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## AN APPLICATION OF NONSTANDARD FINITE-DIFFERENCE SCHEME FOR SOLVING AUTONOMOUS AND NON-AUTONOMOUS MATHEMATICAL MODEL FOR WOLBACHIA-CARRYING MOSQUITO POPULATION DYNAMICS

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**Abstract.** The use of *Wolbachia* bacterium has been proposed as an alternative strategy against Dengue, Zika and Chikungunya. This requires that *Wolbachia*-carrying mosquitoes should persist in the population. A number of mathematical models has been developed and analysed to understand *Wolbachia*-carrying mosquito population dynamics. However, their analytical solutions are not easily derived and therefore, a numerical approach is required. In this paper, we develop a nonstandard finite difference scheme (NSFDS) for autonomous and non-autonomous mathematical models of *Wolbachia*-carrying mosquito population. The dynamical properties of discrete systems are then analysed. We also perform numerical simulations of the scheme and compare to other traditional methods. We found that the discrete system preserves properties of the continuous models such as equilibrium points and stability.

**Keywords:** nonstandard finite difference scheme; numeric; *Wolbachia*; mathematical model.

**2010 AMS Subject Classification:** 65L06, 65L20.

### 1. INTRODUCTION

The use of *Wolbachia* bacterium has been proposed as a promising strategy against Dengue, Chikungunya, and West-Nile virus. There are two mechanism by which *Wolbachia* can reduce

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the transmission of vector-borne diseases. First, the *Wolbachia* can reduce the mosquito lifespan up to 50% depending in the *Wolbachia* strain [1], which minimises the chance for mosquitoes to transmit virus. Second, *Wolbachia* can reduce the level of virus in the mosquitos [2]. *Wolbachia* causes an effect known as Cytoplasmic Incompatibility (CI). That is, *Wolbachia*-carrying female mosquitoes can reproduce successfully when mating with both non-*Wolbachia* and *Wolbachia* carrying male mosquitoes. On the other hand, non-*Wolbachia* female mosquitoes can only reproduce successfully when mating with non-*Wolbachia* male mosquitoes [2]. This gives a reproductive advantage for *Wolbachia*-carrying female mosquitoes. Furthermore, CI affects the population dynamics of mosquitoes.

Mathematical model is commonly formulated and analysed to understand complex phenomena including population dynamics [3, 4, 5, 6, 7, 8, 9, 10]. The models are generally in the form of system of nonlinear differential equations, which consist of autonomous or non-autonomous models. Furthermore, analytical solutions of the models are not easily determined and therefore, a numerical approach is generally used [11]. To date, a number of mathematical models has been developed to understand the mosquito population dynamics in the presence of *Wolbachia*-carrying mosquitoes [4, 12, 3]. The model are autonomous and non-autonomous with seasonal forcing on adult mosquito death rate [4]. Analytical solutions of the models are not easily determined and therefore numerical solutions are presented. Although, NSFDS has been formulated for other problems [13, 14, 11, 15, 16, 17, 18, 19, 20, 21, 22, 23], to the best of our knowledge, there is a little work on the construction of NSFDS for model of *Wolbachia*-carrying mosquito population dynamics. The aim of this paper is to construct a NSFDS for autonomous and nonautonomous mathematical model of *Wolbachia*-carrying mosquito population and analyse the scheme properties.

The remainder of the paper is organised as follows. Section 2 overviews mathematical model for *Wolbachia*-carrying mosquito population dynamics. Section 3 presents the construction of numerical scheme for autonomous and nonautonomous models and their numerical simulations. Finally, the conclusion is presented.

## 2. MATHEMATICAL MODEL

In this section, we present a mathematical model for *Wolbachia*-carrying mosquito population dynamics. The model has been proposed by Ndi *et al.* [4]. The mosquito population consists of non *Wolbachia* and *Wolbachia*-carrying mosquitoes. The mosquito population is divided into Aquatic ( $A_N$  and  $A_W$ ), Male ( $M_N$  and  $M_W$ ) and female ( $F_N$  and  $F_W$ ) mosquitoes. The subscripts  $N$  and  $W$  are to differentiate between non-*Wolbachia* and *Wolbachia*-carrying mosquitoes respectively.

The *Wolbachia*-carrying female mosquitoes reproduce when they mate with non-*Wolbachia* and *Wolbachia*-carrying male mosquitoes and their growth is limited by carrying capacity,  $K$ , that is

$$(1) \quad \frac{F_W(M_N + M_W)}{F_N + M_N + M_W + F_W} \left( 1 - \frac{A_N + A_W}{K} \right).$$

Non-*Wolbachia* female mosquitoes can only reproduce when they mate with non-*Wolbachia* males. This is governed by

$$(2) \quad \frac{F_N M_N}{F_N + M_N + M_W + F_W} \left( 1 - \frac{A_N + A_W}{K} \right).$$

Equations (1) and (2) capture the effect of Cytoplasmic Incompatibility (CI). Furthermore, the maternal transmission of *Wolbachia*-carrying mosquitoes is not perfect [2] and hence there is a proportion of *Wolbachia*-carrying aquatic mosquitoes  $(1 - \alpha)$  that mature to be non-*Wolbachia* mosquitoes. A proportion of  $\alpha$  mature to be *Wolbachia*-carrying adult mosquitoes. When aquatic mosquitoes mature, a proportion of  $\varepsilon$  is male and the rest  $(1 - \varepsilon)$  is female. The model is governed by the following system of differential equations.

$$\begin{aligned} \frac{dA_N}{dt} &= \rho_N \frac{M_N F_N}{P} \left( 1 - \frac{(A_N + A_W)}{K} \right) - \mu_{NA} A_N - \gamma_N A_N, \\ \frac{dM_N}{dt} &= \varepsilon_N \gamma_N A_N - \mu_N M_N + \varepsilon_{NW} (1 - \alpha_W) \gamma_W A_W, \\ \frac{dF_N}{dt} &= (1 - \varepsilon_N) \gamma_N A_N - \mu_N F_N + (1 - \varepsilon_{NW}) (1 - \alpha_W) \gamma_W A_W, \end{aligned}$$

$$\begin{aligned}
 \frac{dA_W}{dt} &= \rho_W \frac{F_W (M_W + M_N)}{P} \left( 1 - \frac{(A_N + A_W)}{K} \right) - \mu_{WA} A_W - \gamma_W A_W, \\
 (3) \quad \frac{dM_W}{dt} &= \varepsilon_W \alpha_W \gamma_W A_W - \mu_W M_W, \\
 \frac{dF_W}{dt} &= (1 - \varepsilon_W) \alpha_W \gamma_W A_W - \mu_W F_W.
 \end{aligned}$$

where  $P = F_N + M_N + M_W + F_W$ . We nondimensionalise the model by setting the ratio of male and female mosquitoes to be the same,  $\varepsilon = 1/2$  and  $K = 1$ , the model is reduced to

$$\begin{aligned}
 \frac{dA_N}{dt} &= \rho_N \frac{F_N^2}{2(F_N + F_W)} (1 - (A_N + A_W)) - (\mu_{NA} + \gamma_N) A_N, \\
 \frac{dF_N}{dt} &= \frac{\gamma_N A_N}{2} + \frac{(1 - \alpha) \gamma_W A_W}{2} - \mu_N F_N, \\
 (4) \quad \frac{dA_W}{dt} &= \rho_W \frac{F_W}{2} (1 - (A_N + A_W)) - (\mu_{WA} + \gamma_W) A_W, \\
 \frac{dF_W}{dt} &= \frac{\alpha \gamma_W A_W}{2} - \mu_W F_W.
 \end{aligned}$$

In this paper, a NSFDS is constructed for nondimensionalised model (Equation (4)).

Model (4) has four equilibriums, which are mosquito-free equilibrium, only non-*Wolbachia* mosquito surviving equilibrium, only and non-*Wolbachia* and two coexistence equilibriums.

$$E_1 = (0, 0, 0, 0) \quad E_2 = (A_N^*, F_N^*, 0, 0) \quad \text{and} \quad E_{3,4} = (A_N^+, F_N^+, A_W^+, F_W^+)$$

where

$$A_N^* = 1 - \frac{4\mu_N(\mu_{NA} + \gamma_N)}{\rho_N \gamma_N} \quad \text{and} \quad F_N^* = \frac{\rho_N A_N^*}{2\mu_N}.$$

The expression for  $E_{3,4}$  is not analytically tractable and hence is explored numerically as given in Ndi et al. [4]. Furthermore, only one of them is locally stable.

As the mosquito lifespan is seasonal-dependent, a seasonal forcing is applied to the adult mosquito death rate. Furthermore, a sensitivity analysis showed that adult mosquito death rate is also the influential parameter [4]. The seasonal forcing of the mosquito adult death rate is the following

$$\mu_N(t) = \mu_{N0} (1 + \eta \cos(2\pi(t + \psi))) \quad \text{and} \quad \mu_W(t) = r\mu_N(t)$$

$\mu_{N0}$  is the baseline adult death rate,  $\eta$  is the degree of seasonality,  $\psi$  is the phase and  $t$  is time, and  $r$  is the ratio of death rate of *Wolbachia*-carrying adult mosquitoes to non-*Wolbachia* adult mosquitoes.

TABLE 1. Parameters, description, values and sources of Model 4.

Symbol	Description	Value	Unit	Source
$\rho_N$	Non- <i>Wolbachia</i> reproductive rate	1.25	day <sup>-1</sup>	[4]
$\mu_{NA}$	Non- <i>Wolbachia</i> aquatic death rate	1/7.78	day <sup>-1</sup>	[24]
$\gamma_N$	Non- <i>Wolbachia</i> maturation rate	1/6.67	day <sup>-1</sup>	[25]
$\mu_N$	Non- <i>Wolbachia</i> adult death rate	1/14	day <sup>-1</sup>	[24]
$\mu_{WA}$	<i>Wolbachia</i> aquatic death rate	1/7.78	day <sup>-1</sup>	[2, 26]
$\mu_W$	<i>Wolbachia</i> adult death rate	1/7	day <sup>-1</sup>	[2, 26]
$\rho_W$	<i>Wolbachia</i> reproductive rate	$1.25\rho_N$	day <sup>-1</sup>	[4]
$\gamma_W$	<i>Wolbachia</i> maturation rate	1/6.67	day <sup>-1</sup>	[25]
$\alpha$	The proportion of <i>Wolbachia</i> -infected offspring from a <i>Wolbachia</i> -infected mother	0.9	N/A	[2, 26]

### 3. SCHEME CONSTRUCTION

This section deals with numerical construction of the proposed scheme. A nonstandard numerical scheme is based on the two fundamental rules[27, 28, 29], which are

- (1) Nonlocal approximation is used; for example

$$x^2 \rightarrow x^n x^{n+1}$$

$$x^3 \rightarrow 2(x^n)^3 - (x^n)^2 x^{n+1}$$

- (2) Discretisation of derivatives is not traditional and use the nonnegative function  $\phi(h) = h + O(h^2)$

Let us define the derivatives as follows

$$(5) \quad \frac{df(t)}{dt} = \frac{f(t+h) - f(t)}{\phi(h)} + O(\phi(h)) \quad \text{as } h \rightarrow \infty$$

where  $\phi(h)$  is real-valued function on  $\mathfrak{R}$  called denominator function that satisfies the following properties [30]

- (1)  $\phi(h) = h + O(h^2)$ ,
- (2)  $0 < \phi(h) < 1$  for all  $h > 0$ .

The above definition is consistent with the traditional derivatives. There is no general rule for determining  $\phi(h)$  but we can find ideas in [30].

**3.1. Scheme construction for autonomous model.** Let define  $(A_N)^n$ ,  $(F_N)^n$ ,  $(A_W)^n$ ,  $(F_W)^n$  the approximation of  $A_N(nh)$ ,  $F_N(nh)$ ,  $A_W(nh)$ , and  $F_W(nh)$  respectively where  $n = 0, 1, 2, 3, 4, \dots$  and  $h > 0$  the step size of the scheme. The approximation scheme for the Model (4) is given as follows,

$$(6) \quad \begin{aligned} \frac{A_N^{n+1} - A_N^n}{\phi_1(h)} &= \rho_N \frac{F_N^n F_N^{n+1}}{2(F_N^n + F_W^n)} (1 - (A_N^n + A_W^n)) - (\gamma_N + \mu_{NA}) A_N^{n+1}, \\ \frac{F_N^{n+1} - F_N^n}{\phi_2(h)} &= \frac{\gamma_N}{2} A_N^n + \frac{(1 - \alpha)}{2} \gamma_W A_W^n - \mu_N F_N^{n+1}, \\ \frac{A_W^{n+1} - A_W^n}{\phi_3(h)} &= \frac{\rho_W F_W^n}{2} (1 - (A_N^n + A_W^n)) - (\gamma_W + \mu_{WA}) A_W^{n+1}, \\ \frac{F_W^{n+1} - F_W^n}{\phi_4(h)} &= \frac{\alpha \gamma_W}{2} A_W^n - \mu_W F_W^{n+1}. \end{aligned}$$

Rearrange the Equation (6) to obtain

$$(7) \quad \begin{aligned} A_N^{n+1} &= \frac{2(F_N^n + F_W^n)A_N^n + \phi_1(h) [\rho_N F_N^{n+1} F_N^n (1 - (A_N^n + A_W^n))]}{2(F_N^n + F_W^n)(1 + \phi_1(h)(\mu_{NA} + \gamma_N))}, \\ F_N^{n+1} &= \frac{2F_N^n + \phi_2(h) [\gamma_N A_N^n + (1 - \alpha)\gamma_W A_W^n]}{2(1 + \phi_2(h)\mu_N)}, \\ A_W^{n+1} &= \frac{2A_W^n + \phi_3(h)\rho_W F_W^n (1 - (A_N^n + A_W^n))}{2(1 + \phi_3(h)(\rho_W + \mu_{WA}))}, \\ F_W^{n+1} &= \frac{2F_W^n + \phi_4(h)\alpha\gamma_W A_W^n}{2(1 + \phi_4(h)\mu_W)}. \end{aligned}$$

We choose the denominator function as

$$\phi_i(h) = \frac{e^{M_i h} - 1}{M_i}$$

where  $i = 1, 2, 3, 4$  and  $(M_1, M_2, M_3, M_4) = (\gamma_N + \mu_{NA}, \mu_N, \gamma_W + \mu_{WA}, \mu_W)$ .

**3.2. Scheme analysis.** This section presents the equilibrium points of the scheme and the stability of the model. We determine the equilibrium points that satisfy the conditions of  $A_N^{n+1} = A_N^n, F_N^{n+1} = F_N^n, A_W^{n+1} = A_W^n, F_W^{n+1} = F_W^n$ .

$$\begin{aligned}
 A_N^n &= \frac{2(F_N^n + F_W^n)A_N^n + \phi_1(h) [\rho_N F_N^n F_W^n (1 - (A_N^n + A_W^n))]}{2(F_N^n + F_W^n)(1 + \phi_1(h)(\mu_{NA} + \gamma_N))}, \\
 F_N^n &= \frac{2F_N^n + \phi_2(h) [\gamma_N A_N^n + (1 - \alpha)\gamma_W A_W^n]}{2(1 + \phi_2(h)\mu_N)}, \\
 A_W^n &= \frac{2A_W^n + \phi_3(h)\rho_W F_W^n (1 - (A_N^n + A_W^n))}{2(1 + \phi_3(h)(\rho_W + \mu_{WA}))}, \\
 F_W^n &= \frac{2F_W^n + \phi_4(h)\alpha\gamma_W A_W^n}{2(1 + \phi_4(h)\mu_W)}.
 \end{aligned}
 \tag{8}$$

If *Wolbachia*-carrying mosquitoes do not persist in the population, then the equilibrium points is

$$E_2 = \left( 1 - \frac{4\mu_N(\gamma_N + \mu_{NA})}{\rho_N \gamma_N}, \frac{1}{2} \left( \frac{\gamma_N}{\mu_N} - \frac{4(\gamma_N + \mu_{NA})}{\rho_N} \right), 0, 0 \right)$$

The equilibrium  $E_2$  corresponds exactly to that of Model (4).

The other equilibrium points ( $E_3$  and  $E_4$ ) can be found numerically since the mathematical expressions are not analytically tractable.

To determine the stability, we construct a Jacobian matrix and find the eigenvalues. For discrete model, the equilibrium points are stable if the root of characteristic polynomial is less than unity.

**Theorem 1.** Let  $x = (A_N, F_N, A_W, F_W)$  and  $\hat{x} = (\hat{A}_N, \hat{F}_N, \hat{A}_W, \hat{F}_W)$  be an equilibrium point of the difference equation

$$x^{n+1} = F(x^n, x^{n-1}, \dots, x^{n-k}), n = 0, 1, 2, \dots
 \tag{9}$$

where the function  $F$  is a continuously differentiable function defined on some open neighborhood of an equilibrium point  $\hat{x}$ . If all the roots of the characteristic polynomial have absolute value less than one, then the equilibrium point  $\hat{x}$  is locally asymptotically stable. If at least one root of the characteristic polynomial has absolute value greater than one, then the equilibrium point  $\hat{x}$  is unstable.

Constructing the Jacobian matrix  $J(\hat{x})$ , which is

$$\begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ -\frac{\phi_2(h)\gamma_N}{2(1+\phi_2(h)\mu_N)} & 1 - \frac{1}{(1+\phi_2(h)\mu_N)} & -\frac{\phi_2(h)(1-\alpha)\gamma_W}{2(1+\phi_2(h)\mu_N)} & 0 \\ \frac{\phi_3(h)\rho_W F_W}{2(1+\phi_3(h)(\gamma_W+\mu_{WA}))} & 0 & 1 - \frac{-\phi_3(h)\rho_W F_W+2}{2(1+\phi_3(h)(\gamma_W+\mu_{WA}))} & -\frac{\phi_3(h)\rho_W(1-(A_N+A_W))}{2(1+\phi_3(h)(\gamma_W+\mu_{WA}))} \\ 0 & 0 & -\frac{\phi_4(h)\alpha\gamma_W}{2(1+\phi_4(h)\mu_W)} & 1 - \frac{2}{2(1+\phi_4(h)\mu_W)} \end{bmatrix}$$

where

$$J_{11} = 1 - \frac{-\phi_1(h)\hat{F}_N^2 \rho_N + 2(\hat{F}_N + \hat{F}_W)}{2((\hat{F}_N + \hat{F}_W)(1 + \phi_1(h)(\mu_{NA} + \gamma_N)))},$$

$$J_{12} = -D_1 + D_2,$$

$$J_{13} = \frac{\phi_1(h)\rho_N \hat{F}_N^2}{2((\hat{F}_N + \hat{F}_W)(1 + \phi_1(h)(\mu_{NA} + \gamma_N)))},$$

$$J_{14} = -D_3 + D_4,$$

with

$$D_1 = \frac{2\hat{A}_N + 2\phi_1(h)\rho_N \hat{F}_N(1 - \hat{A}_N - \hat{A}_W)}{2((\hat{F}_N + \hat{F}_W)(1 + \phi_1(h)(\mu_{NA} + \gamma_N)))},$$

$$D_2 = \frac{((2(\hat{F}_N + \hat{F}_W))\hat{A}_N + \phi_1(h)\rho_N \hat{F}_N^2(1 - \hat{A}_N - \hat{A}_W))}{2((\hat{F}_N + \hat{F}_W)^2(1 + \phi_1(h)(\mu_{NA} + \gamma_N)))},$$

$$D_3 = \frac{\hat{A}_N}{((\hat{F}_N + \hat{F}_W)(1 + \phi_1(h)(\mu_{NA} + \gamma_N)))},$$

$$D_4 = \frac{((2(\hat{F}_N + \hat{F}_W))\hat{A}_N + \phi_1(h)\rho_N \hat{F}_N^2(1 - \hat{A}_N - \hat{A}_W))}{2((\hat{F}_N + \hat{F}_W)^2(1 + \phi_1(h)(\mu_{NA} + \gamma_N)))}.$$

The eigenvalues of the model is found numerically by solving  $|J - \lambda I|$ . The results are given in Table 2.

TABLE 2. Eigenvalues for  $E_2$  and  $E_3$  for discrete model with different values of  $h$ . The parameter values for  $E_2$  is given in Table 2

$h$	$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ for $E_2$	$(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ for $E_3$
0.1	(0.06702, 0.00388, 0.03487, -0.00068)	(0.0686, 0.0290, 0.0036, 0.0051)
0.01	(0.00686, 0.00039, 0.00356, -0.000069)	(0.00703, 0.00297, 0.00036, 0.00051)
0.001	(0.00068, 0.00003, 0.00035, -0.0000069)	(0.00070, 0.00029, 0.00003, 0.00005)



We show the results that the absolute of the eigenvalues is always less than unity and therefore the equilibrium points are locally stable. Note that the obtained equations should be computed in sequence but it starts from  $F_N^{n+1}$ ,  $A_N^{n+1}$ ,  $A_W^{n+1}$ ,  $F_W^{n+1}$  since the outcome of the  $F_N^{n+1}$  is needed for the calculation of  $A_N^{n+1}$ , which is then used to calculate  $A_W^{n+1}$  and then  $F_W^{n+1}$ . The numerical simulations are given in Section 3.3

**3.3. Numerical simulations.** To illustrate the results, we perform numerical simulations of the scheme and compare to the Euler method. The parameter values are given in Table 1 and the numerical simulations are given in Figures 1 and 2

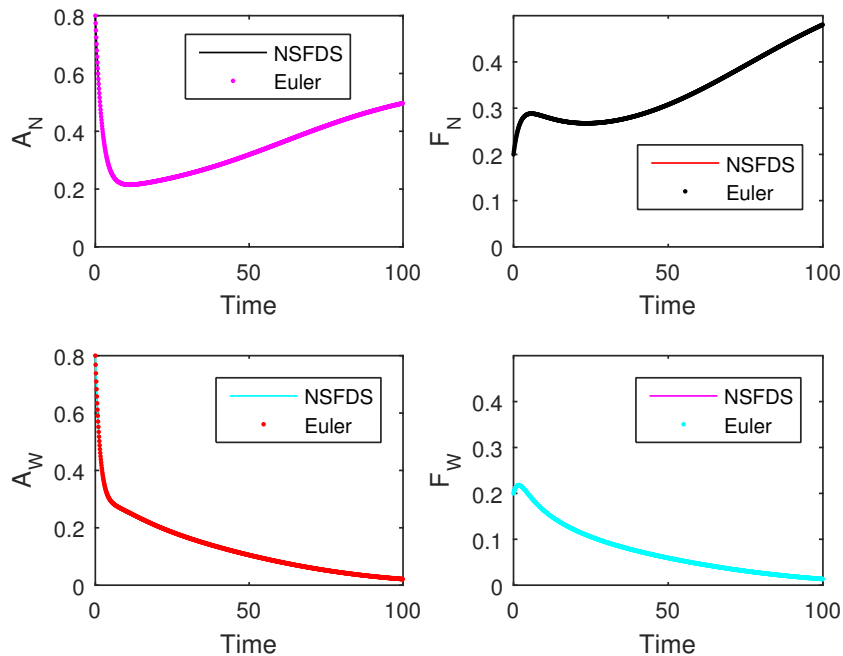


FIGURE 1. Numerical simulations of NSFDS and the Euler method for the non-*Wolbachia* only equilibrium ( $E_2$ ) with  $h = 0.01$ .

Figure 1 shows the numerical simulations for the model for non-*Wolbachia* only steady state. It shows that the numerical simulations using NSFDS and the Euler method are in good agreement.

Figure 2 shows the numerical simulations of the model for the coexistence equilibrium. It shows that NSFDS and the Euler method gives the similar results. We also simulate for a very large time and found that the solutions remain positive.

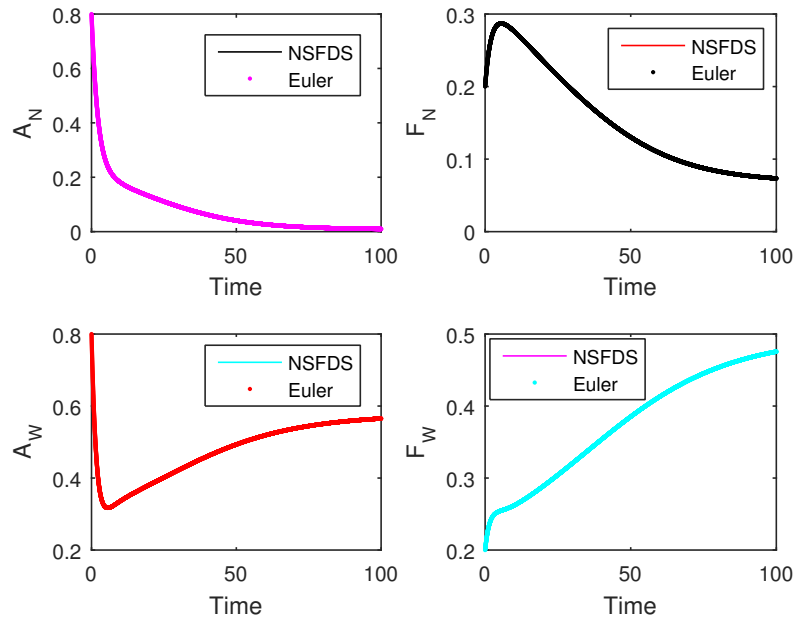


FIGURE 2. Numerical simulations of the NSFDS and the Euler method for co-existence equilibrium ( $E_3$ ) with  $h = 0.01$ .

**3.4. Scheme construction for non-autonomous model.** The numerical scheme for nonautonomous model is the same as that of the autonomous model except for the adult mosquito death rate. The adult mosquito death rate is governed by

$$(10) \quad \mu_N^{n+1} = \mu_{N0}(1 + \eta \cos(2\pi((nh) + \psi))) \quad \text{and} \quad \mu_W^{n+1} = r\mu_N^{n+1},$$

where  $r$  is the ratio of death rate of *Wolbachia*-carrying adult mosquitoes to that of non-*Wolbachia* adult mosquitoes. The numerical scheme for the nonautonomous model is the following.

$$(11) \quad \begin{aligned} \frac{A_N^{n+1} - A_N^n}{\phi_1(h)} &= \rho_N \frac{F_N^n F_N^{n+1}}{2(F_N^n + F_W^n)} (1 - (A_N^n + A_W^n)) - (\gamma_N + \mu_{NA}) A_N^{n+1}, \\ \frac{F_N^{n+1} - F_N^n}{\phi_2(h)} &= \frac{\gamma_N}{2} A_N^n + \frac{(1 - \alpha)}{2} \gamma_W A_W^n - \mu_N^n F_N^{n+1}, \\ \frac{A_W^{n+1} - A_W^n}{\phi_3(h)} &= \frac{\rho_W F_W^n}{2} (1 - (A_N^n + A_W^n)) - (\gamma_W + \mu_{WA}) A_W^{n+1}, \\ \frac{F_W^{n+1} - F_W^n}{\phi_4(h)} &= \frac{\alpha \gamma_W}{2} A_W^n - \mu_W^n F_W^{n+1} \end{aligned}$$

Rearranging Equation (11) gives  $A_N^{n+1}$ ,  $F_N^{n+1}$ ,  $A_W^{n+1}$ ,  $F_W^{n+1}$ , where their expressions are the same as Equation (7) except for  $\mu_N$  and  $\mu_W$ . The  $\mu_N$  and  $\mu_W$  are replaced by Equations 10. This gives the periodic solutions of the model. The simulations of the model is given in Section 3.5. The calculation of the obtained equations begins by calculating the parameter  $\mu_N^{n+1}$ . The following steps are the same as in autonomous model.

**3.5. Numerical Simulations.** In this section, we present numerical simulations of NSFDS for the nonautonomous model and compare the results to that of the Euler method and matlab ode45 routine. The simulation is given in Figure 3.

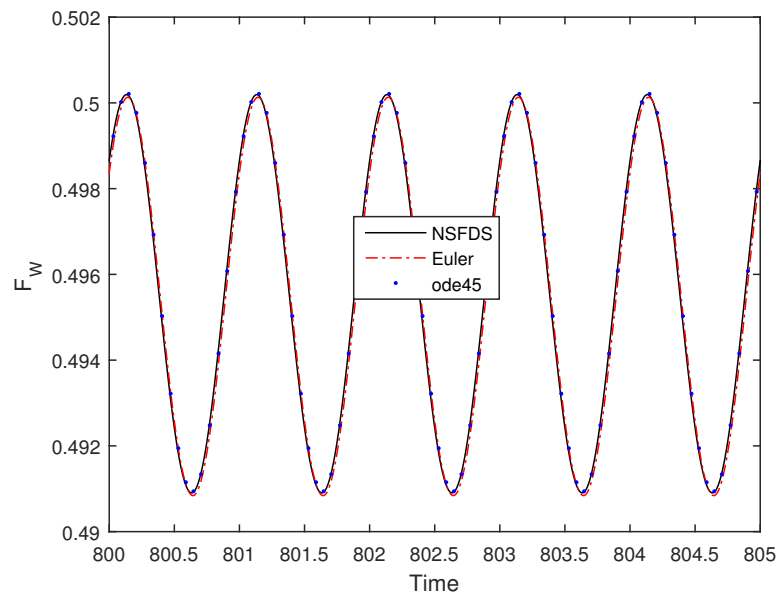


FIGURE 3. Numerical simulations of nonstandard finite difference method and Euler method with  $h = 0.01$  for *Wolbachia*-carrying mosquito population. For the sake of clarity, we only plot from day 800 to 805.

Figure 3 shows that the numerical simulations for the *Wolbachia*-carrying mosquito population using NSFDS, the Euler method and the ode45 routine are in a good agreement. We also conducted numerical simulations for a very large time to examine the positivity of the periodic solutions. We found that the solutions remain positive all the time.

#### 4. CONCLUSIONS

We formulated a nonstandard finite difference schemes for autonomous and non-autonomous model of *Wolbachia*-carrying mosquito population. A seasonal forcing is added in adult mosquito death rate as this is influential parameter of the model. We found that the scheme is locally stable and the numerical simulations are in good agreement with Euler method and Matlab ode45 routine. For large time window, the numerical simulations show that the solutions remain positive as long as  $\phi_i(h) < 1$ . However, the convergence of the scheme depends on the step size  $h$ . Furthermore, one may be interested to use other denominator functions such as hyperbolic tangent function. This may improve the accuracy of the scheme.

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#### CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

#### REFERENCES

- [1] A.A. Hoffmann, B.L. Montgomery, J. Popovici, et al. Successful establishment of *Wolbachia* in *Aedes* populations to suppress dengue transmission, *Nature*. 476 (2011), 454–457.
- [2] T. Walker, P.H. Johnson, L.A. Moreira, I. Iturbe-Ormaetxe, F.D. Frentiu, C.J. McMeniman, Y.S. Leong, Y. Dong, J. Axford, P. Kriesner, A.L. Lloyd, S.A. Ritchie, S.L. O’Neill, A.A. Hoffmann, The wMel *Wolbachia* strain blocks dengue and invades caged *Aedes aegypti* populations, *Nature*. 476 (2011), 450–453.
- [3] X. Zhang, S. Tang, Q. Liu, R.A. Cheke, H. Zhu, Models to assess the effects of non-identical sex ratio augmentations of *Wolbachia* -carrying mosquitoes on the control of dengue disease, *Math. Biosci.* 299 (2018), 58–72.
- [4] M. Z. Ndi, R. I. Hickson, G. N. Mercer, Modelling the introduction of *Wolbachia* into *Aedes aegypti* to reduce dengue transmission, *ANZIAM J.* 53 (2012), 213-227.
- [5] M.Z. Ndi, E. Carnia, A.K. Supriatna, Mathematical models for the spread of rumors: a review. In: *Issues and Trends in Interdisciplinary Behavior and Social Science*, pp. 65–73. CRC Press, Boca Raton, 2018.
- [6] Fatmawati, M.A. Khan, M. Azizah, Windarto, S. Ullah, A fractional model for the dynamics of competition between commercial and rural banks in Indonesia, *Chaos Solitons Fractals*. 122 (2019), 32–46.

- [7] D. Aldila, M.R. Agustin, A Mathematical Model Of Dengue-Chikungunya Co-Infection In A Closed Population, *J. Phys.: Conf. Ser.* 974 (2018), 012001.
- [8] M.Z. Ndi, Modelling the Use of Vaccine and Wolbachia on Dengue Transmission Dynamics, *TropicalMed.* 5 (2020), 78.
- [9] M.Z. Ndi, A.R. Mage, J.J. Messakh, B.S. Djahi, Optimal vaccination strategy for dengue transmission in Kupang city, Indonesia, *Heliyon.* 6 (2020), e05345.
- [10] M.Z. Ndi, Z. Amarti, E.D. Wiraningsih, A.K. Supriatna, Rabies epidemic model with uncertainty in parameters: crisp and fuzzy approaches, *IOP Conf. Ser.: Mater. Sci. Eng.* 332 (2018), 012031.
- [11] A. Suryanto, W. M. Kusumawinahyu, I. Darti, I. Yanti, Dynamically consistent discrete epidemic model with modified saturated incidence rate, *Comput. Appl. Math.* 32 (2013), 373–383.
- [12] M. H. T. Chan, P. S. Kim, Modelling a Wolbachia invasion using a slow fast dispersal reaction diffusion approach, *Bull. Math. Biol.* 75 (9) (2013), 1501–1523.
- [13] J. Xu, J. Hou, Y. Geng, S. Zhang, Dynamic consistent NSFD scheme for a viral infection model with cellular infection and general nonlinear incidence, *Adv. Differ. Equ.* 2018 (2018), 108.
- [14] A. Korpusik, A nonstandard finite difference scheme for a basic model of cellular immune response to viral infection, *Commun. Nonlinear Sci. Numer. Simul.* 43 (2017), 369 – 384.
- [15] M. Yakıt Ongun, İ. Turhan, A Numerical Comparison for a Discrete HIV Infection of CD4 + T-Cell Model Derived from Nonstandard Numerical Scheme, *Journal of Applied Mathematics.* 2013 (2013), 375094.
- [16] L. Jodar, R. J. Villanueva, A. J. Arenas, G. C. Gonzalez, Nonstandard numerical methods for a mathematical model for influenza disease, *Math. Computers Simul.* 79 (3) (2008), 622 – 633.
- [17] A. J. Arenas, J. A. Morano, J. C. Cortés, Non-standard numerical method for a mathematical model of RSV epidemiological transmission, *Computers Math. Appl.* 56 (3) (2008), 670 – 678.
- [18] R.J. Villanueva, A.J. Arenas, G. González-Parra, A Nonstandard Dynamically Consistent Numerical Scheme Applied to Obesity Dynamics, *J. Appl. Math.* 2008 (2008), 640154.
- [19] D. T. Wood, H. V. Kojouharov, D. T. Dimitrov, Universal approaches to approximate biological systems with nonstandard finite difference methods, *Math. Computers Simul.* 133 (2017), 337 – 350.
- [20] M.T. Hoang, O.F. Egbeowo, Nonstandard finite difference schemes for solving an SIS epidemic model with standard incidence, *Rend. Circ. Mat. Palermo, II. Ser.* 69 (2020), 753–769.
- [21] I. Darti, A. Suryanto, M. Hartono, Global stability of a discrete sir epidemic model with saturated incidence rate and death induced by the disease, *Commun. Math. Biol. Neurosci.* 2020 (2020), 33.
- [22] R. Anguelov, T. Berge, M. Chapwanya, J.K. Djoko, P. Kama, J.M.-S. Lubuma, Y. Terefe, Nonstandard finite difference method revisited and application to the Ebola virus disease transmission dynamics, *J. Differ. Equ. Appl.* 26 (2020), 818–854.

- [23] M.T. Hoang, O.F. Egbelowo, Nonstandard finite difference schemes for solving an SIS epidemic model with standard incidence, *Rend. Circ. Mat. Palermo, II. Ser.* 69 (2020), 753–769.
- [24] H. L. Yang, M. L. G. Macoris, K. C. Galvani, M. T. C. Andrighetti, Follow up estimation of *Aedes aegypti* entomological parameters and mathematical modellings, *Biosystems*, 103 (3) (2011), 360–371.
- [25] N. A. Maidana, H. M. Yang, Describing the geographic spread of dengue disease by traveling waves, *Math. Biosci.* 215 (2008), 64 – 77.
- [26] H. L. Yeap, P. Mee, T. Walker, et al. Dynamics of the “Popcorn” *Wolbachia* infection in outbred *Aedes aegypti* informs prospects for mosquito vector control, *Genetics*, 187 (2) (2011), 583–595.
- [27] R. Anguelov, J. M.-S. Lubuma, Contributions to the mathematics of the nonstandard finite difference method and applications, *Numer. Meth. Part. Differ. Equ.* 17 (5) (2001), 518–543.
- [28] R. E. Mickens, *Applications of nonstandard finite difference schemes*, World Scientific, Singapore, 2000.
- [29] R. E. Mickens, *Nonstandard finite difference models of differential equations*, World Scientific, Singapore, 1994.
- [30] R. E. Mickens, Calculation of denominator functions for nonstandard finite difference schemes for differential equations satisfying a positivity condition, *Numer. Meth. Part. Differ. Equ.* 23 (3) (2007), 672–691.