Available online at http://scik.org
J. Math. Comput. Sci. 11 (2021), No. 3, 3223-3231
https://doi.org/10.28919/jmcs/5297
ISSN: 1927-5307

STABILITY OF VARIOUS FUNCTIONAL EQUATION IN COMPLETE METRIC SPACE

RAJEEV KUMAR*

Department of Mathematics, Pt. Neki Ram Sharma Govt. College, Rohtak-124001, Haryana, India

Abstract. This paper discusses Hyers-Ulam stability for functional equation on a complete metric space and also discusses stability result for one variable functional equation i.e., Gamma functional equation on complete metric group.

Keywords: iterative functional equation; fuzzy functional equation; gamma functional equation; metric group.

2010 AMS Subject Classification: Primary 39B12, 39B52, 39B82; Secondary 46C05.

1. Introduction

Hyers-Ulam stability is a basic sense of stability for functional equation. Usually the functional equation

$$H_1(\psi) = H_2(\psi) \tag{1.1}$$

is said to have the Hyers-Ulam stability if for an approximate solution ψ_S such that

$$|H_1(\psi_S)(l) = H_2(\psi_S)(l)| \leq \delta \tag{1.2}$$

*Corresponding author

E-mail address: bhar999999@yahoo.com

Received December 03, 2020

for some fixed constant $\delta \geqslant 0$ there exist a solution ψ of equation (1.1) such that

$$|\psi(l) - \psi_S(l)| \leq \varepsilon \tag{1.3}$$

for some positive constant $\in$. Sometimes we call ψ_S a $\delta-$ approximate solution of equation (1.1) and ψ is $\varepsilon-$ close to ψ_S.

Such an idea of stability was given in 1940 by Ulam [14] for Cauchy equation

$$\psi(l+m) = \psi(l) + \psi(m)$$

and his problem was solved by Hyers [4] in 1941.

Later, the Hyers-Ulam stability was studied extensively ([1-3]). This concept is also generalized in [6, 11].

In 1965, Zadeh [15] initialized the theory of fuzzy sets. Through the classical learning of Zadeh, there has been a large work to find fuzzy illustration of academic notions.

Iterative functional equation given in [5, 7, 16], is one of most important form of functional equations and also referred to as equation of rank one, in which iterates of the unknown function are linked in a linear combination. In energetic systems, many problems like embedding flows and dynamics of a quadratic mapping can be minimized to an iterative equation. We mention here some classical functional equation as

- Gamma Functional Equation

$$f(l+1) = (l+1)f(l)$$

In section 2, We deal with the Hyers-Ulam stability of the Fuzzy functional equation

$$\psi(l) = a(l)F(l, \psi(l)) \tag{1.4}$$

and this equation was firstly discussed by P.V. Subrahmanyam and S.K.Sudarsanam [12] in 2011. In section 3, we deal with the Hyers-Ulam stability of the functional equation

$$\psi(l+1) = l\psi(l) \tag{1.5}$$

on complete metric group (G, ρ) where $\psi : S \to G$ is the unknown function. And this equation was discussed by T. Trif [13] in 2002.

2. Stability of Fuzzy Functional Equation

Theorem 2.1. *Let (L,ρ) be a Complete metric space and $F : S\times L \to L$ be a mapping where S be a non empty set. Suppose that*

$$\rho(aF(l,u),aF(l,v)) \leq a\lambda\rho(u,v), \quad 0\leq\lambda<1, \tag{2.1}$$

and

$$\psi_S : S \to L$$

for all $l \in S$ and for all $u,v \in L$ such that

$$\rho(\psi_S(l),a(l)F(l,\psi_S(l))) \leq \delta \tag{2.2}$$

for all $l\in S$ and $\delta > 0$.

Then there is a unique function $\psi : S \to L$ such that $\psi(l) = a(l)F(l,\psi(l))$ for all $l \in S$ and

$$\rho(\psi(l),\psi_S(l)) \leq \frac{\delta}{1-\lambda} \tag{2.3}$$

for all $l \in S$.

Proof. Let $Y = b : \{S \to L; sup\{\rho(b(l),\psi_S(l)), l \in S\} < \infty\}$.
For $b,c \in Y$ define

$$d(b,c) = sup\{\rho(b(l),c(l)); l\in S\}$$

.

Then $\psi_S \in Y$, d is a metric on Y and convergence with respect to d means uniform convergence on S with respect to ρ implies the completeness of Y with respect to d.

For $b \in Y$ define $T(b) : S \to L$ by

$$T(b)(l) = a(l)F(l,b(l)),\ l\in S.$$

Then T maps Y into Y. If $b,c\in Y$ then for all $l\in S$,

$$\rho(T(b)(l),T(c)(l)) = \rho(aF(l,b(l)),aF(l,c(l)))$$
$$\leq \lambda a\rho(b(l),c(l))$$

$$\leq \lambda a d(b,c) \tag{2.4}$$

by (1.5).Thus,

$$d(T(b),T(c)) \leq \lambda a d(b,c), \qquad \forall\, b,c \in Y.$$

According to the well-known proof of Banach's fixed point theorem, there exist a unique ψ in Y such that $\psi = T(\psi)$ and

$$d(\psi,\psi_S) \leq d(\psi,T(\psi_S)) + d(T(\psi_S),\psi_S)$$

$$\leq d(T(\psi),T(\psi_S)) + \delta$$

$$\leq \lambda a d(\psi,\psi_S) + \delta \quad ,$$

so that $d(\psi,\psi_S) \to \dfrac{\delta}{1-\lambda}$. That is , there exists a unique solution ψ of equation (1.4) such that the inequality (2.2) hold.

□

An example of functional equation

$$\psi(l^5) = \psi(f(l)) \tag{2.5}$$

Applying above theorem, we can give the Hyers-Ulam stability of the equation.

Theorem 2.2. *: Suppose that $f : R \to R$ and $\psi_S : R \to [1,+\infty)$ satisfies*

$$|\psi_S(l) - \psi_S(f(l))^{\frac{1}{5}}| \leq \delta,\ \forall\, l \in R,$$

for a constant $\delta > 0$.
Then there is a unique solution $\psi : R \to [1,+\infty)$ of equation (2.5) such that

$$|\psi(l) - \psi_S(l)| \leq \frac{5}{4}\delta$$

for all $l \in R$.

Proof. : Consider the equivalent form of equation (2.5)

$$\psi(l) = \psi(f(l))^{\frac{1}{5}} \tag{2.6}$$

Regard $[1,+\infty)$ as a complete metric space and let $F(l,u) = u^{\frac{1}{5}}$ where $l \in R$, $u \geq 1$. Then F maps $R \times [1,+\infty)$. By the mean value theorem,

$$|F(l,u) - F(l,v)| = |u^{\frac{1}{5}} - v^{\frac{1}{5}} \leq \frac{1}{5}|u-v|$$

for all $l \in R$ and for all $u, v \geq 1$. Thus, the Hyers-Ulam stability of the equation (2.6) is implied by above theorem and the result is proved.

□

3. Stability of Gamma Functional Equation

In this part, Let R_+^S be the class of all functions $\varepsilon : S \to R_+$ where S be a non empty set and (G,ρ) be a complete metric group with the metric ρ invariant to left translations, i.e.,

$$\rho(l.m, l.n) = \rho(m,n), \ \forall \ l,m,n \in G. \tag{3.1}$$

An example of metric invariant to left translations is the metric induced by a norm.

Definition 3.1. Let $C \subseteq R_+^S$ be nonempty and $\top$ be an operator mapping C *into* R_+^S. We say that the equation (1.5) is $\top$ - stable provided for every $\varepsilon \in C$ and with

$$\rho(\psi(l+1), l\psi(l)) \leq \varepsilon(l), \ \forall \ l \in S$$

there exists a (unique, respectively) solution $\psi_o : S \to G$ of the equation (1.5) such that

$$\rho(\psi(l), \psi_o(l)) \leq \top\varepsilon(l), \ \forall \ l \in S.$$

If ε is a constant function then the equation (1.5) is said to be stable in Hyers-Ulam sense.

Theorem 3.2. *Let $\varepsilon : S \to R_+$ be a function with the property*

$$\Sigma_{q=0}^{\infty} \ \varepsilon((l+1)^q) = \Psi(l), \ \forall \ l \in S, \tag{3.2}$$

where $\Psi : S \to R_+$. Then for every function $\psi : S \to G$ satisfying the inequality

$$\rho(\psi(l+1), l\psi(l)) \leq \varepsilon(l), \ \forall \ l \in S, \tag{3.3}$$

there exists a unique solution $\psi_o : S \to G$ of the functional equation (1.5) such that

$$\rho(\psi(l),\psi_o(l)) \leq \Psi(l),\ \forall\, l \in S. \tag{3.4}$$

Proof. Existence. Let $\psi : S \to G$ be a function satisfying (3.3). Then the following relation holds :

$$\rho(\psi(l+1)^q, \Pi_{k=1}^{q}(l+1)^{k-1}.\psi(l)) \leq \Sigma_{k=1}^{q}\varepsilon((l+1)^{k-1}) \tag{3.5}$$

for all $l \in S$ and $q \in N$. We prove (3.5) by induction on q. Since the group (G,ρ) is not generally commutative, we let

$$\Pi_{k=p}^{q} t_k = t_k.t_{k-1}...t_p,$$

where $t_k \in G$ for $p \leq k \leq q$.

For $q = 1$ the relation (3.5) holds in view of (3.3). We suppose that (3.5) holds for some $q \in N$ and for all $l \in S$, and we prove that

$$\rho(\psi((l+1)^{q+1}), \Pi_{k=1}^{q+1}(l+1)^{k-1}.\psi(l)) \leq \Sigma_{k=1}^{q+1}\varepsilon((l+1)^{k-1}),\ l \in S.$$

Indeed, it follows from (3.3) and (3.5) that

$$\begin{aligned}\rho(\psi((l+1)^{q+1}), \Pi_{k=1}^{q+1}(l+1)^{k-1}.\psi(l)) &\leq \rho(\psi((l+1)^{q+1}), (l+1)^q.\psi((l+1)^q)) \\ &\quad + \rho((l+1)^q.\psi((l+1)^q), \Pi_{k=1}^{q+1}(l+1)^{k-1}.\psi(l)) \\ &\leq \varepsilon((l+1)^q) + \rho(\psi((l+1)^q), \Pi_{k=1}^{q}((l+1)^{k-1}).\psi(l)) \\ &\leq \Sigma_{k=1}^{q+1}\varepsilon((l+1)^{k-1}),\ l \in S.\end{aligned}$$

Hence (3.5) holds for all $l \in S$ and $q \in N$.

Now let $(\varepsilon_q)_{q\geq 1}$ be the sequence of functions defined by

$$\varepsilon_q(l) = (\Pi_{k=1}^{q}(l+1)^{k-1})^{-1}.\psi((l+1)^q),\ l \in S, q \in N \tag{3.6}$$

We prove that $(\varepsilon_q)_{q\geq 1}$ is a Cauchy sequence in (G,ρ) for all $l \in S$, where t^{-1} means the inverse of the element t in the group G. Using (3.1) and (3.5), we have

$$\rho(\varepsilon_{q+p}(l),\varepsilon_q(l)) = \rho((\Pi_{k=1}^{q+p}(l+1)^{k-1})^{-1}.\psi((l+1)^{q+p}), (\Pi_{k=1}^{q}(l+1)^{k-1})^{-1}.\psi((l+1)^q))$$

$$= \rho((\Pi_{k=q+1}^{q+p}(l+1)^{k-1})^{-1}.\psi((l+1)^{q+p}), \psi((l+1)^{q}))$$

$$\leq \Sigma_{k=1}^{p}\varepsilon((l+1)^{k-1}.(l+1)^{q}) \leq \Sigma_{k=0}^{\infty}\varepsilon((l+1)^{q+k}) \tag{3.7}$$

for $l \in S$ and $q, p \in N$.

Now $r_q(l) = \Sigma_{k=0}^{\infty}\varepsilon((l+1)^{q+k}),\ q \in N$, is the remainder of order q of the convergent series (3.2), so $lim_{q\to\infty} r_q(l) = 0$ for all $l \in S$. We conclude that $(\varepsilon_q)_{q\geq 1}$is a Cauchy sequence, it is convergent since G is a complete metric group. Define the function ψ_o by

$$\psi_o(l) = lim_{q\to\infty}\varepsilon_q(l),\ l \in S.$$

The relation (3.7), for $p = 1$, leads to

$$\rho(\varepsilon_{q+1}(l), \varepsilon_q(l)) \leq \Sigma_{k=0}^{\infty}\varepsilon((l+1)^{q+k}),\ l \in S, q \in N. \tag{3.8}$$

Taking account of $\varepsilon_{q+1}(l) = l^{-1}.\varepsilon_q(l+1)$ and letting $q \to \infty$ in (3.8) it follows that

$$\rho(l^{-1}.\psi_o(l+1), \psi_o(l)) = 0$$

which is equivalent to $\psi_o(l+1) = l\psi_o(l),\ l \in S$, i.e., ψ_o is a solution of the equation (1.5).

On the other hand, the relations (3.1) and (3.5) lead to

$$\rho(\varepsilon_q(l), \psi(l)) \leq \Sigma_{k=1}^{q}\varepsilon((l+1)^{k-1}) \tag{3.9}$$

for all $l \in S$ and $q \in N$, therefore letting $q \to \infty$ in (3.9), we get

$$\rho(\psi_o(l), \psi(l)) \leq \Psi(l),$$

which completes the proof of the existence.

Uniqueness. Assume that for a function ψ satisfying (3.3) there exists two solutions $\psi_1.\psi_2$ of the equation (1.5) satisfying

$$\rho(\psi(l), \psi_i(l)) \leq \Psi(l),\ l \in S,\ i \in 1,2$$

and $\psi_1 \neq \psi_2$. Taking into account that ψ_1, ψ_2 satisfy (1.5), it follows easily that

$$\psi_i((l+1)^q) = \Pi_{k=1}^{q}((l+1)^{k-1}).\psi_i(l),\ l \in S, q \in N,\ i \in 1,2$$

and hence

$$\rho(\psi_1(l),\psi_2(l)) = \rho((\Pi_{k=1}^{q}(l+1)^{k-1})^{-1}.\psi_1((l+1)^q),(\Pi_{k=1}^{q}(l+1)^{k-1})^{-1}.\psi_2((l+1)^q))$$

$$= \rho(\psi_1((l+1)^q),\psi_2((l+1)^q))$$

$$\leq \rho(\psi_1((l+1)^q),\psi((l+1)^q)) + \rho(\psi((l+1)^q),\psi_2((l+1)^q))$$

$$\leq 2\Psi((l+1)^q),\ l \in S, q \in N.$$

Since $lim_{q\to\infty}\Psi((l+1)^q) = lim_{q\to\infty}r_q(l) = 0,\ l \in S$ it follows that $\psi_1(l) = \psi_2(l)$, which completes the proof.

□

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] K. Baron, W. Jarczyk, Recent results on functional equations in a single variable, perspectives and open problems, Aequationes Math. 61(1-2) (2001), 1-48.

[2] A. Bodaghi, Intuitionistic fuzzy stability of the generalized forms of cubic and quartic functional equations, J. Intell. Fuzzy Syst. 30 (2016), 2309-2317.

[3] C. Borelli, On Hyers–Ulam stability of Hosszú's functional equation, Result. Math. 26 (1994), 221-224.

[4] D.H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. 27 (1941), 222–224.

[5] M. De la Sen, A. Ibeas, On the global stability of an iterative scheme in a probabilistic Menger space, J. Inequal. Appl. 2015 (2015), 243.

[6] G.H. Kim, Y.-H. Lee, Stability of an additive-quadratic-quartic functional equation, Demonstr. Math. 53 (2020), 1–7.

[7] M. Kuczma, B. Choczewski, R. Ger, Iterative functional equations, Cambridge University Press, Cambridge, 1990.

[8] Y.H. Lee, K.W. Jun, A generalization of the Hyers–Ulam–Rassias stability of Jensen's equation J. Math. Anal. Appl., 238 (1999), 305-315.

[9] M. Mursaleen, S.A. Mohiuddine, K.J. Ansari, On the stability of fuzzy set-valued functional equations, Cogent Math. 4 (2017), 128-157.

[10] K. Nikodem, The stability of the Pexider equations, Ann. Math. Sil. 5 (1991), 91-93.

[11] C. Park, A. Bodaghi, Two multi-cubic functional equations and some results on the stability in modular spaces, J. Ineq. Appl. 2020 (2020), 6.

[12] P.V. Subrahmanyam, S.K. Sudarsanam, On the fuzzy functional equation $x(t) = a(t)h(t,x(t)) + y(t)$, Int. J. Unc. Fuzz. Knowl. Based Syst. 2 (1994), 197–204.

[13] T. Trif, On the stability of a general gamma-type functional equation, Publ. Math. Debrecen. 60 (2002), 47-61.

[14] S.M. Ulam, A Collection of Mathematical Problems, Interscience Publishers, New York, (1940).

[15] L.A. Zadeh, Fuzzy sets, Inform. Control, 8 (1965), 338-353.

[16] W. Zhang, Discussion on the differentiable solutions of the iterated equation $\Sigma_{i=1}^{n} \lambda_i f^i(x) = F(x)$, Nonlinear Anal., Theory Meth. Appl. 15(4) (1990), 387-398.