V_1 = \frac{\partial}{\partial x_1}, V_2 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_2} + \frac{\partial}{\partial y_3}), V_3 = \frac{\partial}{\partial x_3}, V_4 = \frac{\partial}{\partial y_1}, V_5 = \frac{\partial}{\partial z}\}, \\ (\ker f_*)^{\perp} &= span\{H_1 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial y_3}), H_2 = \frac{\partial}{\partial y_2}\}. \end{aligned}$$

Thus it follows that $D_1 = span\{V_1, V_4\}$ and $D_2 = span\{V_2, V_3\}$. Thus, the map f is a conformal semi-slant submersion with the semi-slant angle $\theta = \frac{\pi}{4}$ and dilation $\lambda = e^3$.

Example 3. *Let $\mathbb{R}^7$ has a Cosymplectic structure as in Example 1. Let $f : \mathbb{R}^7 \to \mathbb{R}^2$ be a submersion defined by*

$$f(x_1, x_2, x_3, y_1, y_2, y_3, z) = e^{\pi}(\frac{\sqrt{3}x_1 - y_2}{2}, y_1).$$

Then, by direct calculations, we obtain the Jacobian matrix of f as

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

After some straightforward computations, we derive

$$\begin{aligned} (\ker f_*) &= span\{V_1 = \frac{1}{2}(\frac{\partial}{\partial x_1} + \sqrt{3}\frac{\partial}{\partial y_2}), V_2 = \frac{\partial}{\partial x_2}, V_3 = \frac{\partial}{\partial x_3}, V_4 = \frac{\partial}{\partial y_3}, V_5 = \frac{\partial}{\partial z}\}, \\ (\ker f_*)^{\perp} &= span\{H_1 = \frac{1}{2}(\sqrt{3}\frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_2}), H_2 = \frac{\partial}{\partial y_1}\}. \end{aligned}$$

Thus it follows that $D_1 = span\{V_3, V_4\}$ and $D_2 = span\{V_1, V_2\}$. Thus, the map f is a conformal semi-slant submersion with the semi-slant angle $\theta = \frac{\pi}{6}$ and dilation $\lambda = e^{\pi}$.

Let f be a conformal semi-slant submersion from Cosymplectic manifold $(M_1, \phi, \xi, \eta, g_1)$ onto Riemannian manifold (M_2, g_2). For $X_1 \in \Gamma(\ker f_*)$, we have

$$X_1 = PX_1 + QX_1 + \eta(X_1)\xi, \tag{17}$$

where $PX_1 \in \Gamma(D_1)$ and $QX_1 \in \Gamma(D_2)$.

For $X_2 \in \Gamma(\ker f_*)$, we have

$$\phi X_2 = \psi X_2 + \omega X_2, \tag{18}$$

where ψX_2 and ωX_2 are vertical and horizontal components of ϕX_2 respectively.

Also for $X_3 \in \Gamma(\ker f_*)^{\perp}$, we have

$$\phi X_3 = BX_3 + CX_3, \tag{19}$$

where BX_3 and CX_3 are vertical and horizontal components of ϕX_3 respectively.

Then, $\Gamma(\ker f_*)^{\perp}$ decomposed as

$$\Gamma(\ker f_*)^{\perp} = \omega D_2 \oplus \mu, \tag{20}$$

where μ is the orthogonal complement of ωD_2 in $\Gamma(\ker f_*)^{\perp}$ and it is invariant with respect to ϕ.

Let $f : (M_1, \phi, \xi, \eta, g_1) \to (M_2, g_2)$ is conformal semi-slant submersion from Cosymplectic manifold $(M_1, \phi, \xi, \eta, g_1)$ onto Riemannian manifold (M_2, g_2). Thus the using equations $(3), (18)$ and (19), we get

$$g_1(\psi X_1, X_2) = -g_1(X_1, \psi X_2),\ g_1(V_1, CV_2) = -g_1(CV_1, V_2), \tag{21}$$

for all $X_1, X_2 \in \Gamma(\ker f_*)$ and $V_1, V_2 \in \Gamma(\ker f_*)^{\perp}$.

Then the using equations $(1), (18), (19)$ and (20), we get

$$\psi D_1 = D_1,\ \omega D_1 = 0,\ \psi D_2 \subset D_2,\ B(\Gamma(\ker f_*)^{\perp}) = D_2. \tag{22}$$

Lemma 3. *Let (M_1,ϕ,ξ,η,g_1) be a Cosymplectic manifold and (M_2,g_2) be a Riemannian manifold. If $f:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ is conformal semi-slant submersion, then*

$$\psi^2X_1+B\omega X_1=-X_1+\eta(X_1)\otimes\xi,\ \omega\psi X_1+C\omega X_1=0,$$

$$\psi BX_2+BCX_2=0,\ \omega BX_2+C^2X_2=-X_2,$$

for all $X_1\in\Gamma(\ker f_*)$ and $X_2\in\Gamma(\ker f_*)^{\perp}$.

We define the co-variant derivatives of ψ and ω as follows:

$$(\nabla_{X_1}\psi)X_2=\widehat{\nabla}_{X_1}\psi X_2-\psi\widehat{\nabla}_{X_1}X_2, \tag{23}$$

$$(\nabla_{X_1}\omega)X_2=\mathscr{H}\nabla_{X_1}\omega X_2-\omega\widehat{\nabla}_{X_1}X_2, \tag{24}$$

for all $X_1,X_2\in\Gamma(\ker f_*)$, where $\widehat{\nabla}_{X_1}X_2=\mathscr{V}\widehat{\nabla}_{X_1}X_2$.

Lemma 4. *Let (M_1,ϕ,ξ,η,g_1) be a Cosymplectic manifold and (M_2,g_2) be a Riemannian manifold. If $f:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ is conformal semi-slant submersion, then*

(1)

$$(\nabla_{X_1}\psi)X_2=B\mathscr{T}_{X_1}X_2-\mathscr{T}_{X_1}\omega X_2,$$

$$(\nabla_{X_1}\omega)X_2=C\mathscr{T}_{X_1}X_2-\mathscr{T}_{X_1}\psi X_2,$$

for all $X_1,X_2\in\Gamma(\ker f_*)$.

(2)

$$\mathscr{T}_{X_1}BV_1+\mathscr{H}\nabla_{X_1}CV_1=C\mathscr{H}\nabla_{X_1}V_1+\omega\mathscr{T}_{X_1}V_1,$$

$$\widehat{\nabla}_{X_1}BV_1+\mathscr{T}_{X_1}CV_1=B\mathscr{H}\nabla_{X_1}V_1+\psi\nabla_{X_1}V_1,$$

for all $X_1\in\Gamma(\ker f_*)$ and $V_1\in\Gamma(\ker f_*)^{\perp}$.

(3)

$$\mathscr{V}\nabla_{V_1}\psi X_1+\mathscr{A}_{V_1}\omega X_1=B\mathscr{A}_{V_1}X_1+\psi\mathscr{V}\nabla_{V_1}X_1,$$

$$\mathscr{A}_{V_1}\psi X_1+\mathscr{H}\nabla_{V_1}\omega X_1=C\mathscr{A}_{V_1}X_1+\omega\mathscr{V}\nabla_{V_1}X_1,$$

for all $X_1\in\Gamma(\ker f_*)$ and $V_1\in\Gamma(\ker f_*)^{\perp}$.

(4)

$$\mathscr{A}_{V_1}BV_2+\mathscr{H}\nabla_{V_1}CV_2=C\mathscr{H}\nabla_{V_1}V_2+\omega\mathscr{A}_{V_1}V_2,$$

$$\mathscr{V}\nabla_{V_1}BV_2+\mathscr{A}_{V_1}CV_2=B\mathscr{H}\nabla_{V_1}V_2+\psi\mathscr{A}_{V_1}V_2,$$

for all $V_1,V_2\in\Gamma(\ker f_*)^\perp$.

Lemma 5. *Let $f:(M_1,\phi,\xi,\eta,g_1)\to(M_2,g_2)$ is conformal semi-slant submersion from an almost contact metric manifold (M_1,ϕ,ξ,η,g_1) onto Riemannian manifold (M_2,g_2). Then f is a proper conformal semi-slant submersion if and only if there exists a constant $\lambda\in[-1,0]$ such that*

$$\psi^2X_1=\lambda X_1,\ \textit{for all}\ X_1\in\Gamma(D_2),$$

where $\lambda=-\cos^2\theta$.

Proof. For any non-zero vector field $X_1\in\Gamma(D_2)$, we have

$$\cos\theta=\frac{\parallel\psi X_1\parallel}{\parallel\phi X_1\parallel},\tag{25}$$

and

$$\cos\theta=\frac{g_1(\phi X_1,\psi X_1)}{\parallel\psi X_1\parallel\parallel\phi X_1\parallel},\tag{26}$$

where $\theta(X_1)$ is the semi-slant angle.

Using equations $(1),(2),(18)$ and (26), we get

$$\cos\theta=\frac{-g_1(X_1,\psi^2X_1)}{\parallel\psi X_1\parallel\parallel\phi X_1\parallel}.\tag{27}$$

From equations (25) and (27), we have

$$\psi^2X_1=-\cos^2\theta.X_1.$$

If $\lambda=-\cos^2\theta$, then

$$\psi^2X_1=\lambda X_1,$$

for all $X_1\in\Gamma(D_2)$. □

From Lemma 5 and equations $(2),(18)$ and (19), then we easily have

Corollary 1. *Let f is conformal semi-slant submersion from an almost contact metric manifold (M_1,ϕ,ξ,η,g_1) onto a Riemannian manifold (M_2,g_2), then*

$$g_1(\psi X_1,\psi X_2)=\cos^2\theta g_1(X_1,X_2),\tag{28}$$

$$g_1(\omega X_1, \omega X_2) = \sin^2\theta g_1(X_1, X_2), \tag{29}$$

for all $X_1, X_2 \in \Gamma(D_2)$.

Lemma 6. *Let f is conformal semi-slant submersion from a Cosymplectic manifold* $(M_1, \phi, \xi, \eta, g_1)$ *onto a Riemannian manifold* (M_2, g_2) *with the slant angle* $\theta \in [0, \frac{\pi}{2}]$. *If* ω *is parallel with respect to* ∇ *on* D_2, *then we have*

$$\mathscr{T}_{\psi X_1}\psi X_1 = -\cos^2\theta . \mathscr{T}_{X_1}X_1, \textit{ for all } X_1 \in \Gamma(D_2).$$

Proof. If ω is parallel, then from Lemma (4), we have

$$C\mathscr{T}_{X_1}X_2 = \mathscr{T}_{X_1}\psi X_2, \text{ for all } X_1, X_2 \in \Gamma(D_2). \tag{30}$$

Interchanging the role of X_1 and X_2, we have

$$C\mathscr{T}_{X_2}X_1 = \mathscr{T}_{X_2}\psi X_1, \text{ for all } X_1, X_2 \in \Gamma(D_2) \tag{31}$$

Since $\mathscr{T}$ is symmetric, from equations (30) and (31), we get

$$\mathscr{T}_{\psi X_1}\psi X_1 = -\cos^2\theta . \mathscr{T}_{X_1}X_1, \text{ for all } X_1 \in \Gamma(D_2).$$

□

Theorem 1. *Let f be a conformal semi-slant submersion from a Cosymplectic manifold* $(M_1, \phi, \xi, \eta, g_1)$ *onto a Riemannian manifold* (M_2, g_2). *Then the invariant distribution* D_1 *is integrable if and only if*

$$\begin{aligned} & \frac{1}{\lambda^2} g_2((\nabla f_*)(X_1, \phi X_2) - (\nabla f_*)(X_2, \phi X_1), f_*(\omega V_1)) \\ = \;& g_1(\psi(\widehat{\nabla}_{X_1}\phi X_2 - \widehat{\nabla}_{X_2}\phi X_1), V_1), \end{aligned}$$

for all $X_1, X_2 \in \Gamma(D_1)$ *and* $V_1 \in \Gamma(D_2)$.

Proof. We note that D_1 is integrable if and only if $g_1([X_1, X_2], V_1) = 0$, $g_1([X_1, X_2], V_2) = 0$, and $g_1([X_1, X_2], \xi) = 0$, for all $X_1, X_2 \in \Gamma(D_1), V_1 \in \Gamma(D_2)$ and $V_2 \in (\ker f_*)^\perp$. Since $\ker f_*$ is integrable $g_1([X_1, X_2], V_2) = 0$. Thus, D_1 is integrable if and only if $g_1([X_1, X_2], V_1) = 0$ and $g_1([X_1, X_2], \xi) = 0$.

Now, for any $X_1, X_2 \in \Gamma(D_1)$, we have

$$\begin{aligned} g_1([X_1, X_2], \xi) &= g_1(\nabla_{X_1} X_2, \xi) - g_1(\nabla_{X_2} X_1, \xi) \\ &= -g_1(X_2, \nabla_{X_1} \xi) + g_1(X_1, \nabla_{X_2} \xi) \end{aligned}$$

Using equation (5), we have

$$g_1([X_1, X_2], \xi) = 0.$$

Now, from equations $(2), (3), (4), (6), (9)$ and (18), we have

$$\begin{aligned} & g_1([X_1, X_2], V_1) \\ = \; & g_1(\widehat{\nabla}_{X_1} \phi X_2 - \widehat{\nabla}_{X_2} \phi X_1, \psi V_1) + g_1(\mathscr{H} \nabla_{X_1} \phi X_2, \omega V_1) - g_1(\mathscr{H} \nabla_{X_2} \phi X_1, \omega V_1). \end{aligned}$$

Since f is conformal submersion, using equation (14) and Lemma 2, we have

$$\begin{aligned} & g_1([X_1, X_2], V_1) \\ = \; & -\frac{1}{\lambda^2} g_2((\nabla f_*)(X_1, \phi X_2) - (\nabla f_*)(X_2, \phi X_1), f_*(\omega V_1)) - g_1(\psi(\widehat{\nabla}_{X_1} \phi X_2 - \widehat{\nabla}_{X_2} \phi X_1), V_1). \end{aligned}$$

Then D_1 is integrable $\Leftrightarrow$

$$\begin{aligned} & \frac{1}{\lambda^2} g_2((\nabla f_*)(X_1, \phi X_2) - (\nabla f_*)(X_2, \phi X_1), f_*(\omega V_1)) \\ = \; & g_1(\psi(\widehat{\nabla}_{X_1} \phi X_2 - \widehat{\nabla}_{X_2} \phi X_1), V_1). \end{aligned}$$

□

Theorem 2. *Let f be conformal semi-slant submersion from a Cosymplectic manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Then the slant distribution D_2 is integrable if and only if*

$$\mathscr{T}_{V_2} \omega V_1 - \mathscr{T}_{V_1} \omega V_2 + \psi(\mathscr{T}_{V_1} \omega \psi V_2 - \mathscr{T}_{V_2} \omega \psi V_1) \in \Gamma(D_2),$$

for all $V_1, V_2 \in \Gamma(D_2)$.

Proof. We note that D_2 is integrable if and only if $g_1([V_1, V_2], X_1) = 0$, $g_1([V_1, V_2], X_2) = 0$ and $g_1([V_1, V_2], \xi) = 0$, for all $V_1, V_2 \in \Gamma(D_2), X_1 \in \Gamma(D_1)$ and $X_2 \in (\ker f_*)^{\perp}$. Since $\ker f_*$ is integrable then $g_1([V_1, V_2], X_2) = 0$, and also, $g_1([V_1, V_2], \xi) = 0$. Thus, D_2 is integrable if and only if $g_1([V_1,V_2],X_1)=0$.

From equations $(2),(3),(4),(6)$ and (18), we have

$$\begin{aligned} g_1([V_1,V_2],\phi X_1) &= g_1([V_1,V_2],\phi X_1), \\ &= -g_1(\nabla_{V_1}\psi V_2,X_1)-g_1(\nabla_{V_1}\omega V_2,X_1)+g_1(\nabla_{V_2}\psi V_1,X_1)+g_1(\nabla_{V_2}\omega V_1,X_1), \end{aligned}$$

Next, using equation (9) and Lemma 5, we have

$$\sin^2\theta g_1([V_1,V_2],\phi X_1)=g_1(\mathscr{T}_{V_2}\omega V_1-\mathscr{T}_{V_1}\omega V_2,X_1)+g_1(\psi(\mathscr{T}_{V_1}\omega\psi V_2-\mathscr{T}_{V_2}\omega\psi V_1),X_1),$$

Thus D_2 is integrable $\Leftrightarrow$

$$\mathscr{T}_{V_2}\omega V_1-\mathscr{T}_{V_1}\omega V_2+\psi(\mathscr{T}_{V_1}\omega\psi V_2-\mathscr{T}_{V_2}\omega\psi V_1)\in\Gamma(D_2).$$

□

Theorem 3. *Let f be conformal semi-slant submersion from a Cosymplectic manifold (M_1,ϕ,ξ,η,g_1) onto a Riemannian manifold (M_2,g_2). Then the distribution $(\ker f_*)^\perp$ is integrable if and only if*

$$\begin{aligned} &\frac{1}{\lambda^2}g_2(\nabla_{X_2}f_*(CX_1)-\nabla_{X_1}f_*(CX_2),f_*(\omega V_1)) \\ = \;& g_1(\mathscr{A}_{X_1}BX_2-\mathscr{A}_{X_2}BX_1,\omega V_1)+g_1(\mathscr{V}\nabla_{X_1}BX_2-\mathscr{V}\nabla_{X_2}BX_1+\mathscr{A}_{X_1}CX_2-\mathscr{A}_{X_2}CX_1,\psi V_1)- \\ & g_1(X_1,\omega V_1)g_1(CX_2,grad\ln\lambda)+g_1(X_2,\omega V_1)g_1(CX_1,grad\ln\lambda) \\ & +2g_1(X_1,CX_2)g_1(grad\ln\lambda,\omega V_1), \end{aligned}$$

for all $X_1,X_2\in\Gamma(\ker f_*)^\perp$ and $V_1\in\Gamma(\ker f_*)$.

Proof. For all $X_1,X_2\in\Gamma(\ker f_*)^\perp$ and $V_1\in\Gamma(\ker f_*)$, using equations $(2)-(5),(11),(12),(18)$ and (19), we get

$$\begin{aligned} & g_1([X_1,X_2],V_1) \\ = \;& g_1(\nabla_{X_1}X_2,V_1)-g_1(\nabla_{X_2}X_1,V_1) \\ = \;& g_1(\mathscr{A}_{X_1}BX_2-\mathscr{A}_{X_2}BX_1,\omega V_1)+g_1(\mathscr{V}\nabla_{X_1}BX_2-\mathscr{V}\nabla_{X_2}BX_1+\mathscr{A}_{X_1}CX_2-\mathscr{A}_{X_2}CX_1,\psi V_1)+ \\ & g_1(\mathscr{H}\nabla_{X_1}CX_2-\mathscr{H}\nabla_{X_2}CX_1,\omega V_1). \end{aligned}$$

Since f is conformal submersion and using equation (6) and Lemma 2, we have

$$
\begin{aligned}
& g_1([X_1,X_2],V_1) \\
= \;& g_1(\mathscr{A}_{X_1}BX_2 - \mathscr{A}_{X_2}BX_1, \omega V_1) + g_1(\mathscr{V}\nabla_{X_1}BX_2 - \mathscr{V}\nabla_{X_2}BX_1 + \mathscr{A}_{X_1}CX_2 - \mathscr{A}_{X_2}CX_1, \psi V_1) + \\
& \frac{1}{\lambda^2} g_2(\nabla_{X_1} f_*(CX_2), f_*(\omega V_1)) - \frac{1}{\lambda^2} g_2((\nabla f_*)(X_1, CX_2), f_*(\omega V_1)) - \\
& \frac{1}{\lambda^2} g_2(\nabla_{X_2} f_*(CX_1), f_*(\omega V_1)) + \frac{1}{\lambda^2} g_2((\nabla f_*)(X_2, CX_1), f_*(\omega V_1)), \\
= \;& g_1(\mathscr{A}_{X_1}BX_2 - \mathscr{A}_{X_2}BX_1, \omega V_1) + g_1(\mathscr{V}\nabla_{X_1}BX_2 - \mathscr{V}\nabla_{X_2}BX_1 + \mathscr{A}_{X_1}CX_2 - \mathscr{A}_{X_2}CX_1, \psi V_1) + \\
& \frac{1}{\lambda^2} g_2(\nabla_{X_1} f_*(CX_2), f_*(\omega V_1)) - \frac{1}{\lambda^2} g_2(\nabla_{X_2} f_*(CX_1), f_*(\omega V_1)) - \\
& \frac{1}{\lambda^2} g_2(X_1(\ln\lambda) f_*(CX_2) + X_2(\ln\lambda) f_*(CX_1) - g_1(X_1, CX_2) f_*(grad \ln\lambda), f_*(\omega V_1)) + \\
& \frac{1}{\lambda^2} g_2(X_1(\ln\lambda) f_*(CX_2) + X_2(\ln\lambda) f_*(CX_1) - g_1(X_1, CX_2) f_*(grad \ln\lambda), f_*(\omega V_1)), \\
= \;& g_1(\mathscr{A}_{X_1}BX_2 - \mathscr{A}_{X_2}BX_1, \omega V_1) + g_1(\mathscr{V}\nabla_{X_1}BX_2 - \mathscr{V}\nabla_{X_2}BX_1 + \mathscr{A}_{X_1}CX_2 - \mathscr{A}_{X_2}CX_1, \psi V_1) + \\
& \frac{1}{\lambda^2} g_2(\nabla_{X_1} f_*(CX_2), f_*(\omega V_1)) - \frac{1}{\lambda^2} g_2(\nabla_{X_2} f_*(CX_1), f_*(\omega V_1)) - \\
& g_1(X_1, \omega V_1) g_1(CX_2, grad \ln\lambda) + g_1(X_2, \omega V_1) g_1(CX_1, grad \ln\lambda) \\
& +2 g_1(X_1, CX_2) g_1(grad \ln\lambda, \omega V_1),
\end{aligned}
$$

which completes the proof. □

Theorem 4. *Let f be a conformal semi-slant submersion from a Cosymplectic manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Then the distribution D_1 defines totally geodesic foliation on M_1 if and only if*

$$
\begin{aligned}
& \frac{1}{\lambda^2} g_2((\nabla f_*)(V_1, \phi V_2), f_*(\omega X_1)) \\
= \;& g_1(\mathscr{V}\nabla_{V_1} \phi V_2, \psi X_1),
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{1}{\lambda^2} g_2((\nabla f_*)(V_1, \phi V_2), f_*(CX_2)) \\
= \;& g_1(V_2, \mathscr{V}\nabla_{V_1} \psi BX_2) + g_1(V_2, \mathscr{T}_{V_1} \omega BX_2),
\end{aligned}
$$

for all $V_1, V_2 \in \Gamma(D_1)$, $X_1 \in \Gamma(D_2)$ and $X_2 \in \Gamma(\ker f_*)^{\perp}$.

Proof. The distribution D_1 defines a totally geodesic foliation on M_1 if and only if $g_1(\nabla_{V_1}V_2,X_1)=0$, $g_1(\nabla_{V_1}V_2,X_2)=0$ and $g_1(\nabla_{V_1}V_2,\xi)=0$, for all $V_1,V_2\in\Gamma(D_1)$, $X_1\in\Gamma(D_2)$ and $X_2\in\Gamma(\ker f_*)^\perp$.

After a straightforward computation, we obtain $g_1(\nabla_{V_1}V_2,\xi)=0$, for all $V_1,V_2\in\Gamma(D_1)$. Now, for all $V_1,V_2\in\Gamma(D_1)$, $X_1\in\Gamma(D_2)$ and using equations $(2)-(5),(9),(10),(18)$ and (19), we have

$$\begin{aligned}
g_1(\nabla_{V_1}V_2,X_1) &= g_1(\nabla_{V_1}\phi V_2,\phi X_1),\\
&= g_1(\nabla_{V_1}\phi V_2,\psi X_1)+g_1(\nabla_{V_1}\phi V_2,\omega X_1),\\
&= g_1(\mathscr{V}\nabla_{V_1}\phi V_2,\phi X_1)+g_1(\mathscr{T}_{V_1}\phi V_2,\omega X_1).
\end{aligned}$$

Since f is conformal submersions and using equation (6) and Lemma 2, we have

$$\begin{aligned}
g_1(\nabla_{V_1}V_2,X_1) &= g_1(\mathscr{V}\nabla_{V_1}\phi V_2,\phi X_1)+\frac{1}{\lambda^2}g_2(f_*(\mathscr{T}_{V_1}\phi V_2),f_*(\omega X_1)),\\
&= g_1(\mathscr{V}\nabla_{V_1}\phi V_2,\phi X_1)-\frac{1}{\lambda^2}g_2((\nabla f_*)(V_1,\phi V_2),f_*(\omega X_1)).
\end{aligned}$$

On the other hand, for all $V_1,V_2\in\Gamma(D_1)$, $X_2\in\Gamma(\ker f_*)^\perp$ and using equations $(2)-(5)$, (9), (10), (18) and (19), we have

$$\begin{aligned}
g_1(\nabla_{V_1}V_2,X_2) &= g_1(\nabla_{V_1}\phi V_2,\phi X_2),\\
&= g_1(\nabla_{V_1}\phi V_2,BX_2)+g_1(\nabla_{V_1}\phi V_2,CX_2),\\
&= -g_1(\nabla_{V_1}V_2,\phi BX_2)+g_1(\mathscr{T}_{V_1}\phi V_2,CX_2),\\
&= g_1(V_2,\mathscr{V}\nabla_{V_1}\psi BX_2)+g_1(V_2,\mathscr{T}_{V_1}\omega BX_2)+g_1(\mathscr{T}_{V_1}\phi V_2,CX_2).
\end{aligned}$$

Since f is conformal submersions and using equation (6) and Lemma 2, we have

$$\begin{aligned}
&g_1(\nabla_{V_1}V_2,X_2)\\
&= g_1(V_2,\mathscr{V}\nabla_{V_1}\psi BX_2)+g_1(V_2,\mathscr{T}_{V_1}\omega BX_2)+\frac{1}{\lambda^2}g_2(f_*(\mathscr{T}_{V_1}\phi V_2),f_*(CX_2)),\\
&= g_1(V_2,\mathscr{V}\nabla_{V_1}\psi BX_2)+g_1(V_2,\mathscr{T}_{V_1}\omega BX_2)-\frac{1}{\lambda^2}g_2((\nabla f_*)(V_1,\phi V_2),f_*(CX_2)),
\end{aligned}$$

which completes the proof. □

Theorem 5. *Let f be a conformal semi-slant submersion from a Cosymplectic manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Then the distribution D_2 defines totally geodesic foliation on M_1 if and only if*

$$\begin{aligned} & \frac{1}{\lambda^2} g_2((\nabla f_*)(X_1, X_2), f_*(\omega Y_1)) \\ = \;& g_1(\omega\psi X_2, \mathscr{T}_{X_1}\phi Y_1), \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\lambda^2} g_2(\nabla_{\omega X_2} f_*(\omega X_1), f_*(\phi C Y_2)) - \frac{1}{\lambda^2} g_2((\nabla f_*)(X_1, \omega\psi X_2), f_*(Y_2)) \\ = \;& g_1(\mathscr{T}_{X_1}\omega X_2, BY_2) - g_1(\mathscr{A}_{\omega X_2}\psi X_1, \phi C Y_2) - g_1(\omega X_1, \omega X_2) g_1(grad \ln\lambda, \phi C Y_2), \end{aligned}$$

for all $X_1, X_2 \in \Gamma(D_2)$, $Y_1 \in \Gamma(D_1)$ and $Y_2 \in \Gamma(\ker f_*)^\perp$.

Proof. The distribution D_2 defines a totally geodesic foliation on M_1 if and only if $g_1(\nabla_{X_1} X_2, Y_1) = 0$, $g_1(\nabla_{X_1} X_2, Y_2) = 0$ and $g_1(\nabla_{X_1} X_2, \xi) = 0$, for all $X_1, X_2 \in \Gamma(D_2)$, $Y_1 \in \Gamma(D_1)$ and $Y_2 \in \Gamma(\ker f_*)^\perp$.

By using equations $(2)-(5), (9), (18)$, (19) and Lemma 2, we have

$$\begin{aligned} & g_1(\nabla_{X_1} X_2, \phi Y_1) \\ = \;& -g_1(\phi\nabla_{X_1} X_2, Y_1), \\ = \;& -g_1(\nabla_{X_1}\phi X_2, Y_1), \\ = \;& g_1(\omega\psi X_2, \nabla_{X_1}\phi Y_1) + g_1(\nabla_{X_1}\psi^2 X_2, \phi Y_1) + g_1(\omega X_2, \nabla_{X_1} Y_1), \end{aligned}$$

Since f is conformal submersion and using equation (6) and Lemma 2, we have

$$\begin{aligned} & \sin^2\theta g_1(\nabla_{X_1} X_2, \phi Y_1) \\ = \;& g_1(\omega\psi X_2, \nabla_{X_1}\phi Y_1) - \frac{1}{\lambda^2} g_2((\nabla f_*)(X_1, Y_1), f_*(\omega X_2)). \end{aligned}$$

On the other hand using equations $(2)-(5), (9)-(12), (18)$, (19) and Lemma 5, we have

$$\begin{aligned} & g_1(\nabla_{X_1} X_2, Y_2) \\ = \;& g_1(\nabla_{X_1}\psi X_2, \phi Y_2) + g_1(\nabla_{X_1}\omega X_2, \phi Y_2), \\ = \;& -g_1(\nabla_{X_1}\phi\psi X_2, Y_2) + g_1(\nabla_{X_1}\omega X_2, \phi Y_2), \\ = \;& -g_1(\nabla_{X_1}\psi^2 X_2, Y_2) - g_1(\nabla_{X_1}\omega\psi X_2, Y_2) + g_1(\nabla_{X_1}\omega X_2, \phi Y_2), \end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sin^2\theta g_1(\nabla_{X_1}X_2,Y_2)\\
=\ & -g_1(\nabla_{X_1}\omega\psi X_2,\phi Y_2)+g_1(\nabla_{X_1}\omega X_2,BY_2)+g_1(\nabla_{X_1}\omega X_2,CY_2),\\
=\ & g_1(\mathscr{T}_{X_1}\omega X_2,BY_2)-g_1(\mathscr{H}\nabla_{X_1}\omega\psi X_2,Y_2)-g_1(\mathscr{A}_{\omega X_2}\psi X_1,\phi CY_2)-\\
& g_1(\mathscr{H}\nabla_{\omega X_2}\omega X_1,\phi CY_2).
\end{aligned}$$

Since f is conformal submersions and using equation (6) and Lemma 2, we have

$$\begin{aligned}
&\sin^2\theta g_1(\nabla_{X_1}X_2,Y_2)\\
=\ & g_1(\mathscr{T}_{X_1}\omega X_2,BY_2)-g_1(\mathscr{A}_{\omega X_2}\psi X_1,\phi CY_2)-\frac{1}{\lambda^2}g_2(f_*(\mathscr{H}\nabla_{X_1}\omega\psi X_2),f_*(Y_2))-\\
& \frac{1}{\lambda^2}g_2(f_*(\mathscr{H}\nabla_{\omega X_2}\omega X_1),f_*(\phi CY_2)),\\
=\ & g_1(\mathscr{T}_{X_1}\omega X_2,BY_2)-g_1(\mathscr{A}_{\omega X_2}\psi X_1,\phi CY_2)+\frac{1}{\lambda^2}g_2((\nabla f_*)(X_1,\omega\psi X_2),f_*(Y_2))-\\
& \frac{1}{\lambda^2}g_2(\mathscr{H}\nabla_{\omega X_2}f_*(\omega X_1),f_*(\phi CY_2))-g_1(\omega X_1,\omega X_2)g_1(grad\ln\lambda,\phi CY_2),
\end{aligned}$$

which completes the proof. □

From Theorems 4 and 5, we have the following result:

Theorem 6. *Let f be a conformal semi-slant submersion from a Cosymplectic manifold (M_1,ϕ,ξ,η,g_1) onto a Riemannian manifold (M_2,g_2). Then the fibers of f are locally product manifold if and only if*

$$\frac{1}{\lambda^2}g_2((\nabla f_*)(X_1,\phi X_2),f_*(\omega Y_1))=g_1(\mathscr{V}\nabla_{X_1}\phi X_2,\psi Y_1),$$

$$\frac{1}{\lambda^2}g_2((\nabla f_*)(X_1,\phi X_2),f_*(CY_3))=g_1(X_2,\mathscr{V}\nabla_{X_1}\psi BY_3)+g_1(X_2,\mathscr{T}_{X_1}\omega BY_3),$$

and

$$\frac{1}{\lambda^2}g_2((\nabla f_*)(Y_1,Y_2),f_*(\omega X_1))=g_1(\omega\psi Y_2,\mathscr{T}_{Y_1}\phi X_1),$$

$$\begin{aligned}
&\frac{1}{\lambda^2}g_2(\nabla_{\omega Y_2}f_*(\omega Y_1),f_*(\phi CY_3))-\frac{1}{\lambda^2}g_2((\nabla f_*)(Y_1,\omega\psi Y_2),f_*(Y_3))\\
=\ & g_1(\mathscr{T}_{Y_1}\omega Y_2,BY_3)-g_1(\mathscr{A}_{\omega Y_2}\psi Y_1,\phi CY_3)-g_1(\omega Y_1,\omega Y_2)g_1(grad\ln\lambda,\phi CY_3),
\end{aligned}$$

for all $X_1, X_2 \in \Gamma(D_1)$, $Y_1, Y_2 \in \Gamma(D_2)$ *and* $Y_3 \in \Gamma(\ker f_*)^\perp$.

As the distribution $\ker f_*$ is integrable, it is only studied the integrability of the distribution $(\ker f_*)^\perp$ and afterwards we discuss the geometry of leaves of $(\ker f_*)$ and $(\ker f_*)^\perp$.

Theorem 7. *Let f be a conformal semi-slant submersion from a Cosymplectic manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Then any two conditions below imply the third:*

(i) $(\ker f_*)^\perp$ is integrable.

(ii) f is a horizontally homothetic map.

(iii)

$$\begin{aligned}
&\frac{1}{\lambda^2} g_2(\nabla_{X_2} f_*(CX_1) - \nabla_{X_1} f_*(CX_2), f_*(\omega V_1)) \\
= \;& g_1(\mathscr{A}_{X_1} BX_2 - \mathscr{A}_{X_2} BX_1, \omega V_1) + g_1(\mathscr{V}\nabla_{X_1} BX_2 - \mathscr{V}\nabla_{X_2} BX_1 + \mathscr{A}_{X_1} CX_2 - \mathscr{A}_{X_2} CX_1, \psi V_1),
\end{aligned}$$

for all $X_1, X_2 \in \Gamma(\ker f_*)^\perp$ and $V_1 \in \Gamma(\ker f_*)$.

Proof. For all $X_1, X_2 \in \Gamma(\ker f_*)^\perp$ and $V_1 \in \Gamma(\ker f_*)$, from Theorem 3, we have

$$\begin{aligned}
&\frac{1}{\lambda^2} g_2(\nabla_{X_2} f_*(CX_1) - \nabla_{X_1} f_*(CX_2), f_*(\omega V_1)) \\
= \;& g_1(\mathscr{A}_{X_1} BX_2 - \mathscr{A}_{X_2} BX_1, \omega V_1) + g_1(\mathscr{V}\nabla_{X_1} BX_2 - \mathscr{V}\nabla_{X_2} BX_1 + \mathscr{A}_{X_1} CX_2 - \mathscr{A}_{X_2} CX_1, \psi V_1) - \\
& g_1(X_1, \omega V_1) g_1(CX_2, grad \ln \lambda) + g_1(X_2, \omega V_1) g_1(CX_1, grad \ln \lambda) \\
& + 2 g_1(X_1, CX_2) g_1(grad \ln \lambda, \omega V_1).
\end{aligned}$$

Now, if using (i) and (ii), we get (iii)

$$\begin{aligned}
&\frac{1}{\lambda^2} g_2(\nabla_{X_2} f_*(CX_1) - \nabla_{X_1} f_*(CX_2), f_*(\omega V_1)) \\
= \;& g_1(\mathscr{A}_{X_1} BX_2 - \mathscr{A}_{X_2} BX_1, \omega V_1) + g_1(\mathscr{V}\nabla_{X_1} BX_2 - \mathscr{V}\nabla_{X_2} BX_1 + \mathscr{A}_{X_1} CX_2 - \mathscr{A}_{X_2} CX_1, \psi V_1).
\end{aligned}$$

□

Theorem 8. *Let f be conformal semi-slant submersion from a Cosymplectic manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Then the distributions $(\ker f_*)^\perp$ defines a totally geodesic* *foliation on* M_1 *if and only if*

$$\mathscr{A}_{X_1}CX_2 + \mathscr{V}\nabla_{X_1}BX_2 \in \Gamma(D_2),$$

and

$$\begin{aligned}&\frac{1}{\lambda^2}\{g_2(\nabla_{X_1} f_*(X_2), \omega\phi Y_1) - g_2(\nabla_{X_1} f_*(CX_2), f_*(\omega Y_2))\}\\ = \quad & g_1(A_{X_1}BX_2, \omega Y_1) + g_1(X_1, grad \ln\lambda)g_1(X_2, \omega\phi Y_1) +\\ & g_1(X_2, grad \ln\lambda)g_1(X_1, \omega\phi Y_1) - g_1(X_1, X_2)g_1(grad \ln\lambda, \omega\phi Y_1) -\\ & g_1(CX_2, grad \ln\lambda)g_1(X_1, \omega Y_1) + g_1(X_1, CX_2)g_1(grad \ln\lambda, \omega Y_1),\end{aligned}$$

for all $X_1, X_2 \in \Gamma(\ker f_*)^\perp, Y_2 \in \Gamma(D_1)$ *and* $Y_1 \in \Gamma(D_2)$.

Proof. The distribution $\Gamma(\ker f_*)^\perp$ defines a totally geodesic foliation on M_1 if and only if $g_1(\nabla_{X_1}X_2, Y_2) = 0$, $g_1(\nabla_{X_1}X_2, Y_1) = 0$ and $g_1(\nabla_{X_1}X_2, \xi) = 0$, for all $X_1, X_2 \in \Gamma(\ker f_*)^\perp, Y_2 \in \Gamma(D_1)$ and $Y_1 \in \Gamma(D_2)$.

Let $X_1, X_2 \in \Gamma(\ker f_*)^\perp$. Now using equation (5), We have

$$g_1(\nabla_{X_1}X_2, \xi) \quad = \quad -g_1(X_2, \nabla_{X_1}\xi) = 0.$$

By using equations $(2)-(5), (11), (12), (18)$ and (19), we have

$$\begin{aligned}& g_1(\nabla_{X_1}X_2, Y_2)\\ = \quad & g_1(\nabla_{X_1}\phi X_2, \phi Y_2),\\ = \quad & g_1(\mathscr{V}\nabla_{X_1}BX_2 + \mathscr{A}_{X_1}CX_2, \phi Y_2),\\ = \quad & -g_1(\phi(\mathscr{V}\nabla_{X_1}BX_2 + \mathscr{A}_{X_1}CX_2), Y_2).\end{aligned}$$

On the other hand, using equations $(2)-(5), (11), (12), (18)$, (19) and Lemma 5, we have

$$\begin{aligned}& g_1(\nabla_{X_1}X_2, Y_1)\\ = \quad & g_1(\nabla_{X_1}\phi X_2, \phi Y_1),\\ = \quad & -g_1(\nabla_{X_1}X_2, \psi^2 Y_1) - g_1(\nabla_{X_1}X_2, \omega\psi Y_1) + g_1(\nabla_{X_1}BX_2, \omega Y_1) + g_1(\nabla_{X_1}CX_2, \omega Y_1),\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\sin^2\theta g_1(\nabla_{X_1}X_2,Y_1)\\ =\;& g_1(\mathscr{A}_{X_1}BX_2,\omega Y_1)-g_1(\mathscr{H}\nabla_{X_1}X_2,\omega\psi Y_1)+g_1(\mathscr{H}\nabla_{X_1}CX_2,\omega Y_1).\end{aligned}$$

Since f is conformal submersion and using equation (6) and Lemma 2, we have

$$\begin{aligned}&\sin^2\theta g_1(\nabla_{X_1}X_2,Y_1)\\ =\;& g_1(\mathscr{A}_{X_1}BX_2,\omega Y_1)+\frac{1}{\lambda^2}g_2(f_*(\mathscr{H}\nabla_{X_1}X_2),f_*(\omega\psi Y_1))-\frac{1}{\lambda^2}g_2(f_*(\mathscr{H}\nabla_{X_1}CX_2),f_*(\omega Y_1)),\\ =\;& g_1(\mathscr{A}_{X_1}BX_2,\omega Y_1)-\frac{1}{\lambda^2}g_2(\mathscr{H}\nabla_{X_1}f_*(X_2),f_*(\omega\psi Y_1))+\frac{1}{\lambda^2}g_2(\mathscr{H}\nabla_{X_1}f_*(CX_2),f_*(\omega Y_1))+\\ &\frac{1}{\lambda^2}g_2(X_1(\ln\lambda)f_*(X_2)+X_2(\ln\lambda)f_*(X_1)-g_1(X_1,X_2)f_*(grad\ln\lambda),f_*(\omega\psi Y_1))-\\ &\frac{1}{\lambda^2}g_2(X_1(\ln\lambda)f_*(CX_2)+CX_2(\ln\lambda)f_*(X_1)-g_1(X_1,CX_2)f_*(grad\ln\lambda),f_*(\omega Y_1)),\\ =\;& g_1(\mathscr{A}_{X_1}BX_2,\omega Y_1)-\frac{1}{\lambda^2}g_2(\mathscr{H}\nabla_{X_1}f_*(X_2),f_*(\omega\psi Y_1))+\frac{1}{\lambda^2}g_2(\mathscr{H}\nabla_{X_1}f_*(CX_2),f_*(\omega Y_1))+\\ &g_1(X_1,grad\ln\lambda)g_1(X_2,\omega\psi Y_1)+g_1(X_2,grad\ln\lambda)g_1(X_1,\omega\psi Y_1)\\ &-g_1(X_1,X_2)g_1(grad\ln\lambda,\omega\psi Y_1)-g_1(CX_2,grad\ln\lambda)g_1(X_1,\omega Y_1)\\ &+g_1(X_1,CX_2)g_1(grad\ln\lambda,\omega Y_1),\end{aligned}$$

which completes the proof. □

Theorem 9. *Let f be conformal semi-slant submersion from a Cosymplectic manifold (M_1,ϕ,ξ,η,g_1) onto a Riemannian manifold (M_2,g_2). Then the distributions $(\ker f_*)$ defines a totally geodesic foliation on M_1 if and only if*

$$\begin{aligned}&\frac{1}{\lambda^2}g_2(\nabla_{\omega X_2}f_*(\phi CV_1),f_*(\omega X_1))\\ =\;& -g_1(\omega(\mathscr{V}\nabla_{X_1}\psi X_2+\mathscr{T}_{X_1}\omega X_2),V_1)+g_1(\mathscr{T}_{X_1}\psi X_2,CV_1)-\\ &g_1(\mathscr{A}_{\omega X_2}\phi CV_1,\psi X_1)+g_1(\omega X_1,\omega X_2)(\phi CV_1,grad\ln\lambda),\end{aligned}$$

for all $X_1,X_2\in(\ker f_*)$ and $V_1\in(\ker f_*)^{\perp}$.

Proof. For all $X_1, X_2 \in (\ker f_*)$ and $V_1 \in (\ker f_*)^{\perp}$, using equations $(2)-(5),(9),(10),(18)$ and (19), we have

$$\begin{aligned}
& g_1(\nabla_{X_1}X_2, V_1) \\
= \; & g_1(\nabla_{X_1}\phi X_2, \phi V_1), \\
= \; & g_1(\mathscr{V}\nabla_{X_1}\psi X_2, BV_1) + g_1(\mathscr{T}_{X_1}\omega X_2, BV_1) + \\
& g_1(\nabla_{X_1}\psi X_2, CV_1) + g_1(\nabla_{X_1}\omega X_2, CV_1), \\
= \; & -g_1(\omega(\mathscr{V}\nabla_{X_1}\psi X_2 + \mathscr{T}_{X_1}\omega X_2), V_1) + g_1(\mathscr{T}_{X_1}\psi X_2, CV_1) \\
& -g_1(\nabla_{\omega X_2}\phi CV_1, \psi X_1), \\
= \; & -g_1(\omega(\mathscr{V}\nabla_{X_1}\psi X_2 + \mathscr{T}_{X_1}\omega X_2), V_1) + g_1(\mathscr{T}_{X_1}\psi X_2, CV_1) - \\
& g_1(\nabla_{\omega X_2}\phi CV_1, \psi X_1) - g_1(\nabla_{\omega X_2}\phi CV_1, \omega X_2),
\end{aligned}$$

Since f is conformal submersion and using equation (6) and Lemma 2, we have

$$\begin{aligned}
& g_1(\nabla_{X_1}X_2, V_1) \\
= \; & -g_1(\omega(\mathscr{V}\nabla_{X_1}\psi X_2 + \mathscr{T}_{X_1}\omega X_2), V_1) + g_1(\mathscr{T}_{X_1}\psi X_2, CV_1) - \\
& g_1(\mathscr{A}_{\omega X_2}\phi CV_1, \psi X_1) - \frac{1}{\lambda^2}g_2(\nabla_{\omega X_2}f_*(\phi CV_1), f_*(\omega X_1)) + \\
& \frac{1}{\lambda^2}g_2((\nabla f_*)(\omega X_2, \phi CV_1), f_*(\omega X_1)),
\end{aligned}$$

$$\begin{aligned}
= \; & -g_1(\omega(\mathscr{V}\nabla_{X_1}\psi X_2 + \mathscr{T}_{X_1}\omega X_2), V_1) + g_1(\mathscr{T}_{X_1}\psi X_2, CV_1) - \\
& g_1(\mathscr{A}_{\omega X_2}\phi CV_1, \psi X_1) - \frac{1}{\lambda^2}g_2(\nabla_{\omega X_2}f_*(\phi CV_1), f_*(\omega X_1)) + \\
& g_1(\omega X_1, \omega X_2)(\phi CV_1, grad \ln \lambda),
\end{aligned}$$

which completes the proof. □

Theorem 10. *Let f be a conformal semi-slant submersion from a Cosymplectic manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Then any two conditions below imply the third:*

(i) $(\ker f_*)$ defines a totally geodesic foliation on M_1.

(ii) λ is a constant on $\Gamma(\mu)$.

(iii)

$$\begin{aligned} & \frac{1}{\lambda^2} g_2(\nabla_{\omega X_2} f_*(\phi C V_1), f_*(\omega X_1)) \\ = \quad & -g_1(\omega(\mathscr{V}\nabla_{X_1}\psi X_2 + \mathscr{T}_{X_1}\omega X_2), V_1) + g_1(\mathscr{T}_{X_1}\psi X_2, CV_1) - \\ & g_1(\mathscr{A}_{\omega X_2}\phi C V_1, \psi X_1), \end{aligned}$$

for all $X_1, X_2 \in \Gamma(\ker f_*)$ and $V_1 \in \Gamma(\ker f_*)^\perp$.

Proof. From Theorem 9, we have

$$\begin{aligned} & \frac{1}{\lambda^2} g_2(\nabla_{\omega X_2} f_*(\phi C V_1), f_*(\omega X_1)) \\ = \quad & -g_1(\omega(\mathscr{V}\nabla_{X_1}\psi X_2 + \mathscr{T}_{X_1}\omega X_2), V_1) + g_1(\mathscr{T}_{X_1}\psi X_2, CV_1) - \\ & g_1(\mathscr{A}_{\omega X_2}\phi C V_1, \psi X_1) + g_1(\omega X_1, \omega X_2)(\phi C V_1, grad \ln \lambda), \end{aligned}$$

for all $X_1, X_2 \in \Gamma(\ker f_*)$ and $V_1 \in \Gamma(\ker f_*)^\perp$.

Now, if we have (i) and (iii), then we obtain

$$g_1(\omega X_1, \omega X_2) g_1(H grad \ln \lambda, \phi C V_1) = 0.$$

From above equation, λ is a constant on $\Gamma(\mu)$. Similarly, one can obtain the assertions. □

4. TOTALLY GEODESICNESS OF THE CONFORMAL SEMI-SLANT SUBMERSIONS

At this part, we shall examine the totally geodesicness of a conformal semi-slant submersion. First, we give necessary and sufficient conditions for a conformal semi-slant submersion to be totally geodesic map. Remember that a smooth map f between two Riemannian manifolds is called totally geodesic if $\nabla f_* = 0$.

Theorem 11. *Let f be conformal semi-slant submersion from a Cosymplectic manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Then, f is a totally geodesic map if*

$$\begin{aligned} \nabla_{X_1} f_*(X_2) \quad = \quad & -f_*(C(\mathscr{A}_{X_1}\psi Z_1 + \mathscr{H}\nabla_{X_1}\omega Z_1 + \mathscr{A}_{X_1}\psi Z_2 + \mathscr{H}\nabla_{X_1}\omega Z_2) + \\ & \omega(\mathscr{V}\nabla_{X_1}\psi Z_1 + \mathscr{A}_{X_1}\omega Z_1 + \mathscr{V}\nabla_{X_1}\psi Z_2 + \mathscr{A}_{X_1}\omega Z_2)), \end{aligned}$$

for all $X_1 \in \Gamma(\ker f_*)^\perp$ *and* $X_2 = Z_1 + Z_2$, *where* $Z_1 \in \Gamma(\ker f_*)$ *and* $Z_2 \in \Gamma(\ker f_*)^\perp$.

Proof. From equation (6), we have

$$(\nabla f_*)(X_1, X_2) = \nabla_{X_1} f_*(X_2) - f_*(\nabla_{X_1} X_2),$$

for all $X_1 \in \Gamma(\ker f_*)^\perp$ and $X_2 = Z_1 + Z_2$, where $Z_1 \in \Gamma(\ker f_*)$ and $Z_2 \in \Gamma(\ker f_*)^\perp$.

Using equations $(1)-(5), (9)-(12), (18)$ and (19), we have

$$\begin{aligned}
&(\nabla f_*)(X_1, X_2) \\
= \;& \nabla_{X_1} f_*(X_2) + f_*(\phi^2 \nabla_{X_1} X_2 - \eta(\nabla_{X_1} X_2)\xi), \\
= \;& \nabla_{X_1} f_*(X_2) + f_*(\phi \nabla_{X_1} \phi X_2), \\
= \;& \nabla_{X_1} f_*(X_2) + f_*(\phi \nabla_{X_1} \phi Z_1 + \phi \nabla_{X_1} Z_2), \\
= \;& \nabla_{X_1} f_*(X_2) + f_*(\phi \nabla_{X_1} \phi Z_1 + \phi \nabla_{X_1} Z_2), \\
= \;& \nabla_{X_1} f_*(X_2) + f_*(B\mathscr{A}_{X_1} \psi Z_1 + C\mathscr{A}_{X_1} \psi Z_1 + \psi \mathscr{V} \nabla_{X_1} \psi Z_1 + \omega \mathscr{V} \nabla_{X_1} \psi Z_1 + \\
& B\mathscr{H} \nabla_{X_1} \omega Z_1 + C\mathscr{H} \nabla_{X_1} \omega Z_1 + \psi \mathscr{A}_{X_1} \omega Z_1 + \omega \mathscr{A}_{X_1} \omega Z_1 + \\
& B\mathscr{A}_{X_1} \psi Z_2 + C\mathscr{A}_{X_1} \psi Z_2 + \psi \mathscr{V} \nabla_{X_1} \psi Z_2 + \omega \mathscr{V} \nabla_{X_1} \psi Z_2 + \\
& + B\mathscr{H} \nabla_{X_1} \omega Z_2 + C\mathscr{H} \nabla_{X_1} \omega Z_2 + \psi \mathscr{A}_{X_1} \omega Z_2 + \omega \mathscr{A}_{X_1} \omega Z_2),
\end{aligned}$$

Thus taking into account the vertical parts, we get

$$\begin{aligned}
(\nabla f_*)(X_1, X_2) \;=\;& \nabla_{X_1} f_*(X_2) + f_*(C(\mathscr{A}_{X_1} \psi Z_1 + \mathscr{H} \nabla_{X_1} \omega Z_1 + \mathscr{A}_{X_1} \psi Z_2 + \mathscr{H} \nabla_{X_1} \omega Z_2) + \\
& \omega(\mathscr{V} \nabla_{X_1} \psi Z_1 + \mathscr{A}_{X_1} \omega Z_1 + \mathscr{V} \nabla_{X_1} \psi Z_2 + \mathscr{A}_{X_1} \omega Z_2)),
\end{aligned}$$

which gives our assertion. □

Theorem 12. *Let* f *be conformal semi-slant submersion from a Cosymplectic manifold* $(M_1, \phi, \xi, \eta, g_1)$ *onto a Riemannian manifold* (M_2, g_2). *Then* f *is a totally geodesic map if and only if*

(i)

$$\begin{aligned}&\frac{1}{\lambda^2}\{g_2((\nabla f_*)(X,\omega\psi Y),f_*(Z))-g_2(\nabla_{\omega Y}f_*(\omega X),f_*(\phi CZ))\}\\=&-g_1(\mathscr{T}_X\omega Y,BZ)+g_1(\mathscr{A}_{\omega Y}\psi X,\phi CZ)+g_1(\omega Y,\omega X)g_1(grad\ln\lambda,\phi CZ),\end{aligned}$$

(ii)

$$g_1(\mathscr{V}\nabla_{X_1}\phi X_2,BZ)=-g_1(\mathscr{T}_{X_1}\phi X_2,CZ),$$

(iii)

$$\begin{aligned}&g_1(\mathscr{T}_{Y_1}\psi BZ,V)-g_1(\mathscr{T}_{Y_1}CZ,BV)=\frac{1}{\lambda^2}g_2((\nabla f_*)(Y_1,\omega BZ),f_*(V))\\&-\frac{1}{\lambda^2}g_2((\nabla f_*)(Y_1,CZ),f_*(CV))\end{aligned}$$

(iv) f is a horizontally homothetic map,

for all $X_1,X_2\in\Gamma(D_1)$, $X,Y\in\Gamma(D_2),Y_1\in\Gamma(\ker f_*)$ and $Z,V\in\Gamma(\ker f_*)^{\perp}$.

Proof. (i) From equation (6), we have

$$\frac{1}{\lambda^2}g_2((\nabla f_*)(X,Y),f_*(Z))=-g_1(\nabla_XY,Z),$$

for all $X,Y\in\Gamma(D_2)$ and $Z\in\Gamma(\ker f_*)^{\perp}$.

Using equations $(1)-(5),(10),(18)$, (19) and Lemma 2, we have

$$\begin{aligned}&\frac{1}{\lambda^2}g_2((\nabla f_*)(X,Y),f_*(Z))\\=&-g_1(\nabla_X\phi Y,\phi Z),\\=&\ g_1(\nabla_X\psi^2Y,Z)+g_1(\nabla_X\omega\psi Y,Z)-g_1(\nabla_X\omega Y,BZ)-g_1(\nabla_X\omega Y,CZ),\\=&-\cos^2\theta g_1(\nabla_XY,Z)+g_1(\mathscr{H}\nabla_X\omega\psi Y,Z)-g_1(\mathscr{T}_X\omega Y,BZ)+\\&g_1(\nabla_{\omega Y}\psi X,\phi CZ)+g_1(\nabla_{\omega Y}\omega X,\phi CZ).\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\sin^2\theta\frac{1}{\lambda^2}g_2((\nabla f_*)(X,Y),f_*(Z))\\
=\ & g_1(\mathscr{H}\nabla_{\omega X}\psi Y,Z)-g_1(\mathscr{T}_X\omega Y,BZ)+g_1(\mathscr{A}_{\omega Y}\psi X,\phi CZ)\\
&+g_1(\mathscr{H}\nabla_{\omega Y}\omega X,\phi CZ),\\
=\ & -g_1(\mathscr{T}_X\omega Y,BZ)+g_1(\mathscr{A}_{\omega Y}\psi X,\phi CZ)-\frac{1}{\lambda^2}g_2((\nabla f_*)(X,\omega\psi Y),f_*(Z))+\\
&\frac{1}{\lambda^2}g_2(\nabla_{\omega Y}f_*(\omega X),f_*(\phi CZ))+g_1(\omega Y,\omega X)g_1(grad\ln\lambda,\phi CZ).
\end{aligned}$$

(ii) From equation (2.6), we have

$$\frac{1}{\lambda^2}g_2((\nabla f_*)(X_1,X_2),f_*(Z))=-g_1(\nabla_{X_1}X_2,Z),$$

for all $X_1,X_2\in\Gamma(D_1)$ and $Z\in\Gamma(\ker f_*)^\perp$.

Using equations $(1)-(5),(9)$ and (19), we have

$$\begin{aligned}
&\frac{1}{\lambda^2}g_2((\nabla f_*)(X_1,X_2),f_*(Z))\\
=\ & -g_1(\mathscr{V}\nabla_{X_1}\phi X_2,BZ)-g_1(\mathscr{T}_{X_1}\phi X_2,CZ).
\end{aligned}$$

(iii) From equation (6), we have

$$\frac{1}{\lambda^2}g_2((\nabla f_*)(Y_1,Z),f_*(V))=-g_1(\nabla_{Y_1}Z,V),$$

for all $Z,V\in\Gamma(\ker f_*)^\perp$ and $Y_1\in\Gamma(\ker f_*)$.

Using equations $(2)-(5),(9),(10)$ and Lemma 2, we get

$$\begin{aligned}
&\frac{1}{\lambda^2}g_2((\nabla f_*)(Y_1,Z),f_*(V))\\
=\ & -g_1(\nabla_{Y_1}\phi Z,\phi V),\\
=\ & g_1(\phi\nabla_{Y_1}BZ,V)-g_1(\nabla_{Y_1}CZ,BV)-g_1(\nabla_{Y_1}CZ,CV),\\
=\ & g_1(\mathscr{T}_{Y_1}\psi BZ,V)+g_1(\mathscr{H}\nabla_{Y_1}\omega BZ,V)-g_1(\mathscr{T}_{Y_1}CZ,BV)-g_1(\mathscr{H}\nabla_{Y_1}CZ,CV),\\
=\ & g_1(\mathscr{T}_{Y_1}\psi BZ,V)-g_1(\mathscr{T}_{Y_1}CZ,BV)-\frac{1}{\lambda^2}g_2((\nabla f_*)(Y_1,\omega BZ),f_*(V))\\
&+\frac{1}{\lambda^2}g_2((\nabla f_*)(Y_1,CZ),f_*(CV)).
\end{aligned}$$

(iv) For $X_1, X_2 \in \Gamma(\mu)$, from Lemma 2, we get

$$(\nabla f_*)(X_1,X_2) = X_1(\ln\lambda)f_*(X_2) + X_2(\ln\lambda)f_*(X_1) - g_1(X_1,X_2)f_*(grad\ln\lambda).$$

The above equation taking $X_2 = \phi X_1$, we get

$$\begin{aligned}(\nabla f_*)(X_1,\phi X_1) &= X_1(\ln\lambda)f_*(\phi X_1) + \phi X_1(\ln\lambda)f_*(X_1) - g_1(X_1,\phi X_1)f_*(grad\ln\lambda),\\ &= X_1(\ln\lambda)f_*(\phi X_1) + \phi X_1(\ln\lambda)f_*(X_1).\end{aligned}$$

If $(\nabla f_*)(X_1,\phi X_1) = 0$, then we have

$$X_1(\ln\lambda)f_*(\phi X_1) + \phi X_1(\ln\lambda)f_*(X_1) = 0. \tag{32}$$

Taking inner product in equation (32) with $f_*(\phi X_1)$, we get

$$g_1(grad\ln\lambda,X_1)g_1(f_*(\phi X_1),f_*(\phi X_1)) + g_1(grad\ln\lambda,\phi X_1)g_1(f_*X_1,f_*\phi X_1) = 0.$$

From above equation, it follows that λ is a constant on $\Gamma(\mu)$.
In a similar way, for $Z_1, Z_2 \in \Gamma(\ker f_*)$, using Lemma 2, we get

$$(\nabla f_*)(\omega Z_1,\omega Z_2) = \omega Z_1(\ln\lambda)f_*(\omega Z_2) + \omega Z_2(\ln\lambda)f_*(\omega Z_1) - g_1(\omega Z_1,\omega Z_2)f_*(grad\ln\lambda).$$

From above equation, taking $Z_2 = Z_1$, we obtain

$$\begin{aligned}(\nabla f_*)(\omega Z_1,\omega Z_1) &= \omega Z_1(\ln\lambda)f_*(\omega Z_1) + \omega Z_1(\ln\lambda)f_*(\omega Z_1)\\ &\quad - g_1(\omega Z_1,\omega Z_1)f_*(grad\ln\lambda),\\ &= 2\omega Z_1(\ln\lambda)f_*(\omega Z_1) - g_1(\omega Z_1,\omega Z_1)f_*(grad\ln\lambda). \qquad (33)\end{aligned}$$

Taking inner product in (33) with $f_*(\omega Z_1)$ and since f is conformal submersion, we get

$$2g_1(\omega Z_1,grad\ln\lambda)g_2(f_*(\omega Z_1),f_*(\omega Z_1)) - g_1(\omega Z_1,\omega Z_1)g_2(f_*(\omega Z_1),f_*(grad\ln\lambda)) = 0.$$

From above equation it follows that λ is a constant on $\Gamma\omega(\ker f_*)$. So λ is a constant on $\Gamma(\ker f_*)^\perp$. On the other hand if f is a horizontally homothetic map it is obvious that $(\nabla f_*)(Z_1,Z_2) = 0$. Thus proof is complete. □

Definition 2. *Let f be a conformal semi-slant submersion from a Cosymplectic manifold (M_1,ϕ,ξ,η,g_1) onto Riemannian manifold (M_2,g_2). Then f is called a $(\omega D_2,\mu)-$totally geodesic map if*

$$(\nabla f_*)(\omega X_1,X_2)=0,$$

for all $X_1\in\Gamma(D_2)$ and $X_2\in\Gamma(\mu)$.

Theorem 13. *Let f be conformal semi-slant submersion from a Cosymplectic manifold (M_1,ϕ,ξ,η,g_1) onto a Riemannian manifold (M_2,g_2). Then f is called a $(\omega D_2,\mu)-$totally geodesic map if and only if f is horizontally homothetic map.*

Proof. For all $X_1\in\Gamma(D_2)$ and $X_2\in\Gamma(\mu)$, by Lemma 2, we have

$$(\nabla f_*)(\omega X_1,X_2)=\omega X_1(\ln\lambda)f_*(X_2)+X_2(\ln\lambda)f_*(\omega X_1)-g_1(\omega X_1,X_2)f_*(grad\ln\lambda).$$

If f is a horizontally homothetic, then

$$(\nabla f_*)(\omega X_1,X_2)=0$$

Conversely if $(\nabla f_*)(\omega X_1,X_2)=0$, we have

$$\omega X_1(\ln\lambda)f_*(X_2)+X_2(\ln\lambda)f_*(\omega X_1)=0. \tag{34}$$

Since f is conformal submersion and taking inner product in equation (34) with $f_*(\omega X_1)$, we have

$$g_1(\omega X_1,grad\ln\lambda)g_2(f_*(X_2),f_*(\omega X_1))+g_1(X_2,grad\ln\lambda)g_2(f_*(\omega X_1),f_*(\omega X_1))=0.$$

Above equation implies that λ is a constant on $\Gamma(\mu)$. Besides holding inner product in equation (34), with f_*X_2, we get

$$g_1(grad\ln\lambda,\omega X_1)g_2(f_*X_2,f_*X_2)+g_1(grad\ln\lambda,X_2)g_2(f_*\omega X_1,f_*X_2)=0.$$

From above equation it follows that λ is constant on $\Gamma(\omega D_2)$.

Thus λ is constant on $(\Gamma(\ker f_*)^\perp)$. Therefore proof is complete. □

Theorem 14. *Let f be conformal semi-slant submersion from a Cosymplectic manifold $(M_1, \phi, \xi, \eta, g_1)$ onto a Riemannian manifold (M_2, g_2). Then f is a totally geodesic map if and only if*

(a) $\omega\mathscr{V}\nabla_{X_1}\phi X_2 + C\mathscr{T}_{X_1}\phi X_2 = 0$, for all $X_1, X_2 \in \Gamma(D_1)$,

(b) $C(\mathscr{T}_{X_1}\psi Y_1 + \mathscr{A}_{\omega Y_1}X_1) + \omega(\mathscr{V}\nabla_{X_1}\psi Y_1 + \mathscr{T}_{X_1}\omega Y_1) = 0$, for all $X_1 \in \Gamma(D_1)$ and $Y_1 \in \Gamma(D_2)$,

(c) $C(\mathscr{T}_{Z_1}BZ_2 + \mathscr{A}_{Z_1}CZ_2) + \omega(\mathscr{V}\nabla_{Z_1}BZ_2 + \mathscr{T}_{Z_1}CZ_2) = 0$, for all $Z_1 \in \Gamma(\ker f_*)$ and $Z_2 \in \Gamma(\ker f_*)^{\perp}$.

Proof. (a) From equation (6), we have

$$(\nabla f_*)(X_1, X_2) = -f_*(\nabla_{X_1}X_2),$$

for all $X_1, X_2 \in \Gamma(D_1)$.

Using equations $(1), (2), (9), (10), (18)$ and (19), we have

$$\begin{aligned} & (\nabla f_*)(X_1, X_2) \\ = \;& f_*(\phi\nabla_{X_1}\phi X_2), \\ = \;& f_*(\psi\mathscr{V}\nabla_{X_1}\phi X_2 + \omega\mathscr{V}\nabla_{X_1}\phi X_2 + B\mathscr{T}_{X_1}\phi X_2 + C\mathscr{T}_{X_1}\phi X_2). \end{aligned}$$

Since $\psi\mathscr{V}\nabla_{X_1}\phi X_2 + B\mathscr{T}_{X_1}\phi X_2 \in \Gamma(\ker f_*)$, we get

$$(\nabla f_*)(X_1, X_2) = f_*(\omega\mathscr{V}\nabla_{X_1}\phi X_2 + C\mathscr{T}_{X_1}\phi X_2).$$

Thus, since f is a linear isomorphism between $(\ker f_*)^{\perp}$ and TM_1, $(\nabla f_*)(X_1, X_2) = 0 \Leftrightarrow \omega\mathscr{V}\nabla_{X_1}\phi X_2 + C\mathscr{T}_{X_1}\phi X_2 = 0$.

(b) For $X_1 \in \Gamma(D_1)$, $Y_1 \in \Gamma(D_2)$, from equations $(1) - (5)$ and (18), we get

$$(\nabla f_*)(X_1, Y_1) = f_*(\phi\nabla_{X_1}\psi Y_1 + \phi\nabla_{X_1}\omega Y_1).$$

Again using equations $(9) - (12), (18)$ and (19), we get

$$\begin{aligned} (\nabla f_*)(X_1, Y_1) \;=\; & f_*(B\mathscr{T}_{X_1}\psi Y_1 + C\mathscr{T}_{X_1}\psi Y_1 + \psi\mathscr{V}\nabla_{X_1}\psi Y_1 + \omega\mathscr{V}\nabla_{X_1}\psi Y_1 \\ & + B\mathscr{A}_{\omega Y_1}X_1 + C\mathscr{A}_{\omega Y_1}X_1 + \psi\mathscr{T}_{X_1}\omega Y_1 + \omega\mathscr{T}_{X_1}\omega Y_1). \end{aligned}$$

Since $B\mathscr{T}_{X_1}\psi Y_1 + \psi\mathscr{V}\nabla_{X_1}\psi Y_1 + B\mathscr{A}_{\omega Y_1}X_1 + \psi\mathscr{T}_{X_1}\omega Y_1 \in \Gamma(\ker f_*)$, we have

$$\begin{aligned} & (\nabla f_*)(X_1,Y_1) \\ = \;& f_*(C(\mathscr{T}_{X_1}\psi Y_1 + \mathscr{A}_{\omega Y_1}X_1) + \omega(\mathscr{V}\nabla_{X_1}\psi Y_1 + \mathscr{T}_{X_1}\omega Y_1)). \end{aligned}$$

Since f is a linear isomorphism between $(\ker f_*)^\perp$ and TM_2, $(\nabla f_*)(X_1,Y_1) = 0 \Leftrightarrow C(\mathscr{T}_{X_1}\psi Y_1 + \mathscr{A}_{\omega Y_1}X_1) + \omega(\mathscr{V}\nabla_{X_1}\psi Y_1 + \mathscr{T}_{X_1}\omega Y_1) = 0$.

(c) For $Z_1 \in \Gamma(\ker f_*)$ and $Z_2 \in \Gamma(\ker f_*)^\perp$, from equation $(1)-(5)$ and (19), we obtain

$$(\nabla f_*)(Z_1,Z_2) = f_*(\phi(\nabla_{Z_1}BZ_2 + \nabla_{Z_1}CZ_2)).$$

Using equations $(9)-(12),(18)$ and (19), we have

$$\begin{aligned} & (\nabla f_*)(Z_1,Z_2) \\ = \;& f_*(B\mathscr{T}_{Z_1}BZ_2 + C\mathscr{T}_{Z_1}BZ_2 + \psi\mathscr{V}\nabla_{Z_1}BZ_2 + \omega\mathscr{V}\nabla_{Z_1}BZ_2 \\ & + \psi\mathscr{T}_{Z_1}CZ_2 + \omega\mathscr{T}_{Z_1}CZ_2 + B\mathscr{A}_{Z_1}CZ_2 + C\mathscr{A}_{Z_1}CZ_2). \end{aligned}$$

Since $B\mathscr{T}_{Z_1}BZ_2 + \psi\mathscr{V}\nabla_{Z_1}BZ_2 + \psi\mathscr{T}_{Z_1}CZ_2 + B\mathscr{A}_{Z_1}CZ_2 \in \Gamma(\ker f_*)^\perp$, we have

$$\begin{aligned} & (\nabla f_*)(Z_1,Z_2) \\ = \;& f_*(C(\mathscr{T}_{Z_1}BZ_2 + \mathscr{A}_{Z_1}CZ_2) + \omega(\mathscr{V}\nabla_{Z_1}BZ_2 + \mathscr{T}_{Z_1}CZ_2)). \end{aligned}$$

Since f is a linear isomorphism between $(\ker f_*)^\perp$ and TM_2, $(\nabla f_*)(Z_1,Z_2) = 0 \Leftrightarrow C(\mathscr{T}_{Z_1}BZ_2 + \mathscr{A}_{Z_1}CZ_2) + \omega(\mathscr{V}\nabla_{Z_1}BZ_2 + \mathscr{T}_{Z_1}CZ_2) = 0$.

Therefore proof is complete. □

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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