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DYNAMICS OF A NUTRIENT-PLANKTON MODEL WITH DELAY AND TOXICITY

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Abstract. In this paper, a dynamical system exploring the nutrient-plankton interaction has been studied to determine the impact of toxin liberation delay. It is well known that the toxins liberated by phytoplankton species are harmful to the growth and the life cycle of zooplankton species. Moreover, the process of toxin liberation is not immediate, but it follows some time delay. We have observed many significant features of the given model system like boundedness, positivity, Hopf-bifurcation and its direction, etc. From the analysis of this model, it is observed that the toxin liberation delay can include complexity in the system as time delay passes through its critical value. All analytical results are verified through numerical simulations, and some significant findings are interpreted from the ecological point of view.

Keywords: plankton; toxin liberation delay; Hopf-bifurcation; normal form theory; centre manifold theorem.

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1. INTRODUCTION

Ecology is the branch of science, which studies the pattern of the interrelationship of different organisms and their interaction with the environment. The primary concern of ecological research is the deep study of those parameters or factors which influence the interactivity of organisms and their relationship with the environment to get results that help to the endurable progress of the ecosystem. The study of the plankton ecosystem is significant for ecological balance because 71 percent of our biosphere is covered with water. Plankton species (phytoplankton, zooplankton) are the crucial link in the pelagic food chain, generate about 80 percent oxygen of earth's atmosphere and biological wealth for whole water bodies.

The appearance and disappearance in the concentration of the phytoplankton population are known as bloom. The blooms produced by toxin-producing phytoplankton (TPP) are known as Harmful algal blooms (HABs). The HABs are the crucial ecological problem, as it can cause serious harm to see organisms, environments, and economies. Moreover, the toxicant produced by TPP is also very dangerous as this toxicant may regulate the growth of its predator population. Thus, it is fascinating to study the effects of toxicants on biological organisms.

Researchers in [1, 2] have proposed the plankton nutrient interaction models to study the relation and importance of nutrients in the growth and life cycle of plankton species. In [3, 4], Raun et al. have analyzed the complexity of the nutrient-plankton system when the factor of toxicity is also present in the marine ecosystem. [25, 26, 27, 28] have developed some mathematical models involving nutrients and plankton population to study the stability and instability of these systems. Different plankton dynamics have been presented to investigate the effects of toxin and nutrients on the marine ecosystem [5, 6, 7, 8]. The toxins produced by the phytoplankton species help in reducing the predation effect of the zooplankton, which results in the termination of planktonic blooms [9, 10]. It is also notable that zooplankton species try to avoid those regions where the concentration of TPP is dense.

Nothing is instantaneous in the real world, every biological system is delayed by some time lag, which can convert the stable dynamical systems to unstable ones [11, 12, 13, 21, 22, 23, 32, 33]. Therefore, the study of time delay in dynamical systems has become a crucial field of research.

Many mathematical models have been established to discuss the impacts of different time delays (predation, maturation, gestation, and toxin liberation delay, etc.) on the plankton dynamics [11, 12, 13, 14, 15, 29, 30]. Sharma et al. [21, 22, 23] have analyzed some ecological dynamics showing plankton interaction with multiple delays and observed that these dynamics become unstable, and a Hopf-bifurcation occurs at the interior steady state as the time lag crosses its critical value. Ecologists in [16] have studied a delayed model of phytoplankton and zooplankton interaction to investigate the impact of toxin liberation delay in the formation of HABs. [17, 18] have extended the research of [16] and determine the global stability of the dynamical system involving the zooplankton population and TPP. It is observed that sufficient research has been made by scientists to understand the effect of toxin liberation delay (TLD) on the plankton dynamics and its consequences on the environment [21, 22, 23]. But, the impact of toxin liberation delay and instantaneous recycling of dead mass of both phytoplankton and zooplankton on the nutrient-plankton system is rarely observed by us. The main motive of this analytical study is to examine the impact of the (TLD) on the nutrient-plankton dynamics. So, we have proposed a delayed nutrient-plankton dynamical system involving liquefied nutrient (N(t)), phytoplankton (x(t)), and zooplankton (y(t)), respectively. We organize this study as follows. The ecological model, its assumptions, boundedness, positivity, and persistence are discussed in Section 2, followed by the stability analysis of the plankton dynamics in Section 3. In Section 4, the direction and stability of the bifurcating solutions are obtained. All analytical results of our study are verified through numerical simulations in Section 5. The manuscript ends with certain outcomes in the concluding Section 6.

2. THE ECOLOGICAL MODEL

We have proposed our ecological dynamical system by assuming certain assumptions as follow; Let $N(t)$, x(t), and y(t) denote the biomass of liquefied nutrient, phytoplankton, and zooplankton population, respectively, at time t. The phytoplankton depends upon nutrients for their growth with Holling-I type interaction. Some phytoplankton species like Cylindrospermopsis, protoperidinium, and Karenia produce toxins such as Anatoxin-a, Azaspiracid, Brevetoxin, and Microcystins, etc. In these toxins, some are hepatotoxins and others are neurotoxins, which cause mortality of the predator zooplankton. The process of toxin liberation is not immediate.

Instead, it follows some time delay and the ecological importance of TLD lies in the fact that it is the time required by TPP to become mature enough to avoid the harvesting impact of predator zooplankton. The dead mass of phytoplankton and zooplankton species is converted into nutrients. Based upon these assumptions, our ecological model is represented by the following set of simultaneous equations, where the biological interpretation of all the parameters is as follows: N_0 is the natural availability of nutrients in the water, a is the washout rate, and b denotes its uptake rate by phytoplankton. The parameters k_1 and k_2 represent the recycling rate of the dead mass of phytoplankton and zooplankton species, respectively. The death rate of phytoplankton is denoted by b_1 and the natural mortality rate of zooplankton species is represented by α_2. The parameter α_1 is a maximum conversion rate of nutrients for the growth of phytoplankton and β_1 is the maximal conversion rate of phytoplankton for the growth of zooplankton. The parameter β represents the zooplankton's maximal ingestion rate, ρ is the rate of toxin secreted by TPP, γ is the half-saturation constant, and τ is the time delay.

$$(1) \qquad \begin{cases} \frac{dN}{dt} = N_0 - aN - bNx + k_1 b_1 x + k_2 \alpha_2 y \\ \frac{dx}{dt} = \alpha_1 Nx - b_1 x - \frac{\beta xy}{(\gamma+x)} \\ \frac{dy}{dt} = \frac{\beta_1 xy}{(\gamma+x)} - \alpha_2 y - \rho x(t-\tau)y \end{cases}$$

With the initial conditions $N(\varpi) = \zeta_1(\varpi), x(\varpi) = \zeta_2(\varpi), y(\varpi) = \zeta_3(\varpi)$, $\zeta_1(\varpi) \geq 0$, $\zeta_2(\varpi) \geq 0$, $\zeta_3(\varpi) \geq 0$, $\varpi \in \zeta[-\tau, 0]$, $\zeta_1(0) \geq 0$, $\zeta_2(0) \geq 0$, $\zeta_3(0) \geq 0$, where $\zeta_1(\varpi), \zeta_2(\varpi), \zeta_3(\varpi) \in C([-\tau, 0], R^3_+)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into R^3_+ where $R^3_+ = \{(n_1, n_2, n_3) : n_i \geq 0, i = 1, 2, 3\}$.

Positivity and Boundedness

The following theorems show that the system dynamics (1) is biological valid.

Lemma 2.1. The dynamics (1) contains nonnegative and unique results, initially we take $N(0) > 0, x(0) > 0$ and $y(0) > 0$, where $((N(0), x(0), y(0)) \in R^3_+$ in the octant $\sum = \{(N(t), x(t), y(t)) \in R^3_+; V(t) \leq \frac{N_0}{\psi} + \varepsilon\}$

Proof. The given dynamics (1) can be expressed as,

$\frac{dG}{dt}=G(Z)$, where $Z=(x_1,x_2,x_3)^T=(N,x,y)^T\in R^3$, $G(Z)=\begin{pmatrix}G_1(Z)\\ G_2(Z)\\ G_3(Z)\end{pmatrix}$ and Where, $G_1(Z)=N_0-aN-bNx+k_1b_1x+k_2\alpha_2y$, $G_2(Z)=\alpha_1Nx-b_1x-\frac{\beta xy}{(\gamma+x)}$, and $G_3(Z)=\frac{\beta_1xy}{(\gamma+x)}-\alpha_2y-\rho xy$. Since, $G:R^3\to R^3$ is Lipschitz continuous in octant Σ and $Z(0)=Z_0\in R^3$, therefore by fundamental theorem, there exist unique solution of (1). Since, $[G_i(Z)]_{z_i(t)=0,z\in R^3}\geq 0$, then [24, 25] implies that $Z(t)>0\;\forall\; t\geq 0$. From model system (1), $\frac{dN}{dt}_{N=0}\geq 0$, $\frac{dx}{dt}_{x=0}\geq 0$ and $\frac{dy}{dt}_{y=0}\geq 0$. Further, we show the uniform boundedness of the dynamics in $\sum$.

Suppose $V(t)=N(t)+\frac{b}{\alpha_1}x(t)+\frac{b\beta}{\beta_1\alpha_1}y(t)$,

$\frac{dV}{dt}\leq N_0-aN-(\frac{bb_1}{\alpha_1}-k_1b_1)x(t)-(\frac{\beta b\alpha_2}{\beta_1\alpha_1}-k_2\alpha_2)y(t)$

$\frac{dV}{dt}\leq N_0-\psi V(t)$, where $\psi=\min\{a,(\frac{bb_1}{\alpha_1}-k_1b_1),(\frac{\beta b\alpha_2}{\beta_1\alpha_1}-k_2\alpha_2)\}$

Or $\frac{dV}{dt}+\psi V(t)\leq N_0$.

Next, using Comparison theorem [31], we obtain

$0\leq V(t)\leq\frac{N_0}{\psi}+\frac{V(N(0),x(0),y(0))}{e^{\psi t}}$.

As $t\to\infty$, we have $V(t)\leq\frac{N_0}{\psi}$, Therefore, all solutions of (1) are bounded for $0\leq V(t)\leq\frac{N_0}{\psi}$. Hence, all the results of the dynamical system (1) are lying in the octant,

$\sum=\{(N(t),x(t),y(t))\in R^3_+;V(t)\leq\frac{N_0}{\psi}+\varepsilon\}$ for all $\varepsilon>0$.

Lemma 2.2. The plankton dynamics (1) is uniformly persistence if $\exists$ $+ve$ M_1 and M_2 s.t. all results V(t)= (N(t), x(t), y(t)) with $N(t)>0$, $x(t)>0$ and $y(t)>0$ satisfies the following inequality

$M_1\leq\liminf_{t\to\infty}V(t)\leq\limsup_{t\to\infty}V(t)\leq M_2$.

Proof. To show the system (1) is permanent, take, $M_2=a\frac{N_0}{\psi}$,

then by theorem 2.1., $\limsup_{t\to\infty}V(t)\leq M_2$. we have observed that for all $\varepsilon>0$ there exists a $W>0$ s.t. $\forall\; t\geq W$, we have $N(t)<\frac{N_0}{\psi}+\varepsilon$, $x(t)<\frac{N_0}{\psi}+\varepsilon$ and $y(t)<\frac{N_0}{\psi}+\varepsilon$. Again consider

$V(t)=N(t)+\frac{b}{\alpha_1}x(t)+\frac{b\beta}{\beta_1\alpha_1}y(t)$,

$\frac{dV}{dt}\geq N_0-aN-\frac{bb_1}{\alpha_1}x(t)-\frac{b\beta}{\beta_1\alpha_1}(1+\frac{N_0}{\psi}+\varepsilon)y(t)$

$\frac{dV}{dt}\geq N_0-\Theta V(t)$, where $\Theta=\max\{a,\frac{bb_1}{\alpha_1},\frac{b\beta}{\beta_1\alpha_1}(1+\frac{N_0}{\psi}+\varepsilon)\}$

Or $\frac{dV}{dt}+\Theta V(t)\geq N_0$. Now, results of [24] gives us

$\liminf_{t\to\infty}V(t)\geq\frac{N_0}{\Theta}=M_1(say)$ Hence, we get the required result.

3. DYNAMICAL BEHAVIOR OF THE MODEL SYSTEM ABOUT DIFFERENT EQUILIBRIUM POINTS

The system (1) has three steady states namely; The axial equilibria $E_1 = (\frac{N_0}{a}, 0, 0)$, which always exists, a zooplankton free equilibrium E_2=$(\frac{b_1}{\alpha_1}, \frac{N_0\alpha_1 - ab_1}{b_1(b-\alpha_1 k_1)}, 0)$, exists if $N_0 > \frac{ab_1}{\alpha_1}$ and $\alpha_1 < \frac{b}{k_1}$ and the interior equilibrium $E_* = (N_*, x_*, y_*)$ where N_*=$\frac{N_0+k_1 b_1 x_* + k_2\alpha_2 y_*}{a+bx_*}$,
x_*=$\frac{(\beta_1-\alpha_2-\rho\gamma)\pm\sqrt{(\beta_1-\alpha_2-\rho\gamma)^2-4\rho\alpha_2\gamma}}{2\rho}$, and y_*=$\frac{(\alpha_1 N_* - b_1)(\gamma+x_*)}{\beta}$ exists and unique if $\beta_1 > \alpha_2 + \rho\gamma$, $(\beta_1-\alpha_2-\rho\gamma)^2 > 4\rho\alpha_2\gamma$, and $N_* > \dfrac{b_1}{\alpha_1}$.

Proposition 3.1. The predator free equilibria $E_1 = (\frac{N_0}{a}, 0, 0)$ always exists. The steady state E_1 is stable if the condition $N_0 < \frac{ab_1}{\alpha_1}$ holds true.

The zooplankton free equilibria i.e. E_2=$(\frac{b_1}{\alpha_1}, \frac{N_0\alpha_1 - ab_1}{b_1(b-\alpha_1 k_1)}, 0)$ is stable as long as the inequalities $\frac{ab_1}{N_0} < \alpha_1 < \frac{b}{k}$ and $\alpha_1(bN_0 + K_1^2 b^2) > bb_1(k_1 b_1 + a)$ hold true.

The characteristic equation of the (1) at interior steady state E_* is,

$$\lambda^3 + A\lambda^2 + B\lambda + C + (D + E\lambda)e^{-\lambda\tau} = 0 \tag{2}$$

When $\tau = 0$, the equation (2) can be written as,

$$\lambda^3 + A\lambda^2 + (B+E)\lambda + (C+D) = 0 \tag{3}$$

Where, $A = a + bx_* - \alpha_1 N_* + b_1 + \frac{\beta\gamma y_*}{(\gamma+x_*)^2} > 0$, $E + B = (bx_* + a)\big(b_1 - \alpha_1 N_* + \frac{\beta\gamma y_*}{(\gamma+x_*)^2}\big) + \frac{\beta\beta_1\gamma x_* y_*}{(\gamma+x_*)^3} - \alpha_1 x_*(-bN_* + k_1 b_1) - \frac{\beta\rho x_* y_*}{(\gamma+x_*)} > 0$ $C + D = \frac{\beta x_* y_*(a+bx_*)}{(\gamma+x_*)}\{\frac{\beta_1\gamma}{(\gamma+x_*)^2} - \rho\} > 0$. Let (H_1) : $A(B+E) > (C+D)$.

The interior equilibrium E_* is LAS if H_1 holds true (due to Routh-Hurwitz criterion). Now, considering time delay τ as bifurcation parameter, we shall observe its effects on the behavior of the given plankton dynamics (1) around feasible steady state E_*. Next, the equation (2) can be written in a second order exponential polynomial in λ as,

$$I_1(\lambda,\tau) + I_2(\lambda,\tau)e^{-\lambda\tau} = 0 \tag{4}$$

where $I_1(\lambda,\tau) = \lambda^3 + A\lambda^2 + B\lambda + C$ and $I_2(\lambda,\tau) = (D + E\lambda)$.

Now, for applying the criterion given in [13], we calculate following (i)-(v) properties for $\tau > 0$:

(i) $I_1(0,\tau) + I_2(0,\tau) = D + C \neq 0$;

(ii)$I_1(\imath\omega,\tau)+I_2(\imath\omega,\tau) = -\imath\omega^3 - A\omega^2 + B\imath\omega + C + (D+\imath E\omega)e^{-\imath\omega\tau} \neq 0$;

(iii)$\limsup_{|\lambda|\to\infty}[|\frac{I_2(\lambda,\tau)}{I_1(\lambda,\tau)}|] = \limsup_{|\lambda|\to\infty}[|\frac{D+E\lambda}{\lambda^3+A\lambda^2+B\lambda+C}|] = 0 < 1$;

(iv) $G_1(\omega) = |I_1(\imath\omega,\tau)|^2 - |I_2(\imath\omega,\tau)|^2$ is a six degree equation, having at most 6 zeros (finite roots);

(v) Each $+ve$ zero $\omega(\tau)$ of $G_1(\omega(\tau)) = 0$ (whenever it exists) is continuous and differentiable in τ (Implicit function theorem).

Hence, for $\tau > 0$, $\lambda = \imath\omega(\omega > 0)$ is a zero of (2) and by putting $\lambda = \imath\omega(\omega > 0)$ in (2), we obtain,

$$-\imath\omega^3 - A\omega^2 + B\imath\omega + C + (D+\imath E\omega)e^{-\imath\omega\tau} = 0$$

Separating the real and imaginary parts, we get

$$A\omega^2 - C = D\cos\omega\tau + \omega E\sin\omega\tau \tag{5}$$

$$B\omega - \omega^3 = -\omega E\cos\omega\tau + D\sin\omega\tau \tag{6}$$

Eliminating ω from (6) and using $\omega^2 = z$, we have

$$h(z) = z^3 + pz^2 + qz + r = 0 \tag{7}$$

Where, $p = A^2 - 2B$, $q = B^2 - 2AC - E^2$ and $r = C^2 - D^2$.

Let us assume that $r = C^2 - D^2 < 0$, which implies $h(0) < 0$, $h(\infty) = \infty$, and (7) has at least one $+ve$ zero ω_0. Now, we discuss the system shows complex behavior around E_* for $\tau \geq \tau_k$ through proposition 3.2.

Proposition 3.2. Suppose $\tau > 0$, then there exists τ_k such that the positive interior feasible state E_* is LAS (locally asymptotically stable) for $0 < \tau < \tau_k$ and unstable when $\tau > \tau_k$. Furthermore, the system (1) shows excitability (a Hopf-bifurcation) at $\tau = \tau_k$ where the critical value of τ is as follows,

$$\tau_k = \frac{1}{\omega_0}\arccos\frac{D(A\omega_0{}^2 - C) + \omega_0{}^2 E(\omega_0{}^2 - B)}{D^2 + \omega_0{}^2 E} + \frac{2k\pi}{\omega_0}\ k = 0,1,2....$$

provided $\frac{d\upsilon}{d\tau}|_{\upsilon=0} = \frac{\omega^2}{m_1^2+n_1^2}\{\frac{dh(z)}{dz}\}_{z=\omega^2} \neq 0$.

Proof. Let $\tau > 0$, then (7) has at least one $+ve$ zero. Thus, the characteristic equation (2) has purely imaginary zeros $\pm i\omega_0$. After simplifying (5) and (6), we obtain,

$\tau_k = \dfrac{1}{\omega_0} \arccos \dfrac{D(A{\omega_0}^2 - C) + {\omega_0}^2 E({\omega_0}^2 - B)}{D^2 + {\omega_0}^2 E} + \dfrac{2k\pi}{\omega_0}$, $k = 0,1,2....$ Using, $\lambda(\tau) = \upsilon(\tau) + \imath\omega(\tau)$ in (2), we have determined that $\lambda_{\pm}(\tau_0) = \pm i\omega(\tau_0)$ are purely imaginary zeros of (3) at $\tau = \tau_0$, which crosses imaginary axis from right to left or left to right if the transversality condition $\upsilon(\tau_0) < 0$ or $\upsilon(\tau_0) > 0$ is satisfied, respectively. Where

$\upsilon(\tau_0) = sign\{\frac{d(Re(\lambda))}{d\tau}\}_{\lambda=i\omega}$

The transversality condition can be calculated from the following expression,

$m_1 \dfrac{d\upsilon}{d\tau} - n_1 \dfrac{d\omega}{d\tau} = p_1, n_1 \dfrac{d\upsilon}{d\tau} + m_1 \dfrac{d\omega}{d\tau} = p_2$ where, $m_1 = -3\omega^2 + B + E\cos(\omega\tau) - \tau D\cos(\omega\tau) - \tau\omega E\sin(\omega\tau)$, $n_1 = 2\omega A - E\sin(\omega\tau) + \tau D\sin(\omega\tau) - \tau\omega E\cos(\omega\tau)$, $p_1 = \omega(D\sin(\omega\tau) - \omega E\cos(\omega\tau))$

$p_2 = \omega(D\cos(\omega\tau) + \omega E\sin(\omega\tau))$

Solving above equations, we get,

$\dfrac{d\upsilon}{d\tau}|_{\upsilon=0} = \dfrac{m_1 p_1 + n_1 p_2}{m_1^2 + n_1^2}$ or $\dfrac{d\upsilon}{d\tau}|_{\upsilon=0} = \dfrac{\omega^2}{m_1^2 + n_1^2}\{\dfrac{dh(z)}{dz}\}_{z=\omega^2} \neq 0$

4. DIRECTION OF PERIODIC TRAJECTORIES

Presently, the stability, direction and period of bifurcated periodic trajectories will be determined, using theorems given in [19, 20].

By substituting $z_1 = N - N_*$, $z_2 = x - x_*$ and $z_3 = y - y_*$, in the model system (1) and expanding it about $E_*(N_*, x_*, y_*)$ (Taylor's expansion), we get;

$$(8) \qquad \begin{cases} \frac{dz_1}{dt} = a_{100}z_1(t) + a_{010}z_2(t) + \sum\limits_{i+j+k\geq 2} a_{ijk}{z^i}_1(t){z^j}_2(t){z^k}_3(t) = F^1(z_1,z_2,z_3) \\ \frac{dz_2}{dt} = b_{100}z_1(t) + b_{010}z_2(t) + b_{001}z_3(t) + \sum\limits_{i+j+k\geq 2} b_{ijk}{z^i}_1(t){z^j}_2(t){z^k}_3(t) \\ = F^2(z_1,z_2,z_3) \\ \frac{dz_3}{dt} = c_{100}z_2(t) + c_{001}z_2(t-\tau) + \sum\limits_{i+j+k\geq 2} c_{ijk}{x^i}_2(t){z^j}_3(t){z^k}_2(t-\tau) \\ = F^3(z_1,z_2,z_3) \end{cases}$$

$a_{100} = -(a + bx_*)$, $a_{010} = (-bN_* + k_1 b_1)$, $a_{001} = (k_2\alpha_2)$, $b_{100} = \alpha_1 x_*$, $b_{010} = \alpha_1 N_* - b_1 - \frac{\beta\gamma y_*}{(\gamma + x_*)^2}$, $b_{001} = -\frac{\beta x_*}{(\gamma + x_*)}$, $c_{100} = \frac{\gamma\beta y_*}{(\gamma + x_*)^2}$, and $c_{001} = -\rho y_*$ and we have obtained the coefficients of non-linear terms as, $a_{110} = -b$, $b_{110} = \alpha_1$, $b_{011} = -\frac{\gamma\beta}{(\gamma + x_*)^2}$, $b_{020} = \frac{\gamma\beta y_*}{(\gamma + x_*)^3}$, $c_{110} = \frac{\gamma\beta_1}{(\gamma + x_*)^2}$ and $c_{011} = -\rho$, $c_{200} = -\frac{\gamma\beta_1 y_*}{(\gamma + x_*)^3}$.

Let $\tau = \tau_k + \eta_2, \quad \bar{\varsigma}_i(t) = \varsigma_i(\tau t)$ and after simple mathematical calculations, system (8) becomes a functional differential equation in $\mathscr{C} = \mathscr{C}([-1,0],R^3)$ as

$$\dot{\varsigma}(t) = L_{\eta_2}(\varsigma_t) + f(\eta_2, \varsigma_t) \tag{9}$$

where $\varsigma(t) = (\varsigma_1(t), \varsigma_2(t), \varsigma_3(t))^T \in R^3$ and $L_\mu : \mathscr{C} \to R^3, f : R \times \mathscr{C} \to R^3$ are given respectively, by

$$L_{\eta_2}(\rho) = (\tau_k + \eta_2)[A_1\rho(0) + A_2\rho(-1)] \tag{10}$$

and

$$f(\eta_2, \rho) = (\tau_k + \eta_2)\begin{bmatrix} a_{110}\rho_1(0)\rho_2(0) \\ b_{110}\rho_1(0)\rho_2(0) + b_{011}\rho_2(0)\rho_3(0) + b_{020}\rho_2^2(0) \\ c_{011}\rho_2(-1)\rho_3(0) + c_{110}\rho_2(0)\rho_3(0) + c_{200}\rho_2^2(0) \end{bmatrix} \tag{11}$$

$$A_1 = \begin{bmatrix} a_{100} & a_{010} & a_{001} \\ b_{100} & b_{010} & b_{001} \\ 0 & c_{100} & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c_{001} & 0 \end{bmatrix}$$

Thus, $\exists\ \eta(\eta_1,)$, a function of bounded variation for $\eta_1 \in [-1,0]$ (using Riesz representation theorem) such that

$$L_{\eta_2}(\rho) = \int_{-1}^{0} d\eta(\eta_1, \eta_2)\rho(\eta_1)\ for\ \rho \in \mathscr{C} \tag{12}$$

Even, we can select

$$\eta(\eta_1, \eta_2) = (\tau_k + \eta_2)[A_1\delta(\eta_1) - A_2\delta(\eta_1 + 1)] \tag{13}$$

Here δ is taken as Dirac delta function. Where ρ lies in $\mathscr{C}[(-1,0),R^3]$, define

$$A(\eta_2)\rho = \begin{cases} \dfrac{d\rho(\eta_1)}{d\eta_1} & \eta_1 \in [-1,0) \\ \displaystyle\int_{-1}^{0} d\eta(s, \eta_2)\rho(s) & \eta_1 = 0 \end{cases}$$

and

$$R(\eta_2)(\rho) = \begin{cases} 0 & \eta_1 \in [-1,0) \\ f(\eta_2, \rho) & \eta_1 = 0 \end{cases}$$

Then (9) gives

$$\dot{\varsigma}(t) = A(\eta_2)\varsigma_t + R(\eta_2)\varsigma_t \tag{14}$$

where $\varsigma_t(\eta_1) = \varsigma(t+\eta_1)$ whenever $\eta_1 \in [-1,0]$

For $\psi \in \mathscr{C}^1([-1,0),R^3)$, we define

$$A^*\psi = \begin{cases} -\dfrac{d\psi(s)}{ds} & s \in (0,1] \\ \displaystyle\int_{-1}^{0} \psi(-t)d\eta(t,0) & s = 0 \end{cases}$$

Now, the bilinear inner product

$$< \psi(s), \rho(\eta_1) >= \bar{\psi}(0)\rho(0) - \int_{-1}^{0}\int_{\xi=0}^{\eta_1} \bar{\psi}(\xi-\eta_1)d\eta(\eta_1)\rho(\xi)d(\xi)\Phi \tag{15}$$

where $\eta(\eta_1) = \eta(\eta_1, 0)$. Then $A(0)$ and A^* are adjoint operators. From the results of the Section 3., it is clear $\pm\imath\omega_0\tau_k$ are eigen values of $A(0)$ and so of A^*. Which can be determined through Lemma 4.1.

Lemma 4.1. Suppose $\Phi(\eta_1) = (1,q_1,q_2)^T e^{\imath\omega_0\tau_k}$ be the eigenvector of $A(0)$ corresponding to $\imath\omega_0\tau_k$ and $\Phi^* = D(1,q_1^*,q_2^*)e^{\imath\omega_0\tau_k}$ be the eigenvector of A^* corresponding to $-\imath\omega_0\tau_k$.
Then $< \Phi^*, \Phi >= 1$ and $< \Phi^*, \bar{\Phi} >= 0$
where $q_1 = \dfrac{b_{001}a_{010} - a_{001}(b_{010} - \imath\omega_0)}{b_{100}a_{001} - b_{001}(a_{100} - \imath\omega_0)}$, $q_2 = \dfrac{(\imath\omega_0 - a_{100}) - a_{010}q_1}{a_{001}}$,
$q_1^* = \dfrac{-\imath\omega_0 - a_{100}}{b_{100}}$, $q_2^* = \dfrac{(\imath\omega_0 + a_{100})b_{001} - a_{001}b_{100}}{\imath\omega_0 b_{100}}$
and $D = \dfrac{1}{1+\bar{q}_1q_1^* + \bar{q}_2q_2^* + \tau_k e^{\imath\omega_0\tau_k}(c_{001}\bar{q}_1q_2^*)}$.

Further, we shall calculate those coordinates which are helpful to obtain the center manifold C_0 at $\eta_2 = 0$. Suppose ς_t is the solution of (14) at $\eta_2 = 0$. We define,

$$z(t) =< \Phi^*, \varsigma_t >, \quad W(t,\eta_1) = \varsigma_t(\eta_1) - 2Re\{z(t)\Phi(\eta_1)\} \tag{16}$$

On the center manifold C_0, we get $W(t,\eta_1) = W(z(t),\bar{z}(t),\eta_1)$, where

$$W(z(t),\bar{z}(t),\eta_1) = W_{20}(\eta_1)\frac{z^2}{2} + W_{11}(\eta_1)z\bar{z} + W_{02}(\eta_1)\frac{\bar{z}^2}{2} + \ldots\ldots.. \tag{17}$$

We consider z and $\bar{z}$ as local coordinates for C_0 in the direction of Φ^* and $\bar{\Phi}^*$. We will take only real solutions as W is real if ς_t is real. So, for solution $\varsigma_t \in C_0$ of (14), we have $\dot{z}(t) = \imath\omega_0\tau_k z + \bar{\Phi}^*(0)f(0,(W(z,\bar{z},\eta_1)+2Rez\Phi(\eta_1)))$=$\imath\omega_0\tau_k z + \bar{\Phi}^*(0)f_0(z,\bar{z})$.

We can write above equation in the following form,

$$\dot{z}(t) = \imath\omega_0\tau_k z + g(z,\bar{z})$$

where

$$g(z,\bar{z}) = \bar{\Phi}^*(0)f_0(z,\bar{z}) = g_{20}(\eta_1)\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \text{........} \tag{18}$$

Using (11), equation (18) can be expressed as, $g(z,\bar{z}) = \bar{\Phi}^*(0)f_0(z,\bar{z}) = \bar{\Phi}^*(0)f(0,\varsigma_t)$

$$\begin{aligned}
&= \tau_k\bar{D}[\xi_{01}\{W^1_{20}(0)\frac{z^2}{2} + W^1_{11}(0)z\bar{z} + W^1_{02}(0)\frac{\bar{z}^2}{2} + z + \bar{z}\}\{W^2_{20}(0)\frac{z^2}{2}\\
&+ W^2_{11}(0)z\bar{z} + W^2_{02}(0)\frac{\bar{z}^2}{2} + q_1z + \bar{q}_1\bar{z}\} + \xi_{02}\{W^2_{20}(0)\frac{z^2}{2} + W^2_{11}(0)z\bar{z}\\
&+ W^2_{02}(0)\frac{\bar{z}^2}{2} + q_1z + \bar{q}_1\bar{z}\}\{W^3_{20}(0)\frac{z^2}{2} + W^3_{11}(0)z\bar{z} + W^3_{02}(0)\frac{\bar{z}^2}{2}\\
&+ q_2z + \bar{q}_2\bar{z}\} + \xi_{03}\{W^2_{20}(0)\frac{z^2}{2} + W^2_{11}(0)z\bar{z} + W^2_{02}(0)\frac{\bar{z}^2}{2} + q_1z + \bar{q}_1\bar{z}\}^2\\
&+ \xi_{04}\{W^2_{20}(-1)\frac{z^2}{2} + W^2_{11}(-1)z\bar{z} + W^2_{02}(-1)\frac{\bar{z}^2}{2} + q_1e^{-\imath\omega_0\tau_k}z\\
&+ \bar{q}_1e^{\imath\omega_0\tau_k}\bar{z}\}\{W^3_{20}(0)\frac{z^2}{2} + W^3_{11}(0)z\bar{z} + W^3_{02}(0)\frac{\bar{z}^2}{2} + q_2z + \bar{q}_2\bar{z}\}]
\end{aligned} \tag{19}$$

Where, $\xi_{01} = a_{110} + b_{110}\bar{q}^*_1$, $\xi_{02} = b_{011}\bar{q^*_1} + c_{110}\bar{q^*_2}$, $\xi_{03} = b_{020}\bar{q}^*_1 + c_{200}\bar{q}^*_2$, $\xi_{04} = c_{011}\bar{q}^*_2$

Comparison of coefficients with equation (18) gives,

$$\begin{cases}
g_{20} = \bar{D}\tau_k\{2q_1\xi_{01} + 2\xi_{02}q_1q_2 + 2\xi_{03}q_1^2 + 2q_1q_2\xi_{04}e^{-\imath\omega_0\tau_k}\}\\
g_{11} = \bar{D}\tau_k\{2Req_1\xi_{01} + 2Re(q_1\bar{q}_2)\xi_{02} + 2q_1\bar{q}_1\xi_{03} + 2Re(q_1\bar{q}_2e^{-\imath\omega_0\tau_k})\xi_{04}\}\\
g_{02} = \bar{D}\tau_k\{2\bar{q}_1\xi_{01} + 2\bar{q}_1\bar{q}_2\xi_{02} + 2\bar{q}_1{}^2\xi_{03} + 2\bar{q}_1\bar{q}_2\xi_{04}e^{\imath\omega_0\tau_k}\}\\
g_{21} = \bar{D}\tau_k\{\xi_{01}(W^1_{20}(0)\bar{q}_1 + 2q_1W^1_{11}(0) + 2W^2_{11}(0) + W^2_{20}(0)) + \xi_{02}(W^2_{20}(0)\bar{q}_2\\
+2q_2W^2_{11}(0) + 2q_1W^3_{11}(0) + \bar{q}_1W^3_{20}(0)) + \xi_{03}(W^2_{20}(0)\bar{q}_1 + 2q_1W^2_{11}(0)\\
+2q_1W^2_{11}(0) + \bar{q}_1W^2_{20}(0)) + \xi_{04}(\bar{q}_2W^2_{20}(-1) + 2q_2W^2_{11}(-1)\\
+2q_1e^{-\imath\omega_0\tau_k}W^3_{11}(0) + \bar{q}_1e^{\imath\omega_0\tau_k}W^3_{20}(0))
\end{cases} \tag{20}$$

Since, $W_{20}(\eta_1)$ and $W_{11}(\eta_1)$ are in g_{21}, we still to compute them

Doing some simple calculations using (14) and (16), we have

$$W_{20}(\eta_1) = \frac{\imath g_{20}}{\omega_0\tau_k} q(0)e^{\imath\omega_0\tau_k\eta_1} + \frac{\imath\bar{g}_{02}}{3\omega_0\tau_k}\bar{q}(0)e^{-\imath\omega_0\tau_k\eta_1} + \mathcal{F}_1 e^{2\imath\omega_0\tau_k\eta_1} \tag{21}$$

$$W_{11}(\eta_1) = -\frac{\imath g_{11}}{\omega_0\tau_k} q(0)e^{\imath\omega_0\tau_k\eta_1} + \frac{\imath\bar{g}_{11}}{3\omega_0\tau_k}\bar{q}(0)e^{-\imath\omega_0\tau_k\eta_1} + \mathcal{F}_2 \tag{22}$$

Finally, we will seek the values of $\mathcal{F}_1$ and $\mathcal{F}_2$, where $\mathcal{F}_1 = (\mathcal{F}_1^1, \mathcal{F}_1^2, \mathcal{F}_1^3)$ and $\mathcal{F}_2 = (\mathcal{F}_2^1, \mathcal{F}_2^2, \mathcal{F}_2^3)$ are constant vectors.

$$\begin{aligned}
\mathcal{F}_1{}^{(1)} &= \frac{1}{M}\{-2(4\omega_0^2 + 2\imath\omega_0 b_{010} + b_{001}(c_{100} + c_{001}e^{-2\imath\omega_0\tau_k}))\Gamma_{11}\\
&-4a_{010}(\imath\omega_0\Gamma_{12} - 2b_{001}\Gamma_{13}) + a_{001}b_{100}(c_{100} + c_{001}e^{-2\imath\omega_0\tau_k})\}\\
\mathcal{F}_1{}^{(2)} &= \frac{1}{M}\{(2\imath\omega_0 - a_{100})(4\imath\omega_0\Gamma_{12} + 2b_{001}\Gamma_{13}) + b_{100}(4\imath\omega_0\Gamma_{11} - 2a_{001}\Gamma_{13}\\
\mathcal{F}_1{}^{(3)} &= \frac{1}{M}\{(2\imath\omega_0 - a_{100})[2(2\imath\omega_0 - b_{010})\Gamma_{13}\\
&+2(c_{100} + c_{001}e^{-2\imath\omega_0\tau_k})\Gamma_{12}] + 2a_{010}b_{100}\Gamma_{13} + 2b_{100}(c_{100} + c_{001}e^{-2\imath\omega_0\tau_k})\Gamma_{11}\}
\end{aligned}$$

and $M = (2\imath\omega_o - a_{100})\{(-4\omega_0^2 - 2\imath\omega_0 b_{010}) - b_{001}c_{100} - b_{001}c_{001}e^{-2\imath\omega_0\tau_k}\} + 2\imath\omega_0 b_{100}a_{010} + a_{001}b_{100}(c_{100} - c_{001}e^{-2\imath\omega_0\tau_k})$

$$\begin{aligned}
\mathcal{F}_2{}^{(1)} &= \frac{1}{N}(a_{010}b_{001}\Gamma_{23} - b_{001}(c_{100} + c_{001})\Gamma_{21}) + a_{001}(c_{100} + c_{001})\Gamma_{22}\\
&-\Gamma_{23}b_{010}), \mathcal{F}_2{}^{(2)} = -\frac{1}{N}(a_{100}b_{001} - b_{100}a_{001})\Gamma_{23}\\
\mathcal{F}_2{}^{(3)} &= \frac{1}{N}(a_{100}b_{010} - b_{100}a_{010})\Gamma_{23} + (c_{100} + c_{001})(b_{100}\Gamma_{21} - a_{100}\Gamma_{22}))
\end{aligned}$$

where $\Gamma_{11} = a_{110}q_1$, $\Gamma_{12} = b_{110}q_1 + b_{011}q_1q_2 + b_{020}q_1^2$, $\Gamma_{13} = c_{011}q_1q_2e^{-\imath\omega_0\tau_k} + c_{110}q_1q_2 + c_{200}q_1^2$, $\Gamma_{21} = 2Re\{q_1\}a_{110}$, $\Gamma_{22} = 2Re(q_1)b_{110} + 2Re(q_1\bar{q_2})b_{011} + 2b_{020}q_1\bar{q_1}$, $\Gamma_{23} = 2Re(q_1\bar{q_2}e^{-\imath\omega_0\tau_k})c_{011} + 2Re(q_1\bar{q_2})c_{110} + 2c_{200}q_1\bar{q_1}$.

$N = -a_{100}b_{010}(c_{100} + c_{001})$.

Thus we can obtain the following quantities after doing some mathematical calculations;

$$
\begin{aligned}
c_1(0) &= \frac{\iota}{2\omega_0\tau_k}\{g_{20}g_{11} - 2|g_{11}|^2 - \frac{(|g_{02}|)^2}{3}\} + \frac{g_{21}}{2} \\
\mu_2 &= -\frac{Re\{c_1(0)\}}{Re\{\frac{d\lambda(\tau_k)}{d\tau}\}}, \beta_2 = 2Re\{c_1(0)\}, \\
T_2 &= -\frac{Im\{c_1(0)\} + \mu_2 Im\{\frac{d\lambda(\tau_k)}{d\tau}\}}{\omega_0\tau_k}, \qquad (23)
\end{aligned}
$$

Where k be any whole number and g_{ij}'s are calculated in (20).

Lemma 4.2. [20] The Hopf-bifurcation at the critical value $\tau = \tau_0$ is supercritical (subcritical) for $\mu_2 > 0 (< 0)$, periodic solutions are stable (unstable) for $\beta_2 < 0 (> 0)$, and the time period of these periodic solutions increases or decreases if $T_2 > 0$ or $T_2 < 0$.

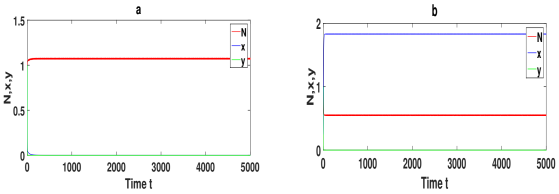

FIGURE 1. System trajectories showing stable behavior around E_1 (fig.a) and E_2 (fig.b)

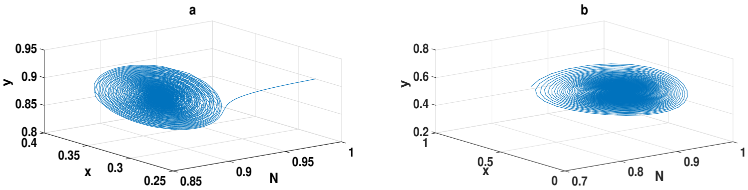

FIGURE 2. Stability of the system around E_* at $\tau = 0$ (fig.a), and at $\tau = 6.3 < 6.4 = \tau_0$(fig.b).

5. NUMERICAL SIMULATION

In the present section, we will verify our analytical results through numerical examples. Firstly, we take a set of parametric values

$[P_1]$: $N_0 = 3$, $a = 2.8$, $b = 1.75$, $k_1 = 0.3$, $b_1 = .55$, $k_2 = 0.01$, $\alpha_2 = 1.25$, $\alpha_1 = .50$, $\beta_1 = 1.20$,

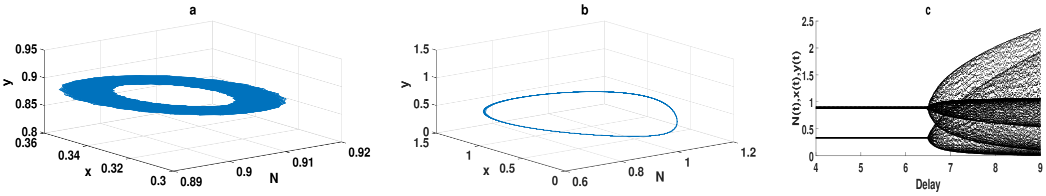

FIGURE 3. Occurrence of Hopf-bifurcation of the dynamical system at critical value $\tau_0 = 6.4$(fig.a), existence of stable limit cycle at $\tau = 7$ (fig.b), and bifurcation diagram of different population w.r.t. τ (fig.c)

$\rho = 0.15$ $\beta = 0.1$, γ=1. For the set $[P_1]$, we obtain the equilibrium point $E_1(1.0718, 0, 0)$ on N-axis which is clearly locally asymptotically stable as condition $N_0 < \frac{ab_1}{\alpha_1}$ i.e $(3 < 3.080)$ stated in proposition 3.1. hold good. Next, when we take the values of the parameters as,

$[P_2] : N_0 = 3$, $a = 2.8$, $b = 1.75$, $k_1 = 0.3$, $b_1 = 0.55$, $k_2 = 0.01$, $\alpha_1 = 1$, $\beta_1 = 0.24$, α_2=1, $\beta = 2$, γ=1, $\rho = 0.05$.

We get the equilibrium point $E_2(0.5505, 1.8306, 0)$ and it can be verified from proposition 3.1. that instability of E_1 implies existence of E_2 as the conditions $\frac{ab_1}{N_0} < \alpha_1 < \frac{b}{k}$ $(0.5133 < 1.50 < 1.9733333)$ and $\alpha_1(bN_0 + K_1^2 b^2) > bb_1(k_1 b_1 + a)$ $(7.9158374 > 2.8538125)$ are satisfied. Fig.1. depicts the stable behavior of the system around E_1 and E_2

Again, by considering the set of parametric values

$[P_3]$: $N_0 = 3$, $a = 2.8$, $b = 1.75$, $k_1 = 0.3$, $b_1 = .55$, $k_2 = 0.01$, $\alpha_2 = 0.25$, $\alpha_1 = 1.50$, $\beta_1 = 1.20$, $\beta = 2$, $\gamma = 1$, $\rho = 0.20$.

The steady state E_* exists and remains LAS as condition $(H_1) : A(B+E) - (C+D) = 0.057797568805302 > 0$ is clearly satisfied in the absence of time lag. The trajectories of the given system tend to $E_*(0.9032, 0.3333, 0.5363)$ as shown Fig.2(a).

Now, we will study the impact of delay on the given dynamical system about the biologically feasible point E_*. Further, we can find a purely imaginary root $\imath\omega_0$ of equation (2) with $\omega_0 = 3.9415$ (using $[P_3]$). The critical value τ_0 of time delay τ can be calculated as $\tau_0 = 6.4$ from equation (3). At this critical value, we can obtain

$$\frac{d\upsilon}{d\tau}\Big|_{\xi=0} = \frac{m_1 p_1 + n_1 p_2}{m_1^2 + n_1^2} = 0.0071 > 0.$$

Thus, the transversality condition is evidently satisfied, which ensures the occurrence of Hopf bifurcation at $\tau_0 = 6.4$. Therefore, positive equilibrium E_* remains stable for $0 \leq \tau \leq 6.4$ and a Hopf-bifurcation appears when τ passes through its thresh hold value $\tau_0 = 6.4$. Fig.2., and Fig.3. show the existence of stability, Hopf-bifurcation and a limit cycle around E_*, respectively. A bifurcation diagram w.r.t. time delay of different populations i.e. nutrient, phytoplankton, and zooplankton is also shown in Fig.3(c). Thus, our numerical results show that when the toxication delay crosses its threshold value, the given system starts to oscillate periodically and exhibits a Hopf-bifurcation. Biologically, these periodic oscillations can be interpreted as the occurrence of plankton bloom. So, it is observed that the delay in the process of toxin liberation can destabilize the system with the existence of planktonic bloom. Further, the quantities calculated in (23) are given by $c_1(0) = -4.223953985824854e+03+2.378151815749307e+03i$, $\mu_2 = 5.988146092405473e+05$, $\beta_2 = -8.447907971649707e+03$, $T_2 = 2.816941143426997e+04$. By using Lemma 4.2., we have determined that the Hopf bifurcation is supercritical as $\mu_2 > 0$, the periodic solutions are stable as $\beta_2 < 0$, and increases as $T_2 > 0$.

6. CONCLUSION AND DISCUSSION

In this manuscript, we study the impact of toxin libration delay on a 3-D nutrient-plankton dynamics. It is assumed that the process of toxin libration by the phytoplankton population is not immediate. Rather, it is followed by some discrete-time variation, which is known as toxin liberation delay. Initially, we have discussed the stability of the non-delayed system dynamics under certain conditions around E_1 and E_2 (preposition 3.1.). It is shown that the steady state E_* is locally asymptotically stable if (H_1) holds good. In the presence of a toxin liberation delay, it is determined that the dynamical system remains stable for $\tau \in [0, 6.4]$ (see Fig.2.). It enters into a Hopf-bifurcation when toxin libration delay τ passes through its threshold value $\tau_0 = 6.4$ (see Fig.3.) with the existence of periodic trajectories around E_*. Lemma 4.2. proves that the Hopf-bifurcation is supercritical. Thus, it is investigated that the predation delay beyond the thresh hold value of τ can include excitability in the dynamical system with the existence of planktonic bloom.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

[1] G.T. Evans, J.S. Parslow, A model of annual plankton cycles, Biol. Oceanogr. 3 (1985), 327-427.

[2] S. Busenberg, K.S. Kishore, P. Austin, G. Wake, The dynamics of a model of a plankton-nutrient interaction. J. Math. Biol. 52 (1990), 677-696.

[3] S. Ruan, Persistence and coexistence in zooplankton-phytoplankton-nutrient models with instantaneous nutrient recycling. J. Math. Biol. 31 (1993), 633-654.

[4] S. Ruan, Oscillations in Plankton models with nutrient recycling. J. Theor. Biol. 208 (2001), 15-26.

[5] S. Chakarborty, S. Roy, J. Chattopadhyay, Nutrient-limiting toxin producing and the dynamics of two phytoplankton in culture media: A mathematical model. J. Ecol. Model. 213 (2008), 191-201.

[6] S. Pal, S. Chatterjee, J. Chattopadhyay, Role of toxin and nutrient for the occurrence and termination of plankton bloom-results drawn from field observations and a mathematical model. J. Biosyst. 90 (2007), 87-100.

[7] S. Khare, J. Dhar, O.P. Misra, Role of toxin producing phytoplankton on a plankton ecosystem. Nonlinear Anal., Hybrid Syst. 4 (2010), 496-502.

[8] S.R.J. Jang, J. Baglama, J. Rick, Nutrient-phytoplankton-zooplankton models with a toxin. Math. Comput. Model. 43 (2006), 105-118.

[9] R.R. Sarkar, J. Chattopadhyay, Occurence of planktonic blooms under environmental fluctuations and its possible control mechanism-mathematical models and experimental observations. J. Theor. Biol. 224 (2003), 501-516.

[10] R.R. Sarkar, S. Pal, J. Chattopadhyay, Role of two toxin-producing plankton and their effect on phytoplankton-zooplankton system-a mathematical study by experimental findings. J. Biosystem. 80 (2005), 11-23.

[11] J.M. Cushing, Integrodifferential equations and delay models in population dynamics. Springer-Verlag, Heidelberg. (1977).

[12] K. Gopalsamy, Stability and oscillations in delay differential equations of population dynamics. Kluwer Academic. (1992).

[13] Y. Kuang, Delay differential equations with applications in population dynamics. Academic Press, New York. (1993).

[14] S. Ruan, The effect of delays on stability and persistence in plankton models. Nonlinear Anal., Theory Meth. Appl. 24 (1995), 575-585.

[15] K.Das, S.Ray, Effect of delay on nutrient cycling in phytoplankton-zooplankton interactions in estuarine system. Ecol. model. 215, 69-76 (2008).

[16] J. Chattopadhyay, R.R. Sarkar, A. Abdllaoui, A delay differential equation model on harmful algal blooms in the presence of toxic substances. IMA J. Math. Appl. Med. Biol. 19 (2002), 137-161.

[17] T. Saha, Malay Bandyopadhyay, Dynamical analysis of toxin producing Phytoplankton-Zooplankton interactions. Nonlinear Anal., Real World Appl. 10 (2009), 314-332.

[18] M. Rehim, M. Imran, Dynamical analysis of a delay model of phytoplankton-zooplankton interaction. Appl. Math. Model. 36 (2012), 638-647.

[19] Y. Song, M. Han, J. Wei, Stability and Hopf bifurcation analysis on a simplified BAM neural network with delays. Physica D. 200 (2005), 185-204.

[20] B.D. Hassard, N.D. Kazarinoff, Y.H. Wan, Theory and Applications of Hopf bifurcation. Cambridge: Cambridge University Press, 1981.

[21] A.K. Sharma, A. Sharma, K. Agnihotri, Bifurcation behaviors analysis of a plankton model with multiple delays. Int. J. Biomath. 9 (2016), 1650086.

[22] A.K. Sharma, A. Sharma, K. Agnihotri, Analysis of a toxin producing phytoplankton-zooplankton interaction with Holling IV type scheme and time delay. Nonlinear Dyn. 81 (2015), 13-25.

[23] A. Sharma, A.k. Sharma, K. Agnihotri, The dynamic of plankton-nutrients interaction with delay. Appl. Math. Comput. 231 (2014), 503-515.

[24] H.L. Smith, Monotone dynamical systems: an introduction to the theory of competitive and cooperative systems. Bull. (New Ser.) Amer. Math. Soc. 33 (1996), 203-209.

[25] Y. Wang, Y. Zhou, F. Brauer, J. M. Heffernan, Viral dynamics model with CTL immune response incorporating antiretroviral therapy. J. Math. Biol. 67 (2013), 901-934.

[26] A. Huppert, B. Blasius, L. Stone, A Model of Phytoplankton Blooms. Amer. Naturalist, 159 (2002), 156-171.

[27] T. G. Hallam, Structural sensibility of grazing formulation in the nutrient controlled plankton model. J. Math. Biol. 5 (1978), 261-280.

[28] A. M. Edwards, J. Brindley, Oscillatory behaviour in three component plankton population model. Dyn. Syst. 11 (1996), 347-370.

[29] A. K. Sharma, A. Sharma, K. Agnihotri, Bifurcation analysis of a plankton model with dicrete delay. Int. J. Math. Comput. Phys. Electric. Comput. Eng. 8 (2014), 133-142.

[30] A. K. sharma, A.sharma, k. Agnihotri, Dynamical analyais of a harvesting model of phytoplankton and zooplankton. Int. J. Math. Comput. Phys. Electric. Comput. Eng. 8 (2014), 1013-1018.

[31] J. K. Hale, Ordinary differential equations.Wiley, New York. (1969).

[32] N. K. Thakur, A. Ojha, Complex dynamics of delay-induced plankton-fish interaction exhibiting defense. SN Appl. Sci. 2 (2020), 1114.

[33] B. Dubey, A. Kumar, Dynamics of prey–predator model with stage structure in prey including maturation and gestation delays, Nonlinear Dyn. 96 (2019), 2653–2679.